Planar Diagrams

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Abstract. We investigate the planar approximation to field theory through the limit of a large internal symmetry group. This yields an alternative and powerful method to count planar diagrams. Results are presented for cubic and quartic vertices, some of which appear to be new. Quantum mechanics treated in this approximation is shown to be equivalent to a free Fermi gas system.

1. Introduction

We present some investigations of the planar approximation to field theory calculated through a limit of a large internal symmetry. Part of the motivation for this work lies in the hope that it might ultimately provide a mean of performing reliable computations in the large coupling phase of non-abelian gauge fields in four dimensions. In addition there are some indications that such topological expansions are related to the dual string models [1]. To support these hopes we may quote the significant simplifications occuring in the large N-limit for the linear or non-linear σ-models which indeed allow to discriminate the phases of broken and unbroken symmetry (even in two dimensions where the symmetry is never broken). On the other hand one has 't Hooft's solution to two-dimensional QCD in this same limit [2]. These promising features suggest to pursue this line of reasoning and develop some new techniques.

A first part of this paper is devoted to preliminary combinatorial aspects [3]. Some of these have already been discussed by Koplik, Neveu and Nussinov [4]. The method that we have used for this "zero-dimensional" field theory, in which every propagator is set equal to unity, is not of combinatorial nature and hopefully allows for extension to genuine calculations of Green functions in a real field theory. This enabled us to solve a few counting problems the solution of which does not seem to be known.

In Section 5, we compute explicitly the contribution of all the planar Feynman diagrams to the ground state energy of a one dimensional gx^4-anharmonic

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oscillator. The solution may also be generalized to include the first non-planar corrections. Amazingly it is found that the problem can be restated as the one of finding the ground state energy of a one-dimensional uninteracting Fermi gas, which is of course trivial.

Let us note finally that in contrast with the true theory the planar sum is analytic near the origin in the complex coupling constant space, which reveals that the large field region of the Feynman path integral has been drastically mutilated.

2. Planar Diagrams and Large N Limit

It is known from the work of 't Hooft that the only diagrams which survive the large N limit of an SU(N) gauge field theory are planar. The planar topology maximizes the number of factors N associated to closed index loops for fixed number of vertices. This feature is not specific of Yang-Mills fields and similar ideas may be applied to study the planar approximation to a φ^3 or a φ^4 (or any interaction $V(\varphi)$) field theory. The method consists in introducing a field theory in which the field is an $N \times N$ matrix $M(x)$ belonging to any of the following three sets characterized by an integer α taking the values 1, 2 or 4

   (i) $\alpha = 1$ real symmetric matrices,
  (ii) $\alpha = 2$ complex hermitian matrices,
 (iii) $\alpha = 4$ complex matrices.

The (Euclidean) Lagrangian is chosen to be

$$\mathscr{L} = \mathrm{tr}(\partial_\mu M \partial_\mu M^\dagger) + \mathrm{tr}(MM^\dagger) + \frac{\alpha g}{2N}\, \mathrm{tr}(MM^\dagger MM^\dagger). \tag{1}$$

The global invariance group is, respectively, SO(N), SU(N), and SU(N) $\times$ SU(N). The limit of interest is to let N go to infinity with fixed g; this selects only planar diagrams. It may be useful to state the Feynman rules derived from the Lagrangian (1). The propagators for the M-fields may be represented by double lines each one corresponding to the separate propagation of its two indices. These lines carry two different colors (in order to distinguish the two SU(N) groups) and have the same orientation in the case $\alpha = 4$. For $\alpha = 2$ the lines must be oriented in opposite directions. No orientation is required for $\alpha = 1$. The large N-limit may be described in terms of a simple φ^4-theory with single lines in which all non planar diagrams are omitted. The remaining diagrams are all those which can be drawn on a plane from rigid vertices and fixed external lines. For completeness let us repeat here the original derivation of 't Hooft, establishing the connection between planarity and large N limit. A general diagram consists of P propagators, V vertices, I closed loops of internal index. If we take an arbitrary interaction

$$g_3 \,\mathrm{tr}\, M^3 + g_4 \,\mathrm{tr}\, M^4 + \ldots,$$

there will be V_3 three-point vertices, V_4 four-point vertices etc..., and

$$V = V_3 + V_4 + \ldots.$$

If we consider for instance a connected vacuum diagram, a simple topological argument gives $2P = 3V_3 + 4V_4 + \ldots$. Each loop of internal index may be considered as a face of a polyhedron, and the Euler relation gives

$$V - P + I = 2 - 2H$$

in which H is the number of holes of the surface on which the polyhedron is drawn (0 for a plane or a sphere, one for a torus, etc. ...). The contribution of the diagrams is proportional to

$$g_3^{V_3} g_4^{V_4} \dots N^I = (g_3 N^{1/2})^{V_3} (g_4 N)^{V_4} \dots N^{2-2H}.$$

Thus provided one takes coupling constants g_p proportional to $N^{1-p/2}$, the vacuum energy divided by N^2 has a finite limit for the diagrams which may be drawn on a planar $(H=0)$ surface. Corrections of order $1/N^2$ are given by diagrams which may be drawn on a torus. If $E_\alpha^{(d)}(g)$ stands for the sum of the connected vacuum diagrams for any of the three theories (1) in d dimensions and if $E^{(d)}(g)$ is the same sum for the planar φ^4-theory then in any dimension

$$\lim_{N \to \alpha} \frac{2}{\alpha N^2} E_\alpha^{(d)}(g) = E^{(d)}(g). \tag{2}$$

The counting rules for the lowest orders are given in Table 1 and Equation (2) may be checked from the Lagrangian (1).

Table 1. Counting rules for the vacuum amplitude $E^{(0)}(g)$ in the planar limit, up to order three

$2g$	$2g^2$	$16g^2$	$\frac{32}{3} g^3$
$64g^3$	$128g^3$	$\frac{256}{3} g^3$	

It is thus sufficient to study the simpler hermitian case $\alpha = 2$, which is analyzed in the following. Note however that the corrections to the leading behaviour may be different in the various cases.

3. Combinatorics of Quartic Vertices

1) Vacuum Diagrams

Setting each diagram equal to unity, apart from the overall weight, is equivalent to treat a field theory in zero dimension, in which space-time is reduced to one or to a finite number of points. It means that

$$\exp - N^2 E^{(0)}(g) = \lim_{N \to \infty} \int d^{N^2} M \exp - \left[\tfrac{1}{2} \operatorname{tr} M^2 + \frac{g}{N} \operatorname{tr} M^4 \right]. \tag{3}$$

The integration measure on hermitian matrices is

$$d^{N^2} M \equiv \prod_i dM_{ii} \prod_{i<j} d(\operatorname{Re} M_{ij}) d(\operatorname{Im} M_{ij}) \tag{4}$$

and it is convenient to express it in terms of the eigenvalues λ_i of M and of the unitary matrix U which diagonalizes the matrix M. This is a well-known problem [5] and the result is

$$d^{N^2} M = \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 dU. \tag{5}$$

Since the integrand (3) depends only on the eigenvalues λ_i, this allows us to integrate over U and, up of to a g-independent normalizing factor we obtain

$$\exp - N^2 E^{(0)}(g) = \lim_{N \to \infty} \int \prod_i d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp - \left[\tfrac{1}{2} \sum \lambda_i^2 + \frac{g}{N} \sum \lambda_i^4 \right]. \tag{6}$$

In the large N-limit the steepest descent method can be used to compute (6), noting that the factor $\prod_{i<j} (\lambda_i - \lambda_j)^2$ in the measure requires that the eigenvalues repel each other and spread evenly around zero. To leading order we have

$$E^{(0)}(g) = \lim \frac{1}{N^2} \left\{ \sum_i \left(\tfrac{1}{2} \lambda_i^2 + \frac{g}{N} \lambda_i^4 \right) - \sum{}' \ln |\lambda_i - \lambda_j| \right\} \tag{7}$$

(in which the primed sum runs over $i \neq j$), and the λ_i are given by the stationary condition

$$\tfrac{1}{2} \lambda_i + 2 \frac{g}{N} \lambda_i^3 = \sum_j{}' \frac{1}{\lambda_i - \lambda_j}. \tag{8}$$

The eigenvalue Equation (8) may be solved in the large N limit by going to a continuous problem. Let us introduce a non decreasing function $\lambda(x)$ such that

$$\lambda_i = \sqrt{N} \lambda(i/N). \tag{9}$$

Then the large N-limit may be explicitly performed and the Equations (7) and (8) are replaced by

$$E^{(0)}(g) = \int_0^1 dx \left[\tfrac{1}{2} \lambda^2(x) + g \lambda^4(x) \right] - \int_0^1 \int_0^1 dx dy \ln |\lambda(x) - \lambda(y)| \tag{10}$$

(up to a constant g-independent term) and

$$\tfrac{1}{2} \lambda(x) + 2g \lambda^3(x) = \fint_0^1 \frac{dy}{\lambda(x) - \lambda(y)} \tag{11}$$

in which $\fint$ stands for the principal part of the integral.

The condition (11) on $\lambda(x)$ suggests to introduce the density of eigenvalues $u(\lambda)$ defined as

$$\frac{dx}{d\lambda} = u(\lambda). \tag{12}$$

The function $u(\lambda)$ should be positive, even, and normalized to

$$\int_{-2a}^{+2a} d\lambda u(\lambda) = 1. \tag{13}$$

The condition (11) becomes an equation for $u(\lambda)$

$$\tfrac{1}{2} \lambda + 2g \lambda^3 = \fint_{-2a}^{+2a} d\mu \frac{u(\mu)}{\lambda - \mu}, \qquad |\lambda| \leq 2a \tag{14}$$

and $u(\mu)$ should vanish outside some support $(-2a, 2a)$, otherwise the equation is inconsistent for large λ. The solution is easily obtained by introducing the analytic function

$$F(\lambda) = \int\limits_{-2a}^{+2a} d\mu \, \frac{u(\mu)}{\lambda - \mu} \tag{15}$$

defined for complex λ outside the real interval $(-2a, 2a)$. Clearly $F(\lambda)$ enjoys the following properties:

(i) it is analytic in the complex λ plane cut along the interval $(-2a, 2a)$,
(ii) it behaves as $1/\lambda$ when $|\lambda|$ goes to infinity, as a consequence of (13),
(iii) it is real for λ real outside $(-2a, 2a)$,
(iv) when λ approaches the interval $(-2a, 2a)$,

$$F(\lambda \pm i\varepsilon) = \tfrac{1}{2}\lambda + 2g\lambda^3 \mp i\pi u(\lambda). \tag{16}$$

There is a unique function which satisfies these requirements which is

$$F(\lambda) = \tfrac{1}{2}\lambda + 2g\lambda^3 - (\tfrac{1}{2} + 4ga^2 + 2g\lambda^2)\sqrt{\lambda^2 - 4a^2} \tag{17a}$$

with

$$12ga^4 + a^2 - 1 = 0. \tag{17b}$$

The square root is defined in the cut λ-plane and is chosen to be positive for λ real larger than $2a$. The odd function $F(\lambda)$ has an even discontinuity

$$u(\lambda) = \frac{1}{\pi} (\tfrac{1}{2} + 4ga^2 + 2g\lambda^2)\sqrt{4a^2 - \lambda^2} \quad |\lambda| \leq 2a \tag{18}$$

with a given by (17b).

In order to obtain $E^{(0)}(g)$, we first transform (10) into

$$E^{(0)}(g) = \int\limits_{-2a}^{+2a} d\lambda u(\lambda)[\tfrac{1}{2}\lambda^2 + g\lambda^4] - \iint\limits_{-2a}^{2a} d\lambda d\mu u(\lambda)u(\mu) \ln|\lambda - \mu| . \tag{19}$$

Then, integrating (14) with respect to λ we replace this expression by

$$E^{(0)}(g) - E^{(0)}(0) = \int\limits_{0}^{2a} d\lambda u(\lambda)(\tfrac{1}{2}\lambda^2 + g\lambda^4 - 2\ln\lambda) - (g = 0)$$

$$= \tfrac{1}{24}(a^2 - 1)(9 - a^2) - \tfrac{1}{2}\log a^2 . \tag{20}$$

This is supplemented by Equation (17b) giving

$$a^2 = \frac{1}{24g}[(1 + 48g)^{1/2} - 1] = 1 - 12g + 2(12g)^2 - 5(12g)^3 + \dots . \tag{21}$$

Correspondingly the first perturbative terms of Equation (20) give

$$E^{(0)}(g) - E^{(0)}(0) = 2g - 18g^2 + 288g^3 - 6048g^4 + 0(g^5)$$

in agreement with Table 1.

The formulae (20) and (21) count the connected planar vacuum diagrams of the φ^4-theory. It is interesting to note that it yields an expression for $E^{(0)}(g)$ which is

analytic in the neighborhood of $g=0$. Its nearest singularity occurs for real negative g at

$$g_c = -1/48 \qquad (22)$$

which is the singularity of $a^2(g)$; the branch point of the logarithm, $a^2 = 0$, corresponds to $g = \infty$. It is easy to derive from (20) the large order behaviour of $E^{(0)}(g)$

$$E^{(0)}(g) = \sum_0^\infty A_k(-g)^{k-1}$$

$$A_k \underset{k \to \infty}{\sim} \frac{1}{2\sqrt{\pi}}(48)^k k^{-7/2}. \qquad (23)$$

Finally let us note that the solution (18) for $u(\lambda)$ generalizes Wigner's semi-circle law for the spacing of eigenvalues of hermitian random matrices with gaussian distributions, which corresponds to the simple case $g=0$, and hence $a^2 = 1$, $u(\lambda) = \frac{1}{\pi}\sqrt{1 - \lambda^2/4}$ [5]. For g not equal to zero $u(\lambda)$ may be interpreted as the distribution of eigenvalues of a non-gaussian random set of hermitian matrices. For any g real, and greater than the critical g_c of Equation (22), $u(\lambda)$ has a square root behaviour near the end points of the interval $(-2a, 2a)$. When g reaches the value g_c, a has increased from 1 to $\sqrt{2}$, and $u(\lambda)$ now vanishes as $(\lambda \mp 2\sqrt{2})^{3/2}$ (Fig. 1).
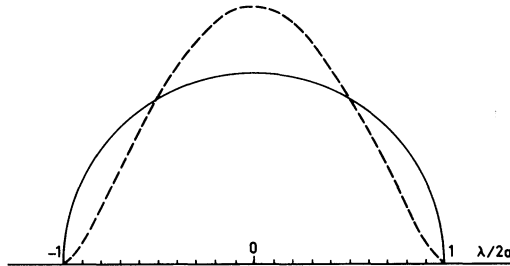

Fig. 1. The level spacing $u(\lambda)$ as a function of $\lambda/2a$; solid line, $g=0$, the semi-circle law $\pi u(\lambda) = \sqrt{1 - \lambda^2}$; dotted line, the critical curve $g_c = -1/48$, $\pi u(\lambda) = \frac{2^{3/2}}{3}\left(1 - \frac{\lambda^2}{8}\right)^{3/2}$, normalized to the same area

2) Green Functions

The same method may be applied to derive the planar limit of the (zero dimensional) Green functions. They are given by the moments of the distribution $u(\lambda)$ since

$$G_{2p}(g) = \langle \operatorname{tr} M^{2p} \rangle = \int_{-2a}^{2a} d\lambda u(\lambda) \lambda^{2p}. \qquad (24)$$

Consequently the generating function

$$\phi(j) = \sum_0^\infty j^{2p} G_{2p} \qquad (25)$$

may be expressed in terms of the function $F(\lambda)$ of Equation (17) by noting that $u(\lambda)$ is the discontinuity of $F(\lambda)$.

The result is

$$\phi(j) = 1/jF(1/j)$$

$$= \frac{1}{2j^2} + \frac{2g}{j^4} - \frac{1}{j^2}\left(\frac{1}{2} + 4ga^2 + \frac{2g}{j^2}\right)\sqrt{1 - 4a^2 j^2}\,. \tag{26}$$

It then follows that

$$G_{2p}(g) = \frac{(2p)!}{p!(p+2)!}\, a^{2p}[2p + 2 - pa^2]\,. \tag{27}$$

Clearly the singularity of G_{2p} in the complex g plane is again given by the analytic structure of $a(g)$, i.e. a branch point at $g = g_c = -1/48$. Note also that G_{2p} does not involve any logarithm and is purely algebraic in g.

An explicit check of the formula (27) for $p = 2$ and 4 can be made at the first few orders in g with the help of the Table 2.

Table 2. Counting factors for G_2 and G_4 to the first few orders

3) *Connected Green Functions*

The usual exponential relation between the generating functionals of connected and disconnected diagrams is invalid in the planar theory. We have rather to use a Lagrange relation expressed as follows. Let $\psi(j)$ generate the connected Green functions

$$\psi(j) = 1 + \sum_1^\infty G_{2p}^{(c)} j^{2p}\,. \tag{28}$$

The Green functions may be obtained in terms of the connected ones if the source j is replaced in $\psi(j)$ by the solution of the implicit equation

$$z(j) = j\psi(z(j))\,. \tag{29}$$

Consequently, if we solve for $z(j)$, then

$$\phi(j) = \psi(z(j)), \tag{30}$$

This defines ψ through a purely algebraic procedure and before giving the solution, it may be useful to note that (29) and (30) summarize the following relations between the connected and disconnected Green functions:

$$G_{2p} = \sum_{\substack{r_q \geqq 0 \\ \sum_q 2q r_q = 2p}} \frac{(2p)!}{\left(2p + 1 - \sum_q r_q\right)!} \frac{(G_2^c)^{r_1}}{r_1!} \frac{(G_4^c)^{r_2}}{r_2!} \cdots \frac{(G_{2q}^c)^{r_q}}{r_q!} \cdots \tag{31}$$

or explicitly

$$G_2 = G_2^c$$
$$G_4 = G_4^c + 2(G_2^c)^2$$
$$G_6 = G_6^c + 6G_4^c G_2^c + 5(G_2^c)^3$$
$$G_8 = G_8^c + 8G_6^c G_2^c + 4(G_4^c)^2 + 28G_4^c(G_2^c)^2 + 14(G_2^c)^4$$

etc. ....

The expression (31) is the solution to the following combinatorial problem: label $2p$ points on the boundary of a circle and join them in non overlapping clusters of r_1 pairs, r_2 quadruplets, ..., $r_q 2q$-*plets*, ... in all possible ways. This gives the coefficients of Equation (31). The algebraic relations (29) and (30) are of course much more tractable. The solution $z(j)$ is very simple, since $z(j) = j\phi(j)$, and thus from (26)

$$z(j) = \frac{1}{2j} + \frac{2g}{j^3} - \frac{1}{j}\left(\frac{1}{2} + 4ga^2 + \frac{2g}{j^2}\right)\sqrt{1 - 4a^2 j^2}. \tag{32}$$

We then solve for j in terms of z and from (29) we obtain

$$\psi(z) = z/j(z) \tag{33}$$

Substituting (33) into (32) and using the (g, a) relation (17b) we end up with the cubic equation for $\psi(z)$

$$3(1 - a^2)\psi^2(\psi - 1) + 9a^4 z^2 \psi - a^2 z^2 [9a^2 z^2 + (2 + a^2)^2] = 0. \tag{34}$$

This equation can be further simplified if we define a new variable

$$y^2 = \tfrac{1}{3} z^2 a^2 (a^2 - 1), \tag{35}$$

then it follows from (34) that $(a^2 - 1)\psi(z)$ can be written uniquely as

$$(a^2 - 1)\psi(z) = a^2 \lambda(y) + \mu(y) \tag{36}$$

with λ and μ independent of a^2. Indeed inserting (35) and (36) into (34) leads to four equations for the two unknown functions $\lambda(y)$ and $\mu(y)$, which reduce to the

two equations

$$
\begin{cases}
\lambda^3 - \lambda^2 + y^2 = 0 & \text{(37a)} \\
\mu = \dfrac{9y^2 - 2\lambda}{3\lambda - 1} & \text{(37b)}
\end{cases}
\qquad \lambda(0) = -\mu(0) = 1 \ .
$$

Solving these equations we obtain

$$
\begin{cases}
G^c_{2p} = -\dfrac{a^{2p}}{3^p}(a^2-1)^{p-1} A_p [3p(a^2-2) - 2(a^2-1)] \\[2mm]
A_1 = 1 \\[2mm]
A_{p+1} = \dfrac{(-1)^p 2^{-p}}{(p+1)!(3p+1)} \displaystyle\sum_{p/2 \leq q \leq p} (-4)^q \dfrac{(p+q)!}{(2q-p)!(p-q)!} \ .
\end{cases}
$$

It is gratifying to observe that due to the factor $(a^2-1)^{p-1}$ the lowest order term of G^c_{2p} is indeed in g^{p-1}. Once again we verify that G^c_{2p} is a simple polynomial in a^2.

4) One Particle Irreducible Functions

Finally we may define one particle irreducible vertex-functions. For convenience we set

$$
\Gamma(x) = \Gamma_2 x^2 + \sum_{p=2}^{\infty} \Gamma_{2p} x^{2p} \ , \tag{38}
$$

$$
\begin{aligned}
\Gamma_2 &= [G^c_2]^{-1} \\
-\Gamma_4 &= G^c_4 [G^c_2]^{-4} \\
-\Gamma_6 &= G^c_6 [G^c_2]^{-6} - 3[G^c_4]^2 [G^c_2]^{-7} \\
-\Gamma_8 &= G^c_8 [G^c_2]^{-8} - 8 G^c_6 G^c_4 [G^c_2]^{-9} + 12 [G^c_4]^3 [G^c_2]^{-10} \ .
\end{aligned}
\tag{39}
$$

Of course the unusual weights appearing in (39) are consequences of the planar topology. The Legendre transformation defined on a generating function ψ which would include the division by the cyclic symmetry factor $2p$ is equivalent to the following relations

$$
\begin{aligned}
\psi(j) &= 1 + \Gamma(x) \\
x &= \frac{1}{j}[\psi(j) - 1] \ .
\end{aligned}
\tag{40}
$$

This is easily seen to be a summary of the previous Equation (39). Thus $\Gamma(x)$ is obtained by substituting in (34) $1+\Gamma$ for ψ and $\dfrac{\Gamma}{x}$ for z. The result is the cubic equation

$$
3x^2(1-a^2)(1+\Gamma)^2 + 9a^4 \Gamma\left(1 + \Gamma - \frac{\Gamma^2}{x^2}\right) - a^2 \Gamma(2+a^2)^2 = 0 \tag{41}
$$

which gives $\Gamma(x)$ by an algebraic formula.

4. Combinatorics for a Cubic Interaction

For completeness we shall briefly present some results for the case of a cubic interaction which makes sense for real symmetric ($\alpha = 1$) or hermitian matrices ($\alpha = 2$). Calculations will be presented in the latter case. The case of a combined cubic and quartic interaction is then a simple extension.

The integral

$$\exp - N^2 E^{(0)}(g) = \int \Pi d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp - \sum_i \left(\frac{\lambda_i^2}{2} + \frac{g}{\sqrt{N}} \lambda_i^3 \right) \tag{42}$$

is only meaningful for complex-g due to the instability of the cubic interaction. However the power series in g is well defined and is the quantity of interest. We repeat the steps of the previous section in the limit of N large using the steepest descent method

$$\lambda_i = \sqrt{N} \lambda \left(\frac{i}{N} \right)$$

$$v(\lambda(x)) = 2 \frac{dx}{d\lambda(x)} \tag{43}$$

$$\lambda + 3g\lambda^2 = \oint \frac{v(\mu) d\mu}{\lambda - \mu}$$

with the normalization condition

$$\int d\lambda v(\lambda) = 2 .$$

The symmetry property $\lambda \to -\lambda$ is lost and v has a non-vanishing support in an interval $2a \leq \lambda \leq 2b$. The solution is defined in terms of the analytic function

$$F(\lambda) = \int_{2a}^{2b} \frac{v(\mu) d\mu}{\lambda - \mu} = \lambda + 3g\lambda^2 - [3g\lambda + 1 + 3g(a+b)][(\lambda - 2a)(\lambda - 2b)]^{1/2} \tag{44}$$

which has been determined by the fact that its real part on the interval $(2a, 2b)$ should be given by (43), and by demanding that the coefficients of λ^2 and λ vanish at infinity. The function $v(\lambda)$ is related to the imaginary part of $F(\lambda)$ since

$$F(\lambda \pm i\varepsilon) = \lambda + 3g\lambda^2 \mp i\pi v(\lambda) \quad 2a < \lambda < 2b$$

$$v(\lambda) = \frac{1}{\pi} [1 + 3g(a+b) + 3g\lambda] \sqrt{(\lambda - 2a)(2b - \lambda)} . \tag{45}$$

It remains now to express that $F(\lambda)$ behaves as $2/\lambda$ for $|\lambda|$ going to infinity. This determines the interval $(2a, 2b)$ in terms of g by the conditions

$$3g(b-a)^2 + 2(a+b)[1 + 3g(a+b)] = 0$$

$$(b-a)^2 [1 + 6g(a+b)] = 4 . \tag{46}$$

Thus for g small

$$\begin{cases} b = 1 - 3g + 18g^2 - 162g^3 + \dots \\ a = -1 - 3g - 18g^2 - 162g^3 + \dots . \end{cases} \tag{47}$$

It is convenient to introduce the single parameter

$$\sigma = 3g(a+b) \tag{48}$$

which, as a consequence of Equation (46) is the solution of

$$18g^2 + \sigma(1+\sigma)(1+2\sigma) = 0 . \tag{49}$$

The expansion of σ as a power series in g^2 reads

$$\sigma = -\frac{1}{4} \sum_1^\infty \frac{(72g^2)^k}{k!} \frac{\Gamma(\frac{1}{2}(3k-1))}{\Gamma(\frac{1}{2}(k+1))} . \tag{50}$$

The vacuum diagrams are then readily obtained and their generating function $E^{(0)}(g)$ is given as

$$E^{(0)}(g) = \frac{1}{2} \int_{2a}^{2b} d\lambda v(\lambda)[\frac{1}{2}\lambda^2 + g\lambda^3] - \frac{1}{4} \int\int d\lambda d\mu v(\lambda)v(\mu) \ln|\lambda - \mu| .$$

With the explicit expression (45) we obtain

$$E^{(0)}(g) = -\frac{1}{3} \frac{\sigma(3\sigma^2 + 6\sigma + 2)}{(1+\sigma)(1+2\sigma)^2} + \frac{1}{2} \ln(1+2\sigma), \tag{51}$$

in which σ is given by (49) and (50).

After tedious calculations, one obtains from (49)–(51)

$$E^{(0)}(g) = -\frac{1}{2} \sum_{k=1}^\infty \frac{(72g^2)^k}{(k+2)!} \frac{\Gamma(3k/2)}{\Gamma(k/2+1)}$$

which gives explicitly the number of connected vacuum diagrams with $2k$ vertices.

The function $E^{(0)}(g)$ is thus analytic near $g=0$ since σ is itself analytic, up to the value of g^2 for which Equation (49) acquires a double root. This gives the closest singularity of $E^{(0)}(g)$ for a value

$$g_c^2 = 1/(108\sqrt{3}) \tag{52}$$

which gives the radius of convergence of the planar perturbation series. Again for this value g_c, $v(\lambda)$, instead of vanishing as a square root at both ends of its support, vanishes as a power $3/2$ at one of these limits.

By the same techniques we can obtain the generating function for the planar Green functions

$$G_p = \frac{1}{2} \int_{2a}^{2b} d\lambda v(\lambda)\lambda^p , \tag{53}$$

$$\phi(j) = \sum_0^\infty j^p G_p = \frac{1}{2j} F\left(\frac{1}{j}\right) \tag{54}$$

which from (44) yields

$$\phi(j) = \frac{1}{2g^2}\left\{\left(\frac{g}{j}\right)^2 + 3\left(\frac{g}{j}\right)^3 - \left(\frac{g}{j}\right)^3\left[3 + \frac{j}{g}(1+\sigma)\right]\right.$$
$$\left.\cdot\left[1 - \frac{26}{3}(j/g) + \tfrac{1}{9}(j/g)^2(3\sigma^2 + 2\sigma)\right]^{1/2}\right\}. \tag{55}$$

This gives the following expressions for the G_p

$$g^p G_p = \left(\frac{\sigma}{3}\right)^p \sum_{k=0}^{[p/2]}\left[-\frac{1}{2}\frac{1+\sigma}{\sigma}\right]^k \frac{p!}{k!(k+1)!(p-2k)!}$$
$$+ \frac{2\sigma}{1+2\sigma}(\sigma/3)^p \sum_{1}^{[(p+1)/2]}\left[-\frac{1}{2}\frac{1+\sigma}{\sigma}\right]^k \frac{p!}{(k-1)!(k+1)!(p-2k+1)!}. \tag{56}$$

The connected Green functions are generated by

$$\psi(j) = 1 + \sum_{1}^{\infty} j^p G_p^{(c)} \tag{57}$$

given by the quadratic equation

$$3g\psi^2 + (j-3g)\psi - \tfrac{1}{2}j\frac{(1+\sigma)(2+3\sigma)}{1+2\sigma} - j^3 = 0. \tag{58}$$

The one-particle irreducible function are more delicate to derive due to the presence of tadpole graphs. To cope with this difficulty we generalize the initial model by including an extra term $\varrho \sum_i \lambda_i$ in Equation (42). By adjusting ϱ it enables one to cancel all tadpole insertions. Consequently the expectation value of λ vanishes and in this new theory the bare propagator remains equal to unity. This has the effect of modifying the Green functions. The new connected ones are generated by ψ' satisfying

$$3g\psi'^2 + (j-3g)\psi' - j(1-\varrho j + j^2) = 0. \tag{59}$$

The demand that G_1 vanishes leads after some algebra to the parametric relation between ϱ and g

$$3g\varrho = -\tau(1-3\tau)$$
$$9g^2 = \tau(1-2\tau)^2. \tag{60}$$

The expansion of τ is

$$\tau = \sum_{1}^{\infty} 2^{n-1}\frac{(3n-2)!}{(n)!(2n-1)!}(9g^2)^n. \tag{61}$$

One then deduces the one-particle irreducible functions $\Gamma_p(g)$

$$\Gamma(x) = \sum_{2}^{\infty} \Gamma_p(g)x^p \tag{62}$$

from the same equations as in Section 3. This gives

$$\Gamma^2(x) - \Gamma(x)[\varrho x + x^2 + 3gx^3] - 3gx^3 = 0. \tag{63}$$

Note that the coefficients of the expansion of $\Gamma_p(g)$ in powers of g give correctly the number of irreducible diagrams with p external lines and a given number of vertices. For instance

$$\Gamma_2 = \frac{(1-2\tau)^2}{1-3\tau}$$
$$= 1 - 9g^2 - 3(9g^2)^2 - 17(9g^2)^3 \ldots \tag{64}$$
$$\Gamma_3 = 3g\frac{(1-2\tau)^2(1-4\tau)}{(1-3\tau)^3} = 3g[1 + 9g^2 + 6(9g^2)^2 + \ldots].$$

As noticed by the authors of [4] it is also simple to count one-particle irreducible diagrams without self-energy insertions. This can be achieved using a similar modification of the theory. Not only do we add the $\varrho\lambda$ term but we also change the quadratic part from $\dfrac{\lambda^2}{2}$ to $(1+m)\dfrac{\lambda^2}{2}$. Again ϱ and m are chosen as functions of g^2 to give $G_1 = 0$ and the complete propagator $G_2 = 1$. The resulting equations for the connected functions are

$$3g\psi''^2 + \psi''[-3g + (1+m)] - j(1+m) - 3gj^2 - j^3 = 0 \tag{65}$$

with

$$\begin{aligned} m &= \alpha(1-2\alpha) \\ 9g^2 &= \alpha(1-\alpha)^3. \end{aligned} \tag{66}$$

This generates modified irreducible functions denoted $\bar{\Gamma}_p(g)$ with

$$\bar{\Gamma}(x) = \sum_2^\infty \bar{\Gamma}_p(g)x^p$$
$$\bar{\Gamma}^2(x) + \bar{\Gamma}(x)[3gx - x^2(1+m) - 3gx^3] - 3gx^3 = 0. \tag{67}$$

Of course by construction $\bar{\Gamma}_2 = 1$, while for instance

$$\bar{\Gamma}_3 = \frac{m}{3g}. \tag{68}$$

Using the parametric relations (66) we obtain

$$3g\bar{\Gamma}_3 = \sum_1^\infty 2(9g^2)^k \frac{(4k-3)!}{(3k-1)!k!} \tag{69}$$

Analogous formulae hold for higher functions. They all have a radius of convergence equal to

$$g_c'^2 = \frac{3}{256} \tag{70}$$

This is of course larger than the value given by Equation (52). Similar techniques can also be applied to the computation of the vacuum energy without tadpole and self energy insertions.

5. One Dimensional Planar Theory

We must now go beyond the mere counting problem and try to sum the planar theory corresponding to interacting systems. The simplest problem, the only one which will be solved in this article, corresponds to one dimension, where we put on the lines a propagator $1/(p^2+1)$ and integrate the momenta from $-\infty$ to $+\infty$. This corresponds to coupled anharmonic oscillators, and as before we introduce hermitian $N \times N$ matrices ($\alpha=2$). The extension to real symmetric or complex matrices is straightforward. We thus consider the problem of determining in the large N limit the vacuum diagrams, i.e., the ground state energy of the Hamiltonian

$$H = -\tfrac{1}{2}\Delta + V$$

$$\Delta = \sum_i \frac{\partial^2}{\partial M_{ii}^2} + \tfrac{1}{2}\sum_{i<j} \frac{\partial^2}{\partial \operatorname{Re} M_{ij}^2} + \frac{\partial^2}{\partial \operatorname{Im} M_{ij}^2} \tag{71}$$

and for a ϕ^4 interaction

$$V = \tfrac{1}{2}\operatorname{tr} M^2 + \frac{g}{N}\operatorname{tr} M^4. \tag{72}$$

We set in this limit

$$H\psi = N^2 E^{(1)}(g)\psi \tag{73}$$

and look for a ground state wave function symmetric under the $U(N)$ group. Thus ψ is a symmetric function of the eigenvalues λ_i of M, if we note that the Laplacian is invariant under the group. The energy $E^{(1)}$ is then the minimum

$$E^{(1)}(g) = \lim_{N\to\infty} \frac{1}{N^2} \operatorname*{Min}_{\psi} \frac{\int d^{N^2}M(\tfrac{1}{2}(\partial\psi)^2 + V\psi^2)}{\int d^{N^2}M\psi^2} \tag{74}$$

over functions invariant under the transformation $\psi(M)\to\psi(UMU^{-1})$. We rewrite (74) by eliminating the "angular" variables U as

$$E^{(1)}(g) = \lim_{N\to\infty} \frac{1}{N^2} \operatorname*{Min}_{\psi} \frac{\int \prod_i d\lambda_i \prod_{i<j}(\lambda_i-\lambda_j)^2 \left[\tfrac{1}{2}\sum_i \left(\frac{\partial\psi}{\partial\lambda_i}\right)^2 + V(\lambda_i)\psi^2\right]}{\int \prod_i d\lambda_i \prod_{i<j}(\lambda_i-\lambda_j)^2\psi^2}, \tag{75a}$$

where we have noted that the gradient term reduces to $\tfrac{1}{2}\sum_i \left(\frac{\partial\psi}{\partial\lambda_i}\right)^2$ and the potential V also appears as a symmetric function of the eigenvalues λ_i. This relation suggests to introduce the antisymmetric function

$$\phi(\lambda_1, ..., \lambda_N) = \left\{\prod_{i<j}(\lambda_i-\lambda_j)\right\}\psi(\lambda_1, ..., \lambda_N) \tag{75b}$$

as if we were dealing with a fermionic problem with N degrees of freedom. The corresponding Schrödinger equation

$$\sum_i \left(-\tfrac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + \tfrac{1}{2}\lambda_i^2 + \frac{g}{N}\lambda_i^4 \right)\phi = N^2 E^{(1)}\phi \tag{76}$$

could of course also be derived directly from (71) and (72) by going over to "polar" coordinates. Note that the substitution (75) has not generated any new terms in the potential, and as a result we have a non interacting Fermi gas where each "particle" is only submitted to the central potential $\dfrac{\lambda^2}{2} + \dfrac{g}{N}\lambda^4$.

We denote $e_1 \leqq e_2 \leqq e_3 \ldots$ the individual energies corresponding to the one particle Hamiltonian $h = -\tfrac{1}{2}\dfrac{\partial^2}{\partial \lambda^2} + \dfrac{\lambda^2}{2} + \dfrac{g}{N}\lambda^4$ and define e_F to be the Fermi level. Then

$$N^2 E^{(1)} = \sum_k e_k \theta(e_F - e_k)$$

$$N = \sum_k \theta(e_F - e_k). \tag{77}$$

This involves no approximation whatsoever. In the limit of large N we may use the semi classical approximation which reads

$$N^2 E^{(1)} = Ne_F - \int \frac{d\lambda\,dp}{2\pi}\,\theta\left(e_F - \frac{p^2}{2} - \frac{\lambda^2}{2} - \frac{g\lambda^4}{N} \right)\left(e_F - \frac{p^2}{2} - \frac{\lambda^2}{2} - \frac{g\lambda^4}{N} \right)$$

$$N = \int \frac{d\lambda\,dp}{2\pi}\,\theta\left(e_F - \frac{p^2}{2} - \frac{\lambda^2}{2} - \frac{g\lambda^4}{N} \right). \tag{78}$$

Integration over p yields

$$N^2 E^{(1)} = Ne_F - \int \frac{d\lambda}{3\pi}\left[2e_F - \lambda^2 - 2g\frac{\lambda^4}{N} \right]^{3/2}\theta\left(2e_F - \lambda^2 - 2g\frac{\lambda^4}{N} \right)$$

$$N = \int \frac{d\lambda}{\pi}\left[2e_F - \lambda^2 - 2g\frac{\lambda^4}{N} \right]^{1/2}\theta\left(2e_F - \lambda^2 - 2g\frac{\lambda^4}{N} \right) \tag{79}$$

rescaling λ as $\sqrt{N}\lambda$ and e_F as $N\varepsilon$ we obtain a pair of equations giving $E^{(1)}(g)$ in the planar limit as

$$E^{(1)}(g) = \varepsilon - \int \frac{d\lambda}{3\pi}[2\varepsilon - \lambda^2 - 2g\lambda^4]^{3/2}\theta(2\varepsilon - \lambda^2 - 2g\lambda^4), \tag{80a}$$

$$1 = \int \frac{d\lambda}{\pi}[2\varepsilon - \lambda^2 - 2g\lambda^4]^{1/2}\theta(2\varepsilon - \lambda^2 - 2g\lambda^4). \tag{80b}$$

It is to some extent surprising to find in such an explicit manner the ground state energy for this approximation. We observe that equation (80b) may be given the following obvious interpretation from the statistical point of view. If the equation (80a) is considered as defining $E^{(1)}$ as a function of ε and g then the Fermi level ε is

obtained from the requirement of stationarity

$$\frac{\partial E^{(1)}}{\partial \varepsilon}(\varepsilon, g) = 0.$$

The expression (80) defines an analytic function of g in the neighborhood of $g=0$. The nearest singularity occurs for negative g when the Fermi level just reaches the degenerate maximum of the potential

$$\varepsilon = -\frac{1}{16 g_c} \quad (g_c < 0)$$

$$1 = \int \frac{d\lambda}{\pi} \left(-\frac{1}{8 g_c} - \lambda^2 - 2 g_c \lambda^4 \right)^{1/2} \theta \left(-\frac{1}{8 g_c} - \lambda^2 - 2 g_c \lambda^4 \right) \tag{81}$$

or

$$g_c = -\frac{\sqrt{2}}{3\pi}.$$

The approximate formulae (80) can be compared with the results of accurate numerical computations on the true anharmonic oscillator [6].

Table 3

g	E_{planar}	E
0.01	0.505	0.507
0.1	0.542	0.559
0.5	0.651	0.696
1.0	0.740	0.804
50	2.217	2.500
1000	5.915	6.694

For very large value of g the agreement gets worse. Asymptotically one finds from (80)

$$E^{(1)}_{\text{planar}}(g) \sim \tfrac{3}{7} g^{1/3} \left[\frac{3}{2\sqrt{\pi}} \Gamma^2(3/4) \right]^{4/3} = 0.58993 \, g^{1/3} \tag{82}$$

whereas the exact result is

$$E^{(1)}(g) \sim 0.66799 \, g^{1/3}.$$

The planar approximation is therefore at most 12 % wrong.

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Note Added in Proof

We collect, some explicit formulas completing the text.

Quartic Vertices. Vacuum diagrams, Equation (20)

$$E^{(0)}(g) - E^{(0)}(0) = \sum_1^\infty (-12g)^k \frac{(2k-1)!}{k!(k+2)!}.$$

Green functions, Equation (27)

$$G_{2p}(g) = \frac{2p!}{p!(p-1)!} \sum_0^\infty (-12g)^k \frac{(2k+p-1)!}{k!(k+p+1)!}.$$

Connected Green functions, following Equation (37)

$$G_{2p}^c(g) = \frac{(3p-1)! 3^{1-p}}{(p-1)!(2p-1)!} \sum_{p=1}^\infty \frac{(-12g)^k (2k+p-1)!}{(k-p+1)!(k+2p)!}.$$

Cubic Vertices. Generating functions of connected Green functions, Equations (57) and (58)

$$\psi = 1 + \sum_{E=2}^\infty j^E \sum_{V=E-2}^\infty (-3g)^V \frac{(2E-2)!}{(E-1)!(E-2)!} \; \frac{2^{\frac{V-E}{2}} + 1 \left(\frac{3V+E}{2} - 1 \right)!}{(E+V)! \left(\frac{V-E}{2} + 1 \right)!}$$

(E = number of external lines; V = number of vertices).
This formula was known to G. 't Hooft [7].