LPTENS-93/20
RU-93-17

Large N Phase Transition in Continuum QCD_2

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We compute the exact partition function for pure continuous Yang-Mills theory on the two-sphere in the large N limit, and find that it exhibits a large N third order phase transition with respect to the area A of the sphere. The weak coupling (small A) partition function is trivial, while in the strong coupling phase (large A) it is expressed in terms of elliptic integrals. We expand the strong coupling result in a double power series in $e^{-g^2 A}$ and $g^2 A$ and show that the terms are the weighted sums of branched coverings proposed by Gross and Taylor. The Wilson loop in the weak coupling phase does not show the simple area law.

We discuss some consequences for higher dimensions.

May 11, 1993

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It is an old and famous idea, that large N QCD is equivalent to a string theory. [1,2] Many approaches have been tried to make this precise. One of the most promising is to interpret the diagrams of the strong coupling expansion of the Euclidean lattice theory as string world sheets. This combines the great advantage of the finite N strong coupling expansion, that confinement is already present at leading order, with the great advantage of large N, that we have a free string theory at leading order. This expansion was proposed for any dimension in [3] and elaborated in [4,5].

Early hopes for the use of the strong coupling expansion were dashed by the discovery of Gross and Witten and of Wadia [6,7], that even the simplest integrals involving the Wilson action, such as

$$\int dU \; e^{N/g^2 \mathrm{tr}\ (U+U^+)} \ \mathrm{tr}\ U \tag{1}$$

(in two-dimensional Yang-Mills theory, this is a Wilson loop enclosing a single plaquette of area 1), were non-analytic in the coupling constant (see [8] for the complete treatment of the problem).

Such behavior is possible because we have $O(N)$ degrees of freedom, and these integrals are dominated by a saddle point. Just as for the infinite volume limit in statistical mechanics, several saddle points may exist, and the large N limit picks the one of lowest action for a given value of the coupling. The consequence in higher dimensions was that integration formulas used in developing the strong coupling expansion were only valid down to a critical g_c^2. The expansion had no validity in the physical regime of weak bare coupling.

It was not then clear whether this transition is a fundamental barrier to combining strong coupling and large N or just a reason not to use the Wilson action in this context. Although we now know much more about large N phase transitions, after having studied them in depth for their application to two-dimensional gravity and string theory (this one was studied in [9]), this point is still not clear. It is therefore interesting to modify the action and see if this modifies or eliminates the transition. The only obvious constraint is the very weak one that for small lattice spacing, the tr F^2 term should be present in the action, with higher dimension terms not unnaturally enhanced.

The point was made by Migdal [10] that in two dimensions, one could exactly integrate out a link common to two plaquettes, and compute an exact renormalization group transformation. The Wilson action is not a fixed point of this transformation; it and generic nearby actions flow to the "heat kernel action," more properly described as a Boltzmann weight for a plaquette with holonomy U:

$$Z_{hk}(U;g^2A) = \sum_R \dim R e^{-g^2AC_2(R)/2N}\chi_R(U) \tag{2}$$

which is the heat kernel on the group manifold $G(U,1;\Delta t = g^2A/2N)$. This was also proposed in [11] for its other nice properties, among them being that it gives exact equivalence between Euclidean and Hamiltonian lattice formulations, and that using the heat kernel action, the above Wilson loop expectation value suffers no large N transition. From this one might conjecture that the transition was a lattice artifact, and that using the heat kernel action gives us continuum answers with no transition. (Of course this action, used for the plaquettes of a regular lattice, is not an RG fixed point in $D > 2$, so even if it worked for $D = 2$ the conjecture would not be proven.)

Already in two dimensions the continuum QCD observables, Wilson loop averages:

$$W(C_1, C_2, ..., C_k) =< \prod_{j=1}^{k} \text{tr } P\exp[i\oint_{C_j} dx_\mu A_\mu(x)] > \tag{3}$$

(where C_j are arbitrarily intersecting and selfintersecting contours on the infinite plane), show a great deal of structure. These were first computed by [12] for the $U(N)$ gauge group with $N \to \infty$, and generalized to any N in [13] and to the lattice version of the theory [14] (see also [15,16]). This calculation was based on the renormalized two-dimensional version of the Makeenko-Migdal loop equations [17] established in [12] and (in modern language) on the invariance of the results under area preserving diffeomorphisms.

The most striking feature of the Wilson averages in the limit $N \to \infty$, observed in [12] was their "stringy" character: each of them was shown to be a sum over all possible surfaces of the minimal area (without folds) spanned on the contour ("soap films"), of an exponent of minus area of the surface (area law) times some polynomial of the areas of domains forming this surface. The geometrical interpretation of the polynomials was not known at the time. It became clearer from the paper of I. Kostov [5] (see Appendix A there) where it was demonstrated that they come from the statistics of surfaces (coverings with boundary) having branch points and cuts, connecting various sheets of a surface. For recent developments in this direction see [18].

Another nontrivial quantity is the partition function on a two dimensional compact manifold of area A and genus G. A formula in terms of the sum over representations of the gauge group,

$$Z_G(g^2A) = \sum_R (\dim R)^{2-2G} e^{-g^2AC_2(R)/2N}. \tag{4}$$

was found in [20]. The sum for the case of the group $U(N)$, say, goes over all Young tableaux characterized by the components of the highest weight $\{n_1, n_2, ..., n_N\}$ which are the integers obeying the inequality:

$$\infty \geq n_1 \geq ... \geq n_N \geq -\infty \tag{5}$$

So we see that it is still a complicated multiple sum which takes some effort to calculate in particular cases.

A nice interpretation of this partition functions in terms of minimal coverings, similar to those of [12] and [5] , was given by Gross and Taylor. [21,22] It was noticed that the partition function can be written in terms of a sum over minimal coverings of a manifold of a genus G and area A, where different sheets of a covering are glued together with elements called branch points, tubes and omega-points. Using an inequality known by mathematicians, about the possibility of minimal covering of the manifold of a genus G by a surface of a genus g, Gross found that many terms of the $1/N$ topological expansion in the free energy $F(A,N) = \frac{1}{N^2}\log Z(A,N)$ are equal to zero. So, for $G = 1$ he found that the $O(N^2)$ (spherical world-sheet) contribution to $F(A,N)$ is zero. However, for the next order (torus topology of coverings) the sum over minimal coverings is infinite and can be given in terms of the Dedekind function:

$$F = -2\log\eta(iA/4\pi) = -\frac{A}{24} - 2\sum_{n\geq 1}\log(1 - e^{-nA/2}) \tag{6}$$

(the constant is of course a choice of ground state energy.)

For the spherical topology of the space $G = 0$, the result is more complicated to obtain. Its geometrical interpretation contains all of the additional elements mentioned above, and a purely geometrical derivation looks tricky.

In this paper we present the explicit result for the leading order (planar) contribution to the free energy of the Yang-Mills theory on the two-dimensional sphere. For large area of the sphere (compared to $1/g^2$) it nicely fits the interpretation in terms of minimal coverings, down to the phase transition point $g^2 A_{crit} = \pi^2$, where the sum over coverings is divergent. In the phase of small $g^2 A$ the result is trivial.

The partition function is

$$Z_{G=0}(A, N=\infty) = \exp[N^2 F(A)] = \sum_R (\dim R)^2 e^{-\frac{A}{2N} C_2(R)} \tag{7}$$

where we measure the area A in units of $1/g^2$, and for the group $U(N)$ we have

$$\begin{aligned} C_2(R) &= \sum_{i=1}^{N} n_i(n_i - 2i + N + 1) \\ \dim R &= \prod_{i>j} (1 - \frac{n_i - n_j}{i - j}) \end{aligned} \tag{8}$$

and the sum over the representations R has to be understood as a multiple sum over N integer variables $n_1, ..., n_N$ obeying the inequality (5). † Now, in the large N limit, nothing prevents us from using continuum variables:

$$n(x) = \frac{n_i}{N}, \qquad x = \frac{i}{N}, \tag{9}$$

obeying now the inequality:

$$n(x) \geq n(y), \quad \text{if} \quad x \leq y. \tag{10}$$

It is convenient to change the variable to

$$h(x) = -n(x) + x - 1/2 \tag{11}$$

and to write formally the following functional integral representation for the partition function:

$$Z_0(A) = \int Dh(x) \exp -N^2 S_{eff}[h(x)] \tag{12}$$

where

$$S_{eff}[h(x)] = -\int_0^1 dx \int_0^1 dy \log|h(x) - h(y)| + \frac{A}{2} \int_0^1 dx h^2(x) - A/24 \tag{13}$$

† Strictly speaking these are only representations whose $U(1)$ charge is correlated with the charge under the center of $SU(N)$, in other words of $SU(N) \times U(1)/\mathbb{Z}_N$. Since a single representation will dominate in the final answer, it is also correct for $U(N)$.

and $h(x)$ obeys, according to (5) and (11) , the inequality

$$\frac{h(x)-h(y)}{x-y} \geq 1. \tag{14}$$

One of the main observations of this paper is the need to respect this condition in calculating (12), which will lead to nontrivial consequences such as a new large N phase transition and the existence of the large area (strong coupling) phase. If we introduce the density of the boxes in the Young tableau in terms of the variable h

$$u(h) = \frac{\partial x(h)}{\partial h} \tag{15}$$

with the normalization

$$\int dh\ u(h) = 1 \tag{16}$$

the condition (14) can be simply rewritten as

$$u(h) \leq 1, \quad \text{for any } h \tag{17}$$

Since we have a large parameter N^2 in front of the S_{eff} in (12) we can try to apply the saddle point approximation, which means that we have to solve the equation on $h(x)$

$$\frac{\delta S_{eff}[h(x)]}{\delta h(x)} = 0 \tag{18}$$

Let us ignore for a moment the constraint (17) . This will lead us immediately to the integral equation

$$\frac{A}{2}h = P\int \frac{ds u(s)}{h-s} \tag{19}$$

which is precisely the same as for the distribution of the eigenvalues in the hermitean gaussian matrix model. The solution is the well-known semi-circle law of Wigner:

$$u(h) = \frac{A}{2\pi}\sqrt{\frac{4}{A}-h^2} \tag{20}$$

From here we obtain immediately for the derivative of the free energy with respect to the area A

$$F'(A) = -\frac{\partial S_{eff}[h^*]}{\partial A} =< \frac{\mathrm{tr}}{2N}h^2 > -\frac{1}{24} = \frac{1}{2A} - \frac{1}{24} \tag{21}$$

or

$$F(A) = \frac{1}{24}A - \frac{1}{2}\log A \tag{22}$$

This result is already clear from (12) where the $\log A$ term can be obtained by simple rescaling of the continuous field h by $\sqrt{A}$, after which the gaussian integral will not depend on A. This result was obtained by the same reasoning in [23], where the inequality (17) was ignored, and hence, the existence of the second, strong coupling phase was not noticed.

Let us explain this result from another point of view. Clearly from (2), we have

$$Z_{G=0}(A) = Z_{hk}(U=1;\ A) = G(U=1,\ U'=1;\ A/2N). \tag{23}$$

For small A, we really are interested in the small time asymptotics of the heat kernel on the group manifold. These are well known for arbitrary group manifold H:

$$Z_{G=0}(A) \sim \left(\frac{N}{2\pi A}\right)^{\dim H/2} + O(e^{-2\pi^2 N/A}) \tag{24}$$

where the correction is the sum of e^{-S} over all closed orbits on H.

We see the origin of the term $-\frac{1}{2}\log A$, and that the corrections are suppressed as $e^{-N/A}$, hence vanishing in the large N limit.

Remembering now that we have the constraint (17), we note that the trivial solution (20) is possible only for small areas A:

$$A \leq A_{crit} = \pi^2 \tag{25}$$

What happens for $A > A_{crit}$? We still have to solve the saddle point equation (18) , but in the presence of the boundary conditions (5) , or, which is the same, (17) . The only thing which may happen is that a finite fraction of the highest weight components $n_1, ... n_N$ will condense at the boundary of the inequality, namely

$$n_{k+1} = n_{k+2} = ... = n_{N-k} = 0 \tag{26}$$

whereas all others are non-zero (see fig.1b and compare it to fig.1a, where a typical Young tableau for the weak coupling phase is presented).

On the language of the density for the continuous variable h it means that

$$\begin{aligned} u(h) &= 1, \qquad for \quad -b \leq h \leq b \\ &= \tilde{u}(x) \qquad elsewhere \end{aligned} \tag{27}$$

where $b = 1/2 - k/N$ is finite in the large N limit and $\tilde{u}$ is some nontrivial function to be found later.

In fig. 2 the Young tableau is presented in both phases for the variable $h(x)$. We are trying to find here a so called two-cut solution, similar to those observed in the hermitean matrix models with double well potentials [24].

Let us substitute the ansatz (27) into (19) . We get the following equation on $\tilde{u}(h)$:

$$A/2h - \log \frac{h-b}{h+b} = P \int \frac{ds \tilde{u}(s)}{h-s} \tag{28}$$

As we see now, the condensate of the zero highest weight components induces an extra logarithmic term in the equation. In the language of the equivalent hermitean one matrix model it would mean that we have the effective matrix potential whose derivative is the l.h.s. of (28) . This potential clearly has two wells separated from each other by the cut of logarithm. The eigenvalues fill these wells always to the top. All the eigenvalues which spill over the top (say, by increasing A) form the condensate.

We introduce, as usual, the function of the complex variable h

$$f(h) = \int ds \frac{\tilde{u}(s)}{h-s} \tag{29}$$

whose imaginary part is $\pi \tilde{u}(h)$. The solution of the corresponding two-cut Cauchy problem [25] is given as a contour integral

$$f(h) = -\frac{1}{2\pi i} \sqrt{(a^2-h^2)(b^2-h^2)} \oint ds \frac{\frac{1}{2}As - \log \frac{h-b}{h+b}}{(h-s)\sqrt{(a^2-s^2)(b^2-s^2)}} \tag{30}$$

where the contour of integration encircles the cuts of the square root but leaves aside the singularities of the nominator and of the pole at $s = h$. The limits a and b have to be found later from the condition of the correct behaviour of $f(h)$ for $h \to \infty$.

Now, by inflating the contour we catch, instead of the cuts of the square root, the singularity at the pole and the cut of the logarithm. This gives

$$f(h) = h\frac{A}{2} - \log \frac{h-b}{h+b} + \sqrt{(a^2-h^2)(b^2-h^2)} \int_{-b}^{b} ds \frac{1}{(h-s)\sqrt{(a^2-s^2)(b^2-s^2)}} \tag{31}$$

Let us note that the imaginary part of the logarithm in the r.h.s. of (31) is exactly equal to the minus π times density of condesate of $h's$ in the interval $[-b, b]$. So it is clear that the last term in (31) represents the full function $u(h)$ defined by (27) . The latter is expressible in terms of the complete elliptic integral of the third kind $\Pi[\theta, x]$:

$$u(h) = \frac{1}{\pi} \frac{b-a}{b+a} \sqrt{\frac{(a+h)(b+h)}{(a-h)(b-h)}} \Pi[\frac{2b}{a+b} \frac{h-a}{h+b}, \frac{2\sqrt{ab}}{a+b}] \tag{32}$$

In order to find the parameters a,b and the whole free energy, it is better to use the asymptotics for the large h of (31)

$$\begin{aligned} f(h) =& h\Big(\frac{A}{2} - \int_{-b}^{b} ds \frac{1}{\sqrt{(a^2-s^2)(b^2-s^2)}}\Big) \\ &+ h^{-1}\big(2b + \int_{-b}^{b} ds \frac{s^2 - \frac{a^2+b^2}{2}}{\sqrt{(a^2-s^2)(b^2-s^2)}}\big) + \\ &+ h^{-3}\big(2b + \int_{-b}^{b} ds \frac{s^4 - \frac{a^2+b^2}{2}s^2 - \frac{(a^2-b^2)^2}{8}}{\sqrt{(a^2-s^2)(b^2-s^2)}}\big) + O(h^{-5}) \end{aligned} \tag{33}$$

and compare it with that which follows from the definition (29) :

$$f(h) = 0 \cdot h - (1-2b)h^{-1} - (F'(A) + 1/24)h^{-3} + O(h^{-5}) \tag{34}$$

This comparison gives (the elliptic integrals are in the appendix) the following results: The first derivative of the free energy in the strong coupling phase, expressed in terms of elliptic integrals with modulus $k = b/a$, is

$$F'(A) = \frac{1}{6}a^2 - \frac{1}{12}a^2k'^2 - \frac{1}{24} + \frac{1}{96}a^4k'^4A \tag{35}$$

where the modulus is related to the area by

$$\frac{1}{4}A = (2E - k'^2K)K, \tag{36}$$

the complementary modulus $k'^2 = 1 - k^2$, and

$$a = 4K/A. \tag{37}$$

This solution represents the strong coupling phase of our theory, namely for the area of sphere $A \geq \pi^2$. It is easy to check that at the point of transition $A_{crit} = \pi^2$ the two solutions coincide completely, even for the distribution $u(h)$ of boxes in the Young tableau. Let us calculate the order of this transition.

Series expansions become easier in terms of theta constants for a torus of complex modulus τ. The equation (36) becomes (the relevant identities are in the appendix)

$$\begin{aligned} A &= 8EK - 4k'^2K^2 \\ &= \frac{\pi^2}{3}(\theta_2^4(0|\tau) + \theta_3^4(0|\tau) + 2E_2(\tau)) \\ &= \pi^2(1 + 8q - 8q^2 + 32q^3 + \ldots) \end{aligned} \tag{38}$$

where $q = e^{i\pi\tau}$, and the critical point is the limit $\tau \to i\infty$.

Now

$$\begin{aligned} F'_{\text{strong}}(A) - F'_{\text{weak}}(A) &= \frac{\pi^2}{3A^2}(\theta_2^4(0|\tau) + \theta_3^4(0|\tau)) + \frac{\pi^4}{6A^3}\theta_4^8(0|\tau) - \frac{1}{2A} - \frac{1}{24} \\ &= \frac{1}{2A} - \frac{2\pi^2}{3A^2}E_2(\tau) + \frac{\pi^4}{6A^3}\theta_4^8(0|\tau) \\ &\to 0 \quad \text{as} \quad \tau \to i\infty \end{aligned} \tag{39}$$

so we see that the transition is higher order. Inverting (38) and substituting,

$$\begin{aligned} F'_{\text{strong}}(A) - F'_{\text{weak}}(A) =& \frac{1}{2A} - \frac{2\pi^2}{3A^2}(1 - \frac{3}{8}\left(\frac{A-A_c}{\pi^2}\right)^2 - \frac{3}{32}\left(\frac{A-A_c}{\pi^2}\right)^3 + \ldots) \\ &+ \frac{\pi^4}{6A^3}(1 - 2\left(\frac{A-A_c}{\pi^2}\right) + \frac{3}{2}\left(\frac{A-A_c}{\pi^2}\right)^2 + \ldots) \\ =& \frac{1}{\pi^2}\left(\frac{A-A_c}{\pi^2}\right)^2 + \ldots \end{aligned} \tag{40}$$

Thus the phase transition is of the third order, like the well known Gross-Witten-Wadia phase transition for the lattice two dimensional multicolour gauge theory. However, in spite of some similarities of these two transitions, the one found in this paper happens already in the continuum version of the theory, so we cannot say that it is a lattice artifact.

The transition also bears some similarity with the Berezinski-Kosterlitz-Thouless transition of condensation of vortices on the world sheet of one-dimensional string theory compactified on a circle [26]. In the language of the corresponding matrix quantum mechanics the point of the phase transition also corresponds there to the disappearance of the gap in the characteristic Young tableau for the $U(N)$ representations of angular matrix variables [27].

We can make contact with the results of Gross and Taylor by expanding the answer about $g^2A = \infty$. Although this is a singular point, the form of the singularity allows a well-defined double expansion in $e^{-A/2}$ and $Ae^{-A/2}$, as will emerge in the following. From [21], a reason to think that the expansion is unambiguous is that each term $e^{-nA/2}$ has a coefficient polynomial in A and of order $2n$, a property we would certainly lose if we expanded the exponentials in some other way. A better argument requires knowing the analytic structure of $F(A)$, to which we turn. The strong coupling limit was $\tau \to 0$; since series expansions of the theta constants are in $e^{i\pi\tau}$, clearly we want to make a modular transformation. This can be done by taking $K \leftrightarrow K'$ and $k \leftrightarrow k'$ and then going to theta functions with modulus $\tau \to i\infty$ in the strong coupling limit. Using Legendre's relation for E' gives

$$\begin{aligned}\frac{1}{4}A &= -k^2K'^2 + 2K'/K(\pi/2 + KK' - EK') \\ &= -\pi i\tau + \frac{(\pi i\tau)^2}{12}(\theta_3^4(0|\tau) + \theta_4^4(0|\tau) - 2E_2(\tau)) \\ &= -\pi i\tau - 2\pi i\tau^2 \frac{\partial}{\partial\tau}\log\theta_4(0|2\tau) \\ &\equiv -\pi i\tau + (2\pi i\tau)^2 R(2\pi i\tau)\end{aligned} \tag{41}$$

Now

$$\begin{aligned}R(2\pi i\tau) &= 2\sum_{n\geq 1}\frac{ne^{2ni\pi\tau}}{1-e^{4ni\pi\tau}} \\ &= 2e^{2i\pi\tau} + 4e^{4i\pi\tau} + 8e^{6i\pi\tau} + 8e^{8i\pi\tau} + 12e^{10i\pi\tau} + \dots,\end{aligned} \tag{42}$$

so in the limit $A \to \infty$ we have $\tau = iA/4\pi$ with corrections exponentially small in A. To get a double expansion in $\exp -A/2$ and $A\exp -A/2$, we express everything in terms of theta functions with modulus 2τ, and solve for $2\pi i\tau$ in terms of A and the exponentially small R:

$$\begin{aligned}2\pi i\tau &= \frac{1}{4R}(1-\sqrt{1+4RA}) \\ &\equiv -\frac{1}{2}A\ s(AR) \\ &= -\frac{1}{2}A(1 - RA + 2R^2A^2 - 5R^3A^3 + 14R^4A^4 + \dots)\end{aligned} \tag{43}$$

We would then successively substitute A for τ in R.

Rewriting $F'(A)$ in the same way, we find

$$\begin{aligned}F'(A) &= -\frac{1}{24} + \frac{1}{24}s(AR)^2(\theta_3^4(0|\tau) - \frac{1}{2}\theta_2^4(0|\tau)) + \frac{1}{1536}s(AR)^4\theta_2^8(0|\tau)A \\ &= -\frac{1}{24} + \frac{1}{48}s(AR)^2(\theta_3^4(0|\tau) + \theta_4^4(0|\tau)) + \frac{1}{1536}s(AR)^4\theta_2^8(0|\tau)A \\ &= -\frac{1}{24} - s(AR)^2\frac{1}{4\pi i}\frac{d}{d\tau}\log\frac{\theta_4(0|2\tau)}{\eta(2\tau)} + \frac{1}{96}s(AR)^4\theta_2^4(0|2\tau)\theta_3^4(0|2\tau)A \\ &\equiv -\frac{1}{24} + s(AR)^2F_0'(2\pi i\tau) + A\ s(AR)^4F_1'(2\pi i\tau).\end{aligned} \tag{44}$$

We see that the analytic structure of F' in terms of $w = e^{-A/2}$ is not so simple; however the branch cut in (43) is away from the origin, and near the origin we have a sum of terms $(\log w)^m f_m(w)$ with each f_m analytic. If we were only given the function $F(w)$, we could isolate these terms by combining its values on the sheets $F(e^{2\pi ik}w)$; at each order in w only finitely many f_m contribute. This would fix the double expansion uniquely.

Using (43) and (44), we can get the terms at a given order A^m in the double expansion to all orders in $e^{-A/2}$ by taking AR small; then (here prime is always $d/dA = -4q\ d/dq$)

$$
\begin{aligned}
F'(A) =& -\frac{1}{24} + F_0'(-A/2) + A(F_1' - 2RF_0') \\
&+ A^2\left(-4RF_1' + 5R^2F_0' - RF_0''\right) \\
&+ O(A^3) \\
=& \sum_{n\geq 1} \frac{(2n-1)e^{-(2n-1)A/2}}{1-e^{-(2n-1)A/2}} \\
&+ A\Bigg(\frac{1}{6}(1 + 8\sum_{n\geq 1}\frac{ne^{-nA/2}(1-(-e^{-A/2})^n)}{1-e^{-nA}})\sum_{m\geq 1}\frac{(2m-1)e^{-(2m-1)A/2}}{1-e^{-(2m-1)A}} \\
&\quad - 4\sum_{m\geq 1}\frac{me^{-mA/2}}{1-e^{-mA}}(\frac{1}{24} + \sum_{n\geq 1}\frac{(2n-1)e^{-(2n-1)A/2}}{1-e^{-(2n-1)A/2}})\Bigg) \\
&+ O(A^2).
\end{aligned}
\tag{45}
$$

We have compared this with a direct expansion of the formula (7) to $O(\exp(-2A))$:

$$
\begin{aligned}
F(A) &= 2e^{-A/2} + (-1 - 2A + \frac{1}{2}A^2)e^{-A} + (\frac{8}{3} + 4A^2 - \frac{8}{3}A^3 + \frac{1}{3}A^4)e^{-3A/2} + \ldots \\
-F'(A) &= e^{-A/2} + (1 - 3A + \frac{1}{2}A^2)e^{-A} + (4 - 8A + 2A^2 - \frac{8}{3}A^3 + \frac{1}{2}A^4)e^{-3A/2} + \ldots,
\end{aligned}
\tag{46}
$$

and with the $O(A^0)$ and $O(A)$ terms to much higher order, and found complete agreement. One can also reproduce this result by the direct expansion of (35) . †

We also checked this expansion from Gross and Taylor's rules, dropping the terms involving "tubes" and "handles" and proportional to powers of A (as appropriate for $U(N)$).

One technical conclusion we can draw from the solution is that e^{-A} and A are not really the natural expansion parameters in the problem. One way to think of this is in terms of the formalism of [19]. The perturbation is a higher derivative operator, and it is perhaps surprising that this even has a non-zero radius of convergence. Evidently it does, and much of the effect of the perturbation can be expressed as a "renormalization" of the modulus of the cylinder from A to the variable τ determined as above.

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† We thank J.-M.Daul, who reproduced this expansion starting from the integral representation (32) up to $O(\exp(-\frac{3}{2}A))$ and found complete agreement.

In Gross and Taylor's language the power-like expansion in A is given by inserting branch points on the string world-sheet. Evidently this expansion diverges at A_c.

We might hope to see the transition from the weak coupling side by noticing that some expectation value obtains an impossible value in the continuation beyond A_c, just as for the GWW transition, the continuation of the strong coupling spectral density to small g was no longer positive. [28] The easy observables to compute here are expectation values of the local operators tr E^k (E is the electric field), given by $\int dhu(h)h^k$. Another possible way to see the transition would be to see the sum of terms $e^{-N/A}$ in (24) diverge for sufficiently large A.

Another quantity revealing of the weak coupling phase is the Wilson loop, separating regions of area A_1 and A_2. This is again simple in terms of the heat kernel:

$$W_{G=0}(A_1, A_2) = \int dU\ G(1,\ U;\ A_1/2N)\ \ \frac{1}{2}\text{tr}\ (U + U^+)\ \ G(U,\ 1;\ A_2/2N). \qquad (47)$$

From (24) we learned that in the weak coupling phase, only the leading classical solution contributes in the heat kernel. Thus we can take the expression of [11] and drop the winding terms:

$$G(1, U; A/2N) = \mathcal{N} \prod_{i<j} \frac{\theta_i - \theta_j}{\sin \frac{1}{2}(\theta_i - \theta_j)} e^{-(N/A)\sum_i \theta_i^2} \qquad (48)$$

where $e^{i\theta_i}$ are the eigenvalues of U. Since the invariant measure is

$$\int dU = \int \prod d\theta_i \prod_{i<j} \sin^2 \frac{1}{2}(\theta_i - \theta_j), \qquad (49)$$

(47) becomes

$$W_{G=0}(A_1, A_2) = \int \prod d\theta_i \prod_{i<j} (\theta_i - \theta_j)^2 e^{-N(\frac{1}{A_1}+\frac{1}{A_2})\sum_i \theta_i^2} \sum_i \cos\theta_i \qquad (50)$$

which is again an expectation value at a semicircular saddle point:

$$\begin{aligned} W_{G=0}(A_1, A_2) &= \left(\frac{2}{\sqrt{x}}\right) J_1(\sqrt{x}) \\ &= 1 - x/8 + \dots \end{aligned} \qquad (51)$$

where $x = A_1 A_2/(A_1 + A_2)$.

This is perhaps a peculiar result. It is positive for all $A_1 + A_2 < A_{crit}$ but becomes oscillatory for (unphysical) large A. Even stranger, its asymptotic behavior is $\cos(\sqrt{A})/A$.

It would be very interesting to calculate the Wilson loop in the strong coupling phase. Knowing the result for the simple loop, the two-dimensional renormalized loop equations of [12] would then determine all loop averages. It is clear that these will obey the same Gross and Taylor rules of the large A_1, A_2 expansion (but the coverings will have the topology of the disc now). In the limit of large total area of the sphere we would reproduce all the results of [12] for the Wilson average on the infinite plane. Only the coverings which do not wind over the sphere will survive, and they are precisely those which were observed in [12] , [5] .

Returning to the general consequences of the phase transition, the conclusion for the relation of the strong coupling expansion to string theory is that string rules derived from the heat kernel action by expanding about $g^2 = \infty$ (in [22], the expansion was in $\exp -g^2$), do not give the correct answer on a small two-sphere. Now if we were two-dimensional, we might not care about this – the string rules DO give the right answer in the $A \to \infty$ limit, and all we seem to need for this is that the overall area of our universe be large. Expectation values for a Wilson loop enclosing an arbitrary area in this universe will be given correctly. Certainly the 't Hooft model of mesons in QCD_2 has no large N transition in infinite Minkowski space.

For four dimensional physicists, however, this transition looks like a real problem. The only precise way we know to define the strong coupling expansion is to start on a lattice, and take the continuum limit as defined by Wilson. For QCD this will require taking the bare coupling to zero in the way prescribed by the RG, so our expansion must make sense at weak coupling. We therefore need to choose an action for which there is (among other constraints) no large N transition. Although we do not know if the transition we find for the heat kernel action persists in $D > 2$, the simple fact that any higher dimensional lattice contains embedded topological two-spheres puts the burden of proof on the other side – to show that somehow cancellations between terms in the higher dimensional series eliminate the transition.

We thank T.Banks, E.Brezin, D.Boulatov, J.-M.Daul, J.Distler, P.Ginsparg, D.Gross, C.Itzykson, I.Kostov, H.Neuberger, S.Shenker and M.Staudacher for enjoyable discussions.

Appendix A. Elliptic Integrals and Theta Functions

The basic integrals required are

$$I_0 = \int_{-b}^{b} \frac{d\lambda}{\sqrt{(a^2-\lambda^2)(b^2-\lambda^2)}} = \frac{2}{a} K(b/a) \tag{A.1}$$

$$I_2 = \int_{-b}^{b} \frac{d\lambda\ \lambda^2}{\sqrt{(a^2-\lambda^2)(b^2-\lambda^2)}} = 2a[K(b/a) - E(b/a)] \tag{A.2}$$

and

$$I_4 = \int_{-b}^{b} \frac{d\lambda\ \lambda^4}{\sqrt{(a^2-\lambda^2)(b^2-\lambda^2)}} = \frac{2}{3} a[(2a^2+b^2)K(b/a) - 2(a^2+b^2)E(b/a)] \tag{A.3}$$

in terms of the standard complete elliptic integrals with modulus $k = b/a$ (e.g. as in [29])

We will write the results in terms of the complementary modulus k' satisfying $k'^2 = 1-k^2$, using $K(k) = K'(k')$, $E(k) = E'(k')$ and re-express them as theta constants, for the strong coupling expansion. From [29] (13.20) we have

$$\begin{aligned}
k &= \theta_2(0|\tau)^2/\theta_3(0|\tau)^2 \\
k' &= \theta_4(0|\tau)^2/\theta_3(0|\tau)^2 \\
K(k) &= \frac{\pi}{2}\theta_3^2(0|\tau) \\
K'(k) &= \frac{-i\pi\tau}{2}\theta_3^2(0|\tau) \\
E(k) &= \frac{\theta_3^4(0|\tau)+\theta_4^4(0|\tau)}{3\theta_3^4(0|\tau)} K(k) - \frac{1}{12K(k)} \frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)}
\end{aligned} \tag{A.4}$$

where $q = e^{i\pi\tau} = \exp(-\pi K'/K)$.

Some standard identities involving these functions:

$$\begin{aligned}
&KE' + K'E - KK' = \frac{1}{2}\pi \\
&\frac{\partial^2}{\partial\nu^2}\theta_n = 4\pi i \frac{\partial}{\partial\tau}\theta_n \qquad \forall n \\
&\theta_1'(0|\tau) = \pi\theta_2(0|\tau)\theta_3(0|\tau)\theta_4(0|\tau) = 2\pi\eta(\tau)^3 \\
&\theta_3^4(0|\tau) = \theta_2^4(0|\tau) + \theta_4^4(0|\tau) \\
&\frac{\theta_1'''(0|\tau)}{\theta_1'(0|\tau)} = 4\pi i \frac{\partial}{\partial\tau}\log\theta_1'(0|\tau) = 12\pi i\frac{\partial}{\partial\tau}\log\eta(\tau) = -\pi^2 E_2(\tau)
\end{aligned} \tag{A.5}$$

where E_2 is the normalized Eisenstein series. We will need as well

$$\begin{aligned} \theta_3^4(0|\tau) &= \frac{4}{\pi i}\frac{\partial}{\partial\tau}\log\frac{\theta_2(0|\tau)}{\theta_4(0|\tau)} \\ \theta_2^4(0|\tau) &= \frac{4}{\pi i}\frac{\partial}{\partial\tau}\log\frac{\theta_3(0|\tau)}{\theta_4(0|\tau)} \end{aligned} \tag{A.6}$$

proven by checking that both sides are modular forms of weight 2 and level 2, and low orders in the q-expansion.

All of the relevant theta constants can be expressed in terms of ones with modulus 2τ:

$$\begin{aligned} \theta_3^4(0|\tau) &= (\theta_3^2(0|2\tau)+\theta_2^2(0|2\tau))^2 \\ \theta_4^4(0|\tau) &= (\theta_3^2(0|2\tau)-\theta_2^2(0|2\tau))^2 \\ \theta_3^4(0|\tau)+\theta_4^4(0|\tau) &= \frac{8}{\pi i}\frac{\partial}{\partial\tau}\log\frac{\theta_2(0|2\tau)\theta_3(0|2\tau)}{\theta_4^2(0|2\tau)} \\ &= -\frac{12}{\pi i}\frac{\partial}{\partial\tau}\log\frac{\theta_4(0|2\tau)}{\eta(2\tau)} \\ E_2(\tau) &= \frac{6}{\pi i}\frac{\partial}{\partial\tau}\log\theta_4(0|2\tau)\eta(2\tau) \end{aligned} \tag{A.7}$$

The first two follow from [29] 13.23.15; the third uses these, (A.6) and Jacobi's identity (line 3 in (A.5)); the fourth follows from substituting the product representations below.

We will then need series expansions of these and their τ-derivatives. These are best derived from the logarithmic derivatives. (Here $q = e^{i\pi\tau}$):

$$\begin{aligned} -\frac{1}{4\pi i}\frac{\partial}{\partial\tau}\log\frac{\theta_4(0|2\tau)}{\eta(2\tau)} &= \frac{1}{24}+\sum_{n\geq 1}\frac{(2n-1)q^{4n-2}}{1-q^{4n-2}} \\ -\frac{1}{4\pi i}\frac{\partial}{\partial\tau}\log\theta_4(0|2\tau) &= \sum_{n\geq 1}\frac{nq^{2n}}{1-q^{4n}} \\ \theta_2^4(0|2\tau) &= 16\sum_{n\geq 1}\frac{(2n-1)q^{4n-2}}{1-q^{8n-4}} \\ \theta_3^4(0|2\tau) &= 1+8\sum_{m\geq 1}\frac{mq^{2m}(1-(-q^2)^m)}{1-q^{4m}} \\ E_2(2\tau) &= 1-24\sum_{m\geq 1}\frac{mq^{4m}}{1-q^{4m}}. \end{aligned} \tag{A.8}$$

Finally, we use product representations like

$$\begin{aligned} \theta_4(0|\tau) &= \prod_{n\geq 1}(1-q^{2n})(1-q^{2n-1})^2 \\ \eta(\tau) &= q^{1/12}\prod_{n\geq 1}(1-q^{2n}). \end{aligned} \tag{A.9}$$

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