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Lectures on the Theory of Functions of a Complex Variable

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Lectures on the Theory of Functions of a Complex Variable

Translated from the Russian
by Eugene Yankovsky

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ЛЕКЦИИ ПО ТЕОРИИ ФУНКЦИЙ КОМПЛЕКСНОГО ПЕРЕМЕННОГО

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Preface

This book is based on more than ten years experience in teaching the theory of functions of a complex variable at the Moscow Physics and Technology Institute. It is a textbook for students of universities and institutes of technology with an advanced mathematical program. We believe that it can also be used for independent study.

We have stressed the methods of the theory that are often used in applied sciences. These methods include series expansions, conformal mapping, application of the theory of residues to evaluating definite integrals, and asymptotic methods. The material is structured in a way that will give the reader the maximum assistance in mastering the basics of the theory. To this end we have provided a wide range of worked-out examples. We hope that these will help the reader acquire a deeper understanding of the theory and experience in problem solving.

The book falls into two parts. The first, consisting of Chapters I-III, V, and VI, contains the necessary minimum of information concerning the theory of functions of a complex variable that every researcher must have at his fingertips. Chapter I is introductory, providing the basic facts about complex numbers and continuous functions of a complex variable. Chapter II gives the main properties of regular functions, including the concept of analytic continuation and the analytical properties of integrals that depend on a parameter. Chapter III is devoted to the Laurent series and singularities of single-valued functions. Chapter V presents the theory of residues and its applications. We consider many important types of integrals of single- and multiple-valued analytic functions, integral transformations that are important in problems of mathematical physics (Fourier's, Mellin's, and Laplace's transforms), and integrals of the beta-function type. Chapter VI is devoted to the properties of conformal mapping and studies in detail the mappings performed by elementary functions. Additional material incorporates the Dirichlet problem, vector fields in a plane, and some physical problems from vector field theory.

The second part, Chapters IV, VII, and VIII, is aimed at a more advanced reader. Chapter IV deals with multiple-valued analytic functions and details the analytical properties and the various formulas for calculating the values, of the more important elementary functions. Special attention is paid to isolating regular branches of multiple-valued functions. In the same chapter we give the analytic theory of linear second-order ordinary differential equations. Chapter VII provides the main elementary asymptotic methods (Laplace's method, the method of stationary phase, the saddle-point method, and Laplace's method of contour integration). Finally, Chapter VIII briefly surveys operational calculus.

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Chapter I

Introduction

1 Complex Numbers

1.1 The definition of a complex number A *complex number* is a pair (x, y) of real numbers x and y for which the concept of equality and the operations of addition and multiplication are defined as follows:

(1) Two complex numbers (x_1, y_1) and (x_2, y_2) are *equal* if and only if $x_1 = x_2$ and $y_1 = y_2$.

(2) The *sum* of two complex numbers (x_1, y_1) and (x_2, y_2) is the complex number $(x_1 + x_2, y_1 + y_2)$.

(3) The *product* of two complex numbers (x_1, y_1) and (x_2, y_2) is the complex number $(x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$.

We use the same mathematical notation for equality, sum, and product of complex numbers as for real numbers. Thus, by definition,

$$(x_1, y_1) = (x_2, y_2) \quad \text{if and only if} \quad x_1 = x_2 \quad \text{and} \quad y_1 = y_2, \tag{1.1}$$

and the sum and product of two complex numbers are, respectively,

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \tag{1.2}$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1). \tag{1.3}$$

In particular, from (1.2) and (1.3) it follows that

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0), \quad (x_1, 0)(x_2, 0) = (x_1x_2, 0),$$

which shows that operations on complex numbers of the $(x, 0)$ type coincide with the corresponding operations on real numbers x. For this reason complex numbers of the $(x, 0)$ type are identified with real numbers, viz. $(x, 0) = x$.

The complex number $(0, 1)$ is known as the *unit imaginary number* and is denoted by the letter i, i.e. $i = (0, 1)$. Let us use (1.3) to calculate the product $i \times i = i^2$. We have

$$i^2 = i \times i = (0, 1)(0, 1) = (-1, 0) = -1.$$

From (1.2) and (1.3) it also follows that

$$(0, y) = (0, 1)(y, 0) = iy,$$

$$(x, y) = (x, 0) + (0, y) = x + iy.$$

Thus, each complex number (x, y) can be written as $x + iy$. It is often said that to write a complex number in the form $x + iy$ is to represent it in *algebraic form*. A complex number of the iy type is said to be *pure imaginary*. In particular, the number 0, i.e. the complex number $(0, 0)$, is the only number that is simultaneously a real number and a pure imaginary number.

The algebraic form of complex numbers enables us to write (1.1)-(1.3) thus:

$$x_1 + iy_1 = x_2 + iy_2 \text{ if and only if } x_1 = x_2 \text{ and } y_1 = y_2, \quad (1.4)$$

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2), \quad (1.5)$$

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1). \quad (1.6)$$

The complex number $x + iy$ is usually denoted by a single symbol z, i.e. $z = x + iy$. The number x is called the *real part* of the complex number $z = x + iy$ and y the *imaginary part*. The notation is

$$x = \operatorname{Re}(x + iy) = \operatorname{Re} z, \quad y = \operatorname{Im}(x + iy) = \operatorname{Im} z.$$

Here and in what follows, if not stated otherwise, it is assumed that x and y are both real.

The complex number $x - iy$ is called the *complex conjugate* of $z = x + iy$ and is denoted by $\bar{z}$, i.e.

$$\bar{z} = \overline{x + iy} = x - iy. \quad (1.7)$$

Obviously, $\overline{(\bar{z})} = z$ for every complex number z. From (1.4) we can see that $\bar{z} = z$ if and only if z is real.

The number $\sqrt{x^2 + y^2}$ is called the *absolute value* (*norm*, *modulus*) of the complex number $z = x + iy$ and is denoted by $|z|$:

$$|z| = |x + iy| = \sqrt{x^2 + y^2}. \quad (1.8)$$

Obviously, $|z| \geqslant 0$, with $|z| = 0$ if and only if $z = 0$.

We note that from (1.7) and (1.8) and from the fact that

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

it follows that

$$|z| = |\bar{z}|, \quad (1.9)$$

$$z\bar{z} = |z|^2. \quad (1.10)$$

Other properties concerning $\bar{z}$ and z are considered below.

1.2 Properties of the operation on complex numbers The operations of *addition* and *multiplication* of complex numbers possess the following properties:

(1) *Commutative laws*:

$$z_1 + z_2 = z_2 + z_1, \quad z_1z_2 = z_2z_1.$$

(2) *Associative laws*:

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \quad (z_1 z_2) z_3 = z_1 (z_2 z_3).$$

(3) *Distributive law*:

$$z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

Let us prove, say, the commutative law for addition. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (1.5) yields

$$z_1 + z_2 = (x_1 + x_2) + i (y_1 + y_2),$$
$$z_2 + z_1 = (x_2 + x_1) + i (y_2 + y_1).$$

But the commutative law for real numbers states that $x_1 + x_2 = x_2 + x_1$ and $y_1 + y_2 = y_2 + y_1$. Hence, $z_1 + z_2 = z_2 + z_1$.

The other laws in (1)-(3) can be proved in a similar manner.

Properties (1)-(3) of complex numbers imply that the operations of addition and multiplication on complex numbers $x + iy$, in the formal sense, do not take into account the fact that i is not real. For instance, there is no need in memorizing (1.5) and (1.6), since they can be obtained from the usual rules of algebra. Say,

$$(x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + ix_1 y_2 + ix_2 y_1 + i^2 y_1 y_2$$

and the fact $i^2 = -1$ imply (1.6).

The numbers zero and unity in the set of complex numbers have the same properties as in the set of real numbers, viz. for every complex number z we have

$$z + 0 = z, \quad z \times 1 = z.$$

We can introduce in the set of complex numbers an operation that is the inverse of addition. The operation is called *subtraction*, as usual. For any two numbers z_1 and z_2 there is always the (unique) number z that satisfies the equation

$$z + z_2 = z_1. \tag{1.11}$$

The number is said to be the *difference* of z_1 and z_2 and is denoted by $z_1 - z_2$. In particular, $0 - z$ is denoted by $-z$.

From (1.4) and (1.5) it follows that, for any two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, Eq. (1.11) has the unique solution $z = (x_1 - x_2) + i (y_1 - y_2)$. Thus,

$$z_1 - z_2 = (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i (y_1 - y_2). \tag{1.12}$$

We will now introduce the operation of division in the set of complex numbers. The operation that is the inverse of multiplication is known as *division*, and the *quotient* of the division of z_1 by z_2 is the (unique) number z that satisfies the equation

$$zz_2 = z_1, \tag{1.13}$$

which is denoted by $z_1 : z_2$ or z_1/z_2 or $z_1 \div z_2$.

We wish to prove that Eq. (1.13) has only one solution for any pair of complex numbers z_1 and z_2 with $z_2 \neq 0$. Multiplying both sides of Eq. (1.13) by $\bar{z}_2$ and employing (1.10), we obtain $z \mid z_2 \mid^2 = z_1\bar{z}_2$, which when multiplied by $1/\mid z_2 \mid^2$ yields $z = z_1\bar{z}_2/\mid z_2 \mid^2$. Thus,

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{\mid z_2 \mid^2}, \quad z_2 \neq 0. \tag{1.14}$$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, we can rewrite (1.14) as follows:

$$\frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{x_2^2+y_2^2} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + i\,\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}.$$

There is no need to memorize this formula; it is sufficient to remember that the result is obtained by multiplying the numerator and denominator by a number that is the complex conjugate of the denominator.

Example 1.

$$\frac{2-3i}{3+4i} = \frac{(2-3i)(3-4i)}{(3+4i)(3-4i)} = \frac{6-8i-9i+12i^2}{3^2+4^2}$$

$$= \frac{6-17i-12}{25} = -\frac{6}{25} - \frac{17}{25}i. \ \square$$

1.3 The geometric interpretation of a complex number Suppose we take a plane with a rectangular system of coordinates assigned to it. The complex number $z = x + iy$ is represented by a point in this plane with coordinates (x, y), and we can denote this point by the letter z (Fig. 1). The correspondence between complex numbers and points in a plane is obviously one-to-one. Real numbers are denoted by points on the horizontal axis (see Fig. 1), while pure imaginary numbers are denoted by points on the vertical axis. For this reason the horizontal axis is called the *real axis* and the vertical axis the *imaginary axis*. The plane whose points represent complex numbers is called the *complex (number) plane* or the *Argand plane* or the *Gauss plane*.

Clearly, the points z and $-z$ are symmetric with respect to point 0, while points z and $\bar{z}$ are symmetric with respect to the real axis, since if $z = x + iy$, then $-z = (-x) + i(-y)$ and $\bar{z} = x + i(-y)$ (Fig. 1).

A complex number z can also be represented by a position vector whose beginning is at point 0 and end at point z (Fig. 1). The correspondence between complex numbers and vectors in the complex plane whose beginnings are at point 0 is also one-to-one. For this reason a vector representing a complex number is designated by the same letter as the number.

From Fig. 1 and (1.8) we can see that the length of vector z is $|z|$ and the following inequalities hold:

$$|\operatorname{Re} z| \leqslant |z|, \quad |\operatorname{Im} z| \leqslant |z|.$$

The vector interpretation graphically illustrates the operations of addition and subtraction of complex numbers. From (1.5) it

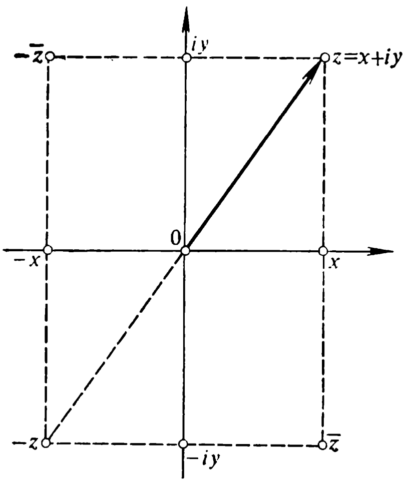


Fig. 1

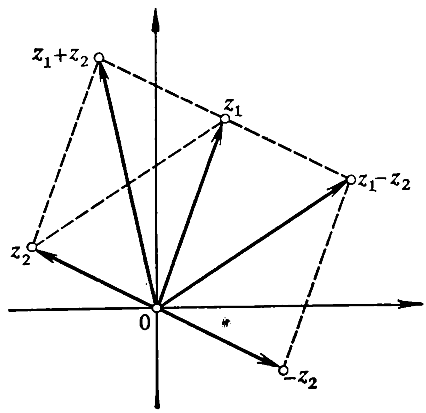


Fig. 2

follows that the number $z_1 + z_2$ is depicted by a vector built according to the common rule of adding vectors z_1 and z_2 (Fig. 2). The vector $z_1 - z_2$ is built as the sum of vectors z_1 and $-z_2$ (Fig. 2).

Figure 2 shows that the *distance between two points*, say z_1 and z_2, *is equal* to the length of vector $z_1 - z_2$, i.e. $|z_1 - z_2|$.

Example 2. The set of points z satisfying the equation $|z - z_0| = R$ is a circle of radius R centered at point z_0, since $|z - z_0|$ is the distance between the points z and z_0. □

Example 3. The points z satisfying the equation $|z - z_1| = |z - z_2|$ constitute a set of points equidistant from points z_1 and z_2, which means that this is the equation of a straight line that is perpendicular to the segment connecting points z_1 and z_2 and passes through the segment's middle. □

Example 4. (a) The set of points z satisfying the equation $|z - z_1| + |z - z_2| = 2a$, where $2a > |z_1 - z_2|$, constitute an ellipse with the foci at z_1 and z_2 and the major semiaxis equal to a, since $|z - z_1| + |z - z_2|$ is the sum of distances from point z to points z_1 and z_2.

(b) Similarly, the equation $||z - z_1| - |z - z_2|| = 2a$, where $2a < |z_1 - z_2|$, is the equation of a hyperbola with the foci at z_1 and z_2 and the real semiaxis equal to a. □

The triangle inequality *For each pair of complex numbers z_1 and z_2 we have*

$$||z_1| - |z_2|| \leqslant |z_1 + z_2| \leqslant |z_1| + |z_2|. \qquad (1.15)$$

Proof. The lengths of the sides of the triangle whose vertices are at 0, z_1, and $z_1 + z_2$ are equal to $|z_1|$, $|z_2|$, and $|z_1 + z_2|$ (Fig. 2). Consequently, the inequalities (1.15) coincide with the inequalities for the lengths of the sides of a triangle known from elementary geometry.

Corollary *For an arbitrary set of complex numbers $z_1, z_2, \ldots, z_n$ we always have*

$$\left|\sum_{k=1}^{n} z_k\right| \leqslant \sum_{k=1}^{n} |z_k|. \qquad (1.16)$$

2.1.4 The polar and exponential forms of a complex number The position of the point $z = x + iy$ in the complex plane is uniquely defined not only by its Cartesian coordinates x and y but also by its polar coordinates r and φ (Fig. 3), where $r = |z|$ is the distance from point 0 to point z, and φ is the angle between the real axis and the position vector z reckoned from the positive direction of the real axis counterclockwise if the angle is positive and clockwise if the angle is negative. This angle is called the *argument* of the complex number z ($z \neq 0$) and is denoted thus: $\varphi = \arg z$. For the number $z = 0$ the argument is undefined, which means that in all further discussions involving the argument concept the complex number z is assumed to be nonzero.

Figure 3 shows that

$$x = r\cos\varphi, \quad y = r\sin\varphi. \qquad (1.17)$$

Hence, any complex number $z \neq 0$ can be represented as follows:

$$z = r(\cos\varphi + i\sin\varphi). \qquad (1.18)$$

This form of a complex number is called *polar form*.

From (1.17) it follows that if $z = x + iy$ and $\varphi = \arg z$, then

$$\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\varphi = \frac{y}{\sqrt{x^2+y^2}}. \qquad (1.19)$$

Figure 3 shows that the converse is true, too, viz. the number φ is the argument of the complex number $z = x + iy$ only if both equations in (1.19) are true. Hence, to find the argument of a complex number we must solve the system of equations (1.19).

This system has an infinite number of solutions given by the formula $\varphi = \varphi_0 + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, where φ_0 is a particular solution of (1.19). Thus, the argument of a complex number is not determined uniquely, i.e. if φ_0 is one of values of the arguments of

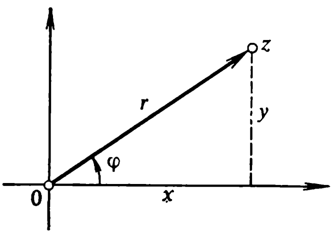


Fig. 3

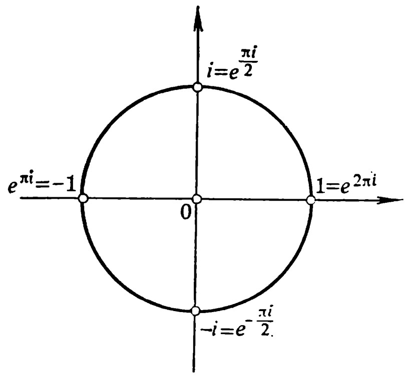


Fig. 4

a complex number z, then all the values can be found via the formula

$$\arg z = \varphi_0 + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{1.20}$$

Equations (1.19) imply that the argument φ of a complex number $z = x + iy$ satisfies the equation

$$\tan \varphi = y/x. \tag{1.21}$$

We note that all solutions of Eq. (1.21) are solutions of Eqs. (1.19).

Example 5. Let us find the argument of the complex number $z = -1 - i$. Since point $z = -1 - i$ lies in the third quadrant and $\tan \varphi = 1$, we conclude that $\arg(-1 - i) = 5\pi/4 + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. □

If $|z| = 1$ and $\varphi = \arg z$, then (1.18) implies that $z = \cos \varphi +$ sin φ. The complex number $\cos \varphi + i \sin \varphi$ is denoted by $e^{i\varphi}$, i.e. for all real φ the function $e^{i\varphi}$ is defined by the *Euler formula*

$$e^{i\varphi} = \cos \varphi + i \sin \varphi. \tag{1.22}$$

In particular, $e^{2\pi i} = 1$, $e^{\pi i} = -1$, $e^{\pi i/2} = i$, and $e^{-\pi i/2} = -i$ (Fig. 4). Note that $|e^{i\varphi}| = 1$ for any real φ. If in (1.22) we substitute $-\varphi$ for φ, we obtain

$$e^{-i\varphi} = \cos \varphi - i \sin \varphi. \tag{1.23}$$

Now, if we add (1.22) and (1.23) and subtract (1.23) from (1.22), we obtain the *Euler formulas*

$$\cos\varphi = \frac{1}{2}(e^{i\varphi} + e^{-i\varphi}),$$
$$\sin\varphi = \frac{1}{2i}(e^{i\varphi} - e^{-i\varphi}), \tag{1.24}$$

which can be used to express trigonometric function in terms of the exponential function.

The function $e^{i\varphi}$ possesses the properties of an ordinary exponential function, as if i was real. The main properties are as follows:

$$e^{i\varphi_1}e^{i\varphi_2} = e^{i(\varphi_1+\varphi_2)}, \tag{1.25}$$

$$\frac{e^{i\varphi_1}}{e^{i\varphi_2}} = e^{i(\varphi_1-\varphi_2)}, \tag{1.26}$$

$$(e^{i\varphi})^n = e^{in\varphi},\quad n=0,\ \pm 1,\ \pm 2,\ \ldots. \tag{1.27}$$

Let us prove that (1.25) is true. We have

$$e^{i\varphi_1}e^{i\varphi_2} = (\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_2)$$
$$= (\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2.$$
$$= \cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2) = e^{i(\varphi_1+\varphi_2)})$$

Equation (1.26) can be verified in a similar manner, and Eq. (1.27) can be found from (1.25) and (1.26) by induction.

Equations (1.27) and (1.22) lead to *de Moivre's formulas*

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi,\quad n = 0,\ \pm 1,\ \pm 2,\ \ldots. \tag{1.28}$$

Example 6. Let us find the sums

$$S_1 = \cos x + \cos(x+\alpha) + \cos(x+2\alpha) + \ldots + \cos(x+n\alpha),$$
$$S_2 = \sin x + \sin(x+\alpha) + \sin(x+2\alpha) + \ldots + \sin(x+n\alpha).$$

We put $S = S_1 + iS_2$. Then the Euler formula (1.22) yields

$$S = (\cos x + i\sin x) + (\cos(x+\alpha) + i\sin(x+\alpha)) + \ldots$$
$$\ldots + (\cos(x+n\alpha) + i\sin(x+n\alpha))$$
$$= e^{ix} + e^{i(x+\alpha)} + \ldots + e^{i(x+n\alpha)}.$$

For the sum of this geometric progression with the ratio $e^{i\alpha}$ we can write $S = [e^{ix}(e^{i(n+1)\alpha} - 1)]/(e^{i\alpha} - 1)$. Since $S_1 = \operatorname{Re} S$ and $S_2 = \operatorname{Im} S$, we can find both sums from this one formula. If we divide the numerator and denominator by $e^{i\alpha/2}$, then the denominator will be equal to $2i\sin(\alpha/2)$ and the numerator to $\cos(x + (n + 1/2)\alpha) - \cos(x - \alpha/2) + i[\sin(x + (n + 1/2)\alpha) - \sin(x - \alpha/2)] = 2\sin[(n+1)\alpha/2][-\sin(x + n\alpha/2) +$

$i \cos (x + n\alpha/2)]$. This leads to the final result

$$S_1 = \frac{\sin \frac{(n+1)\alpha}{2}}{\sin \frac{\alpha}{2}} \cos\left(x + \frac{n\alpha}{2}\right),$$

$$S_2 = \frac{\sin \frac{(n+1)\alpha}{2}}{\sin \frac{\alpha}{2}} \sin\left(x + \frac{n\alpha}{2}\right). \quad \square$$

From (1.18) and (1.22) it follows that any nonzero complex number z can be represented in the form

$$z = re^{i\varphi}, \tag{1.29}$$

where $r = |z|$, and $\varphi = \arg z$. This form of a complex number is called *exponential form.*

Using (1.25) and (1.26), we can easily find formulas for multiplication and division of complex numbers written in exponential form:

$$z_1 z_2 = r_1 e^{i\varphi_1} r_2 e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1+\varphi_2)}, \tag{1.30}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1-\varphi_2)}. \tag{1.31}$$

From (1.30) it follows that the absolute value of the product of two complex numbers is equal to the product of the absolute values of these numbers,

$$|z_1 z_2| = |z_1|\,|z_2|,$$

and the sum of arguments of the cofactors is the argument of the product, i.e.

if $\varphi_1 = \arg z_1$ and $\varphi_2 = \arg z_2$, then $\varphi_1 + \varphi_2 = \arg (z_1 z_2)$. (1.32)

Similarly, from (1.31) it follows that the absolute value of the quotient of two complex numbers is equal to the quotient of the absolute values of these numbers:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad z_2 \neq 0,$$

and the difference of the arguments of the dividend and the divisor is equal to the argument of the quotient, i.e.

if $\varphi_1 = \arg z_1$ and $\varphi_2 = \arg z_2$, then $\varphi_1 - \varphi_2 = \arg (z_1/z_2)$. (1.33)

Example 7.

$$(1 - i\sqrt{3})^3 (1+i)^2 = (2e^{-\pi i/3})^3 (\sqrt{2} e^{\pi i/4})^2$$
$$= 2^3 e^{-\pi i} 2 e^{\pi i/2} = 2^3 (-1) 2i = -16i. \quad \square$$

Note that the geometric interpretation (Fig. 3) leads to a rule for determining whether two complex numbers written in exponential form are equal, viz. if $z_1 = r_1e^{i\varphi_1}$ and $z_2 = r_2e^{i\varphi_2}$, then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\varphi_1 = \varphi_2 + 2k\pi$, where k is an integer. Thus, $z_1 = z_2$ if and only if

$$|z_1| = |z_2| \quad \text{and} \quad \arg z_1 = \arg z_2 + 2k\pi, \tag{1.34}$$

with k an integer.

Many formulas of analytic geometry have a straightforward appearance if we employ complex numbers. For instance, let $A_1 = (x_1, y_1)$, $A_2 = (x_2, y_2)$, and $O = (0, 0)$ be points in the (x, y) plane, and $a_1 = \overrightarrow{OA_1}$ and $a_2 = \overrightarrow{OA_2}$ vectors. These vectors correspond to complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. We have

$$\bar{z}_1z_2 = x_1y_1 + x_2y_2 + i\,(x_1y_2 + x_2y_1).$$

The real part of this expression is the scalar (dot) product of vectors a_1 and a_2,

$$a_1 \cdot a_2 = \operatorname{Re}(\bar{z}_1z_2),$$

while the imaginary part is the oriented area S of the triangle with vertices at points O, A_1, and A_2,

$$S = \operatorname{Im}(\bar{z}_1z_2) = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}.$$

Now suppose that $A = (x, y)$ is a point in the (x, y) plane. We wish to find the coordinates (x', y') of point A in a new coordinate system rotated with respect to the old coordinate system through an angle α (Fig. 5). The point A corresponds to the complex number $x + iy = re^{i\varphi}$. Then $x' + iy' = re^{i(\varphi-\alpha)} = re^{i\varphi}e^{-i\alpha} = (x + iy)^{-}e^{i\alpha}$, i.e.

$$x' + iy' = (x + iy)(\cos\alpha - i\sin\alpha).$$

Equating the real and imaginary parts, we find that

$$x' = x\cos\alpha + y\sin\alpha, \quad y' = -x\sin\alpha + y\cos\alpha.$$

1.5 Root extraction Consider the equation

$$z^n = a, \tag{1.35}$$

where a is a nonzero complex number, and n a positive integer. Let $a = \rho e^{i\theta}$ and $z = re^{i\varphi}$. Then

$$r^ne^{in\varphi} = \rho e^{i\theta}.$$

This equation yields, via property (1.34), $r^n = \rho$ and $n\varphi = \theta + 2k\pi$; whence $r = \sqrt[n]{\rho}$, $\varphi_k = (\theta + 2k\pi)/n$, and

$$z_k = \sqrt[n]{\rho}\,e^{(\theta+2k\pi)\,i/n}, \quad k = 0, \pm1, \pm2, \ldots. \tag{1.36}$$

Let us show that among the complex numbers given by (1.36) there are exactly n different numbers. Note that the numbers z_0, $z_1, \ldots, z_{n-1}$ are all different, since their arguments are

$$\varphi_0 = \frac{\theta}{n}, \quad \varphi_1 = \frac{\theta + 2\pi}{n}, \quad \ldots, \quad \varphi_{n-1} = \frac{\theta + 2\pi(n-1)}{n}$$

and the difference is less than 2π (see (1.34)). Next, $z_n = z_0$, since $|z_n| = |z^0| = \sqrt[n]{\rho}$ and $\varphi_n = \varphi_0 + 2\pi$. Similarly, $z_{n+1} = z_1$, $z_{-1} = z_{n-1}$, etc.

Thus, Eq. (1.35) with $a \neq 0$ has exactly n different roots

$$z_k = \sqrt[n]{\rho}\, e^{(\theta + 2k\pi)\, i/n}, \quad k = 0, 1, \ldots, n-1. \tag{1.37}$$

In the complex plane these roots lie at the vertices of a regular n-gon inscribed into a circle of radius $\sqrt[n]{\rho}$ centered at point 0 (Fig. 6).

Remark. The complex number z is said to be the nth *root* (not to

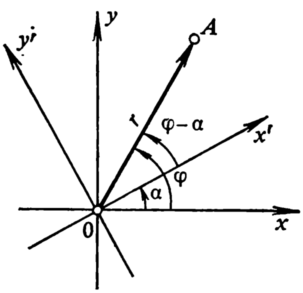


Fig. 5

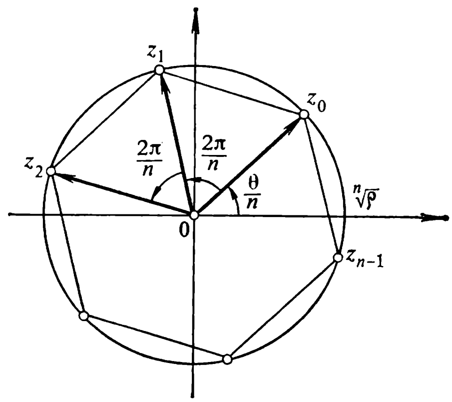


Fig. 6

be confused with the root of number "n"!) of the complex number a (and is denoted by $\sqrt[n]{a}$) if $z^n = a$. We have just shown that for $a \neq 0$ there are exactly n different nth roots of a.

1.6 Other properties of complex numbers If $z = x + iy$, then by definition (1.7) $\bar{z} = x - iy$. As already noted (see (1.9)), the absolute values of complex conjugate numbers are equal, $|z| = |\bar{z}|$. But how are the arguments of such numbers related?

Suppose that $z = re^{i\varphi}$. Then from (1.22) and (1.23) (or Fig. 7) we can see that $\bar{z} = re^{-i\varphi}$. Hence, if $\varphi = \arg z$, then $-\varphi = \arg \bar{z}$.

Note that the operation of complex conjugation is permutative with the arithmetic operations on complex numbers, viz.

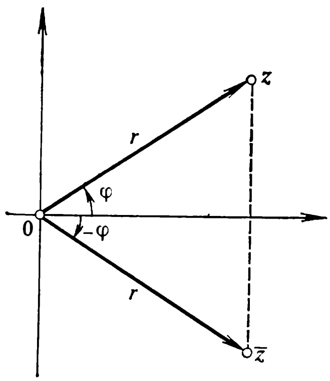


Fig. 7

$$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2,$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \quad z_2 \neq 0,$$

$$\overline{(z^n)} = (\bar{z})^n, \quad n = 0, \pm 1, \pm 2, \ldots,$$
$$z \neq 0 \quad \text{for} \quad n < 0.$$

These relationships can be verified directly.

Example 8. Let $P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ be a polynomial with real coefficients. Then

$$\overline{P(z)} = a_0 (\bar{z})^n + a_1 (\bar{z})^{n-1} + \ldots + a_n = P(\bar{z}).$$

If $P(z_0) = 0$, then $P(\bar{z}_0) = \overline{P(z_0)} = 0$, i.e. if a number z_0 is a root of a polynomial with real coefficients, its complex conjugate $\bar{z}_0$ is also a root of this polynomial. □

2 Sequences and Series of Complex Numbers

2.1 Sequences The definition of the limit of a sequence $\{z_n\}$ of complex numbers $z_1, z_2, \ldots, z_n, \ldots$ is formally the same as that of the limit of a sequence of real numbers.

Definition. A complex number a is said to be the *limit of a sequence* $\{z_n\}$ if for every positive ε there exists a positive integer $N = N(\varepsilon)$ such that

$$|z_n - a| < \varepsilon \tag{2.1}$$

for all $n > N$. We then write $\lim_{n \to \infty} z_n = a$.

In other words, a is said to be the limit of a sequence $\{z_n\}$ if

$$\lim_{n \to \infty} |z_n - a| = 0. \tag{2.2}$$

A sequence that has a limit is said to be *convergent*.

The geometric meaning of the inequality (2.1) is that point z_n lies within a circle of radius ε centered at point a (Fig. 8). This circle, i.e. the set of points z each of which satisfies the inequality $|z - a| < \varepsilon$, where $\varepsilon > 0$, is called the *ε-neighborhood of point a*. Hence, a is the limit of the sequence $\{z_n\}$ if every neighborhood of a contains all the terms of the sequence except, perhaps, a finite number of these terms.

Thus, the definition of the limit of a sequence $\{z_n\}$ coincides with the common definition of the limit of a sequence of points in a plane but formulated in terms of complex numbers.

Each sequence of complex numbers $\{z_n\}$ has two sequences of real numbers, $\{x_n\}$ and $\{y_n\}$, with $z_n = x_n + iy_n$, $n = 1, 2, \ldots$, corresponding to it. This is stated in

Theorem 1 *The fact that there exists a limit* $\lim\limits_{n\to\infty} z_n = a$, *with* $a = \alpha + i\beta$, *is equivalent to*

$$\lim_{n\to\infty} x_n = \alpha, \quad \lim_{n\to\infty} y_n = \beta.$$

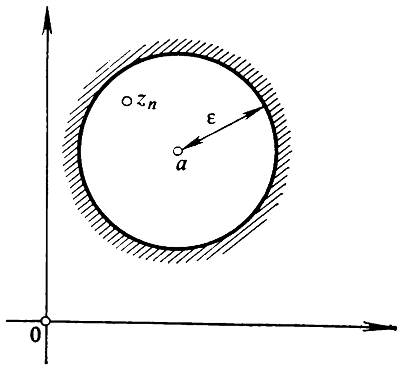


Fig. 8

Proof. Suppose there exists a limit $\lim\limits_{n\to\infty} z_n = a$, i.e. condition (2.2) is met. Then $|x_n - \alpha| \leqslant |z_n - a|$ and $|y_n - \beta| \leqslant |z_n - a|$ imply that $\lim\limits_{n\to\infty} x_n = \alpha$ and $\lim\limits_{n\to\infty} y_n = \beta$. The converse follows from the estimate

$$|z_n - a| = |(x_n - \alpha) + i(y_n - \beta)| \leqslant |x_n - \alpha| + |y_n - \beta|$$

(see (1.15)).

Theorem 1 and the properties of convergent sequences of real numbers yield the following properties of sequences of complex numbers: if $\lim\limits_{n\to\infty} z_n = a$ and $\lim\limits_{n\to\infty} \zeta_n = b$, then

$$\lim_{n\to\infty} (z_n \pm \zeta_n) = a \pm b, \tag{2.3}$$

$$\lim_{n\to\infty} (z_n \zeta_n) = ab, \tag{2.4}$$

$$\lim_{n\to\infty} \frac{z_n}{\zeta_n} = \frac{a}{b} \quad (\zeta_n \neq 0 \text{ at } n = 1, 2, \ldots;\ b \neq 0). \tag{2.5}$$

In the same manner as is done in courses of mathematical analysis we can prove

Cauchy's condition for convergence of a sequence *A sequence* $\{z_n\}$ *converges if and only if for every positive* ε *there is a positive integer* N *such that for all* $n > N$ *and* $m > N$ *we have* $|z_n - z_m| < \varepsilon$.

A sequence of complex numbers $\{z_n\}$ is said to be *bounded* if there is a number R such that $|z_n| < R$ for all positive integers n. The geometric interpretation of the limit of a sequence implies that each convergent sequence is bounded. The converse is generally not true. However, there is

Weierstrass's theorem *Out of every bounded sequence we can select a convergent subsequence.*

To prove this theorem it is sufficient to note that the boundedness of $\{z_n\}$ implies the boundedness of $\{x_n\}$ and $\{y_n\}$, where $z_n = x_n + iy_n$, and then apply Weierstrass's theorem for sequences of real numbers and use Theorem 1.

Let us investigate the properties of sequences of complex numbers that are related to the properties of the sequences of the absolute values and arguments of these numbers.

(1) The definition of the limit of a sequence and the inequality $||z_n| - |a|| \leqslant |z_n - a|$ (see (1.15)) yield the following property:

$$\text{if} \quad \lim_{n\to\infty} z_n = a, \quad \text{then} \quad \lim_{n\to\infty} |z_n| = |a|.$$

(2) The following condition is sufficient for the convergence of a sequence of complex numbers. Let $z_n = r_n e^{i\varphi_n}$, where $r_n = |z_n|$ and $\varphi_n = \arg z_n$. If $\lim\limits_{n\to\infty} r_n = \rho$ and $\lim\limits_{n\to\infty} \varphi_n = \alpha$, then $\lim\limits_{n\to\infty} z_n = \rho e^{i\alpha}$. This property follows from the formula $z_n = r_n \cos \varphi_n + ir_n \sin \varphi_n$ and Theorem 1.

2.2 The extended complex plane The following definition introduces the concept of infinity.

Definition. A sequence of complex numbers $\{z_n\}$ is said to *converge to infinity*,

$$\lim_{n\to\infty} z_n = \infty, \tag{2.6}$$

if

$$\lim_{n\to\infty} |z_n| = \infty. \tag{2.7}$$

Formally, this definition coincides with the appropriate definition for real numbers, since condition (2.7) means that for any positive R there is a positive integer N such that

$$|z_n| > R \tag{2.8}$$

for all $n > N$.

For sequences of complex numbers that tend to infinity the following properties are true:

(1) If $z_n \neq 0$, $n = 1, 2, \ldots$, then $\lim\limits_{n\to\infty} z_n = \infty$ if and only if $\lim\limits_{n\to\infty} (1/z_n) = 0$.

(2) If $\lim\limits_{n\to\infty} z_n = \infty$ and $\lim\limits_{n\to\infty} \zeta_n = a \neq \infty$, then $\lim\limits_{n\to\infty} (z_n + \zeta_n) = \infty$ and $\lim\limits_{n\to\infty} (\zeta_n/z_n) = 0$.

(3) If $\lim_{n\to\infty} z_n = \infty$ and $\lim_{n\to\infty} \zeta_n = a \neq 0$ or ∞, then $\lim_{n\to\infty} (z_n\zeta_n) = \infty$ and $\lim_{n\to\infty} (z_n/\zeta_n) = \infty$.

What is the geometric meaning of (2.7)? The inequality $|z_n| > R$ reflects the fact that point z_n lies outside a circle of radius R centered at 0 (see Fig. 9). The set of points lying outside such a circle is said to be a *neighborhood of infinity*. Hence, the point $z = \infty$, the *point at infinity*, is the limit of a sequence $\{z_n\}$ if any neighborhood of this point contains all the terms of this sequence except, perhaps, a finite number of these terms.

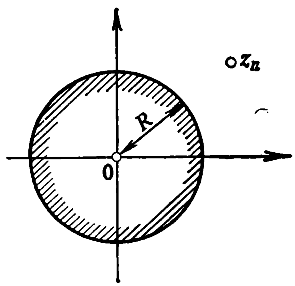


Fig. 9

Thus, to the "number" $z = \infty$ there corresponds a symbolic point at infinity. The complex plane with this point added to it is said to be the *extended complex plane*. Let us see how to interpret this concept geometrically.

Let us take a sphere S that touches the complex plane at point 0 (Fig. 10). Suppose P is the point that is the antipode of 0. To each point z in the complex plane we assign a point M that is the point

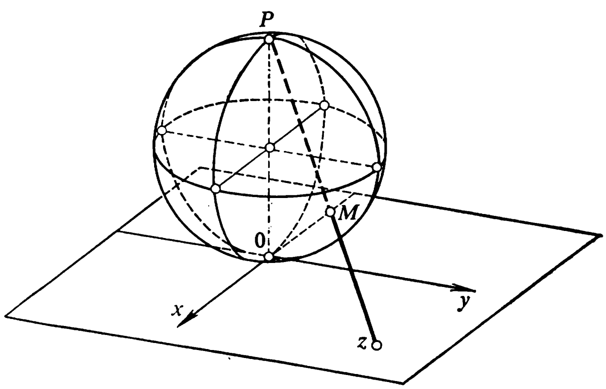


Fig. 10

at which a line connecting points z and P intersects sphere S (Fig. 10). Clearly, a sequence of points $\{z_n\}$ that converges to the point at infinity has corresponding to it a sequence of points on S that converges to P. For this reason the "image" of point $z = \infty$ is P.

Such correspondence between the points in the extended complex plane and those on the sphere S is one-to-one and is called a *stereographic projection*, while sphere S is called *Riemann's sphere.*

Complex numbers (including $z = \infty$) can be depicted by points of Riemann's sphere, and convergent sequences of complex numbers are depicted on Riemann's sphere by convergent sequences of points.

A stereographic projection maps circles into circles and angles between intersecting curves in the plane into the same angles between the images of these curves on Riemann's sphere (e.g. see Privalov [1])

Remark. The concepts of the sum, product, etc. of the complex number z and the symbol ∞ do not work, i.e. the expressions $z + \infty$, $z \times \infty$, $\infty + \infty$, and the like have no meaning. However, the following notation is often used:

$(-\infty, +\infty)$ for the real axis,

$(-i\infty, +i\infty)$ for the imaginary axis,

$(\alpha - i\infty, \alpha + i\infty)$ for the straight line $\operatorname{Re} z = \alpha$,

$(\beta i - \infty, \beta i + \infty)$ for the straight line $\operatorname{Im} z = \beta$.

Theorem *The extended complex plane is compact, i.e. out of any sequence of complex numbers we can select a convergent subsequence (converging, perhaps, to infinity).*

Proof. If the sequence $\{z_n\}$ is bounded, we can always employ Weierstrass's theorem and select a convergent subsequence out of the terms of the sequence, while if a sequence is not bounded, then for any positive integer k there is a positive integer n_k such that $|z_{n_k}| > k$. Hence $\lim\limits_{k\to\infty} z_{n_k} = \infty$.

2.3 Series *Definition.* The series

$$\sum_{k=1}^{\infty} z_k \tag{2.9}$$

is said to be *convergent* if the sequence of its partial sums $\{s_n = \sum\limits_{k=1}^{n} z_k\}$ is convergent. The limit s of the sequence $\{s_n\}$ is called the *sum* of the series (2.9): $s = \sum\limits_{k=1}^{\infty} z_k$.

The series (2.9) is said to be *absolutely convergent* if $\sum\limits_{k=1}^{\infty} |z_k|$ is convergent.

Thus, the investigation of the convergence of a series is reduced to that of the sequence of the partial sums of this series. For instance, the following properties follow from the properties of convergent sequences:

(1) The series $\sum\limits_{k=1}^{\infty} z_k$ is convergent if and only if the convergence of $\sum\limits_{k=1}^{\infty} x_k$ and $\sum\limits_{k=1}^{\infty} y_k$, where $z_k = x + iy_k$, is established. Here

$$\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} x_k + i \sum_{k=1}^{\infty} y_k.$$

(2) If the series $\sum\limits_{k=1}^{\infty} z_k$ is convergent, so is the series $\sum\limits_{k=1}^{\infty} az_k$, where a is a complex number, and

$$\sum_{k=1}^{\infty} az_k = a \sum_{k=1}^{\infty} z_k.$$

(3) If $\sum\limits_{k=1}^{\infty} z_k$ and $\sum\limits_{k=1}^{\infty} \zeta_k$ are convergent series, the series $\sum\limits_{k=1}^{\infty} (z_k + \zeta_k)$ is convergent, too, and

$$\sum_{k=1}^{\infty} (z_k + \zeta_k) = \sum_{k=1}^{\infty} z_k + \sum_{k=1}^{\infty} \zeta_k.$$

(4) If $\sum\limits_{k=1}^{\infty} z_k$ and $\sum\limits_{k=1}^{\infty} \zeta_k$ are convergent series and their sums are equal to s and σ, respectively, then the series $\sum\limits_{k=1}^{\infty} \left(\sum\limits_{n=1}^{k} z_n \zeta_{k-n+1} \right)$ is convergent, too, and its sum is $s\sigma$.

(5) Cauchy's condition for convergence of a series *A series* $\sum\limits_{k=1}^{\infty} z_k$ *is convergent if and only if for any positive* ε *there exists a positive integer* N *such that for all* $n > N$ *and* $m \geqslant n > N$ *the following inequality holds*:

$$\left| \sum_{k=n}^{m} z_k \right| < \varepsilon.$$

(6) A necessary condition for convergence of $\sum\limits_{k=1}^{\infty} z_k$ is $\lim\limits_{k\to\infty} z_k = 0$.

(7) If the series $\sum\limits_{k=1}^{\infty} | z_k |$ is convergent, so is the series $\sum\limits_{k=1}^{\infty} z_k$.

3 Curves and Domains in the Complex Plane

3.1 Complex valued functions of one real variable Suppose that a function defined as $z = \sigma(t)$ on the segment $\alpha \leqslant t \leqslant \beta$ assumes complex values. This complex valued function can be written as

$\sigma(t) = \xi(t) + i\eta(t)$, where $\xi(t) = \operatorname{Re}\sigma(t)$ and $\eta(t) = \operatorname{Im}\sigma(t)$ are real valued functions of the real variable t. Many properties of real valued functions are naturally carried over to complex valued functions.

The *limit* of the function $\sigma(t) = \xi(t) + i\eta(t)$ is defined as

$$\lim_{t\to t_0} \sigma(t) = \lim_{t\to t_0} \xi(t) + i \lim_{t\to t_0} \eta(t). \tag{3.1}$$

Thus, if $\lim\limits_{t\to t_0} \sigma(t)$ exists, then $\lim\limits_{t\to t_0} \xi(t)$ and $\lim\limits_{t\to t_0} \eta(t)$ exist, too.

This definition is equivalent to the following one: $\lim\limits_{t\to t_0} \sigma(t) = a$ if for any positive ε there is a positive δ such that $|\sigma(t) - a| < \varepsilon$ for all t such that $|t - t_0| < \delta$ with $t \neq t_0$.

We can formulate this definition in still another way, viz. $\lim\limits_{t\to t_0} \sigma(t) = a$ if $\lim\limits_{t\to t_0} \sigma(t_n) = a$ for every sequence $\{t_n\}$ such that $\lim\limits_{n\to\infty} t_n = t_0$, $t_n \neq t_0$, at $n = 1, 2, \ldots$.

The limit of a complex valued function has the following properties: if $\lim\limits_{t\to t_0} \sigma_1(t) = a_1$ and $\lim\limits_{t\to t_0} \sigma_2(t) = a_2$ exist, so do

$$\lim_{t\to t_0} [\sigma_1(t) \pm \sigma_2(t)] = a_1 \pm a_2, \quad \lim_{t\to t_0} [\sigma_1(t)\sigma_2(t)] = a_1 a_2,$$

and

$$\lim_{t\to t_0} \frac{\sigma_1(t)}{\sigma_2(t)} = \frac{a_1}{a_2} \text{ if } a_2 \neq 0.$$

The definitions and properties of $\lim\limits_{t\to t_0-0} \sigma(t)$ and $\lim\limits_{t\to t_0+0} \sigma(t)$ are similar.

We will now introduce the notion of the continuity of a complex valued function. A function defined as $\sigma(t) = \xi(t) + i\eta(t)$ is said to be *continuous* at a point or on a segment if at this point or on this segment both $\xi(t)$ and $\eta(t)$ are continuous.

This definition is equivalent to the following one: a function $\sigma(t)$ is called *continuous* at a point t_0 if $\lim\limits_{t\to t_0} \sigma(t) = \sigma(t_0)$, i.e. if for every positive ε there is a positive δ such that $|\sigma(t) - \sigma(t_0)| < \varepsilon$ for all t such that $|t - t_0| < \delta$.

Clearly, the sum, difference, and product of continuous complex valued functions are continuous functions, while the quotient of two continuous complex valued functions is a continuous function at all points where the denominator is nonzero. We note also that a complex valued function $\sigma(t)$ continuous on $[\alpha, \beta]$ is bounded on this segment, i.e. $|\sigma(t)| \leqslant M$ for a positive M and all $t \in [\alpha, \beta]$.

The *derivative* of a complex valued function $\sigma(t) = \xi(t) + i\eta(t)$ is defined as

$$\sigma'(t) = \xi'(t) + i\eta'(t). \tag{3.2}$$

Hence, $\sigma'(t)$ exists if $\xi'(t)$ and $\eta'(t)$ do.
This definition is equivalent to the following:

$$\sigma'(t) = \lim_{\Delta t \to 0} \frac{\sigma(t+\Delta t) - \sigma(t)}{\Delta t}. \tag{3.3}$$

We can easily verify that if the derivatives $\sigma_1'(t)$ and $\sigma_2'(t)$ exist, so do

$$(\sigma_1 \pm \sigma_2)' = \sigma_1' \pm \sigma_2', \quad (\sigma_1\sigma_2)' = \sigma_1'\sigma_2 + \sigma_1\sigma_2',$$

and

$$\left(\frac{\sigma_1}{\sigma_2}\right)' = \frac{\sigma_1'\sigma_2 - \sigma_1\sigma_2'}{\sigma_2^2} \quad \text{if} \quad \sigma_2(t) \neq 0.$$

However, not all the properties of real valued functions are carried over to complex valued functions. For instance, for complex valued functions, Rolle's and Lagrange's theorems are not valid, generally speaking.

Example 1. The function $\sigma(t) = e^{it}$ is differentiable on the segment $[0, 2\pi]$, with $\sigma'(t) = ie^{it}$ and $|\sigma'(t)| = 1$ for all $t \in [0, 2\pi]$. Thus, $\sigma'(t)$ does not vanish at a single point of $[0, 2\pi]$ although $\sigma(0) = \sigma(2\pi) = 1$. □

The *integral* of a complex valued function $\sigma(t) = \xi(t) + i\eta(t)$ is defined as

$$\int_\alpha^\beta \sigma(t)\,dt = \int_\alpha^\beta \xi(t)\,dt + i\int_\alpha^\beta \eta(t)\,dt. \tag{3.4}$$

This is equivalent to the following definition of an integral as the limit of the sequence of Riemann's sums:

$$\int_\alpha^\beta \sigma(t)\,dt = \lim_{\Delta \to 0} \sum_{k=1}^{n} \sigma(\tau_k)\,\Delta t_k, \tag{3.5}$$

where $\alpha = t_0 < t_1 < \ldots < t_n = \beta$, $\Delta t_k = t_k - t_{k-1}$, $t_{k-1} \leqslant \tau_k \leqslant t_k$, and $\Delta = \max\limits_{(k)} \Delta t_k$.

Obviously, a complex valued function that is continuous on a segment is integrable on this segment. It is also clear that the following properties hold:

(1) $\displaystyle\int_\alpha^\beta a\sigma(t)\,dt = a\int_\alpha^\beta \sigma(t)\,dt, \quad a = \text{const}.$

(2) $\displaystyle\int_\alpha^\beta [\sigma_1(t) \pm \sigma_2(t)]\,dt = \int_\alpha^\beta \sigma_1(t)\,dt \pm \int_\alpha^\beta \sigma_2(t)\,dt.$

(3) $\int\limits_{\alpha}^{\beta} \sigma(t)\,dt = -\int\limits_{\beta}^{\alpha} \sigma(t)\,dt.$

(4) $\int\limits_{\alpha}^{\beta} \sigma(t)\,dt = \int\limits_{\alpha}^{\delta} \sigma(t)\,dt + \int\limits_{\delta}^{\beta} \sigma(t)\,dt.$

(5) $\int\limits_{\alpha}^{\beta} \sigma(t)\,dt = \Phi(\beta) - \Phi(\alpha),$

where $\Phi(t)$ is the antiderivative of $\sigma(t)$, i.e. $\Phi'(t) = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$ (the *Newton-Leibniz formula*).

(6) *If a complex valued function* $\sigma(t)$ *is integrable on* $[\alpha, \beta]$, *the function* $|\sigma(t)|$ *is also integrable on this segment and*

$$\left|\int\limits_{\alpha}^{\beta} \sigma(t)\,dt\right| \leqslant \int\limits_{\alpha}^{\beta} |\sigma(t)|\,dt. \tag{3.6}$$

Proof. The fact that $|\sigma(t)|$ is integrable follows from the properties of integrable real valued functions. Using the inequality (1.16), we find that

$$\left|\sum_{k=1}^{n} \sigma(\tau_k)\,\Delta t_k\right| \leqslant \sum_{k=1}^{n} |\sigma(\tau_k)|\,\Delta t_k.$$

Passing to the limit as $\Delta \to 0$, where $\Delta = \max\limits_{(k)} \Delta t_k$, we arrive at (3.6).

Corollary *Property* (3.6) *leads to*

$$\left|\int\limits_{\alpha}^{\beta} \sigma(t)\,dt\right| \leqslant (\beta-\alpha) \max_{\alpha\leqslant t\leqslant\beta} |\sigma(t)|, \quad \beta > \alpha. \tag{3.7}$$

Note that for complex valued functions the mean-value theorem is invalid.

Example 2. $\int\limits_{0}^{2\pi} e^{it}dt = 0$, but the function e^{it} vanishes nowhere on $[0, 2\pi]$. □

Remark. A complex valued function $\sigma(t) = \xi(t) + i\eta(t)$ can be thought of as a vector function $(\xi(t), \eta(t))$. The concepts of the limit, continuity, and derivative for complex valued functions directly correspond to similar concepts for vector functions formulated in terms of complex numbers.

A complex valued function $z = \sigma(t)$ defined on $[\alpha, \beta]$ maps this segment into a set of points in the complex plane. This set of points can be thought of as the "graph" of the function. For instance, if

$z = \sigma(t)$ is a continuous function, its graph constitutes a curve in the complex plane.

3.2 Curves Suppose a continuous complex valued function is defined on $[\alpha, \beta]$, where both α and β are finite. It is then said that this function specifies a *continuous curve*

$$z = \sigma(t), \quad \alpha \leqslant t \leqslant \beta, \tag{3.8}$$

while Eq. (3.8) is called the *parametric equation* of the curve (when speaking of curves, we will always assume them to be continuous).

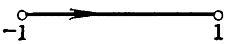


Fig. 11

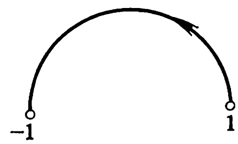


Fig. 12

If $z_1 = \sigma(t_1)$ and $z_2 = \sigma(t_2)$, where $\alpha \leqslant t_1 < t_2 \leqslant \beta$, it is said that point z_2 of curve (3.8) *follows* point z_1 (or point z_1 *precedes* point z_2). Thus, curve (3.8) constitutes an ordered set of points in the complex plane. In other words, a curve in the complex plane is always assumed to be oriented in the direction that corresponds to parameter t growing. The direction in which point z, while moving along the curve, corresponds to increasing values of t is said to be *positive*. Point $a = \sigma(\alpha)$ is called the *initial point* (beginning) and point $b = \sigma(\beta)$ the *terminal point* (end).

Suppose we have a curve γ given by (3.8). Then in the complex plane the points $z = \sigma(t), \alpha \leqslant t \leqslant \beta$, form a set of points $M(\gamma)$. This set differs from the curve, first, in that a curve constitutes an ordered set of points.

Example 3. The curve $z = \cos t$, $\pi \leqslant t \leqslant 2\pi$, is the segment $[-1, 1]$ oriented in the direction from point $z = -1$ to point $z = 1$ (Fig. 11). □

Example 4. The curve $z = e^{it}$, $0 \leqslant t \leqslant \pi$, is the semicircle $|z| = 1$, $\operatorname{Im} z \geqslant 0$ oriented counterclockwise (Fig. 12). □

The other feature that distinguishes curve γ from set $M(\gamma)$ is that there might be two points on the curve corresponding to one point in the plane, namely, if $\sigma(t_1) = \sigma(t_2)$ at $t_1 \neq t_2$, points $z_1 = \sigma(t_1)$ and $z_2 = \sigma(t_2)$ are different from the standpoint of the curve γ but coincide in the complex plane. Such points are called *self-intersection points* of a curve. An exception is when the initial and terminal points of a curve coincide, namely, if $\sigma(\alpha) = \sigma(\beta)$, then this point is not considered a self-intersection.

A curve without self-intersection points is said to be *simple*. A curve whose initial and terminal points coincide is called *closed*. (The curves of Examples 3 and 4 are simple and not closed.)

Example 5. The curve $z = e^{it}$, $0 \leqslant t \leqslant 2\pi$, is the circle $|z| = 1$ oriented counterclockwise with the initial and terminal points at $z = 1$. This is an example of a simple closed curve (or *Jordan curve*) (Fig. 13). □

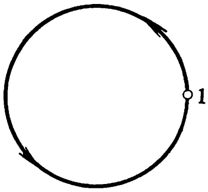


Fig. 13

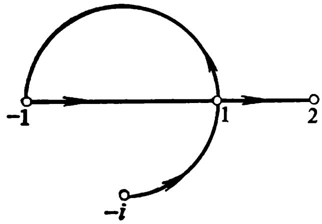


Fig. 14

Fig. 15

Example 6. The curve $z = \sigma(t)$, $-\pi/2 \leqslant t \leqslant 2\pi$, where

$$\sigma(t) = \begin{cases} e^{it}, & -\frac{\pi}{2} \leqslant t \leqslant \pi, \\ \frac{3t}{\pi} - 4, & \pi \leqslant t \leqslant 2\pi, \end{cases}$$

is an open (i.e. not closed) curve with a self-intersection at $z = 1$ (Fig. 14). The points $z_1 = \sigma(0)$ and $z_2 = \sigma(5\pi/3)$ are different points on the given curve although they coincide in the complex plane, $z_1 = z_2 = 1$. □

Example 7. The curve $z = \cos t$, $-\pi \leqslant t \leqslant \pi$, is the segment $[-1, 1]$ traversed twice, first from point $z = -1$ to point $z = 1$ and then from point $z = 1$ to point $z = -1$ (Fig. 15). This is an example of a closed curve each point of which in $(-1, 1)$ is a self-intersection point. □

Remark. Two curves, $z = \sigma_1(t)$, $\alpha_1 \leqslant t \leqslant \beta_1$, and $z = \sigma_2(\tau)$, $\alpha_2 \leqslant \tau \leqslant \beta_2$, are assumed to coincide if there exists a real function $t = s(\tau)$ that is continuous and monotonically increasing on $[\alpha_2, \beta_2]$ and such that $s(\alpha_2) = \alpha_1$, $s(\beta_2) = \beta_1$, and $\sigma_1(s(\tau)) \equiv \sigma_2(\tau)$, $\alpha_2 \leqslant \tau \leqslant \beta_2$.

Two curves that coincide correspond to the same set of points in the complex plane.

Clearly, the equation of a curve $z = \sigma_1(t)$, $\alpha \leqslant t \leqslant \beta$, can be written in the form $z = \sigma_2(t)$, $0 \leqslant \tau \leqslant 1$, by, say, introducing the substitution $t = \alpha + (\beta - \alpha)\tau$, which means that $\sigma_1(t) = \sigma_1(\alpha + (\beta - \alpha)\tau) = \sigma_2(\tau)$. Thus, without loss of generality, we can write the equation of a curve in terms of a complex valued function defined on $[0, 1]$.

Example 8. The equation of the curve considered in Example 3 can be written as $z = t$, $-1 \leqslant t \leqslant 1$, or as $z = 2t - 1$, $0 \leqslant t \leqslant 1$. □

Let us take a curve γ given by the equation $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$. We denote by γ^{-1} the curve that is obtained from γ by reversing the sense of direction on the latter. We can then write the equation of γ^{-1} as $z = \sigma(-t)$, $-\beta \leqslant t \leqslant -\alpha$.

The part of a curve γ traversed from point $z_1 = \sigma(t_1)$ to point $z_2 = \sigma(t_2)$, where t_1 and t_2 belong to $[\alpha, \beta]$, is called an *arc of curve* γ.

Let $\alpha = t_0 < t_1 < t_2 < \ldots < t_n = \beta$ and let γ_k be an arc of γ traversed from point $z_{k-1} = \sigma(t_{k-1})$ to $z_k = \sigma(t_k)$, $k = 1, 2, \ldots, n$. We will then say that the curve is *partitioned into arcs* $\gamma_1, \gamma_2, \ldots, \gamma_n$, or that γ *consists of arcs* $\gamma_1, \gamma_2, \ldots, \gamma_n$. We will reflect this fact by writing $\gamma = \gamma_1\gamma_2 \ldots \gamma_n$. The broken line with the vertices at $z_k = \sigma(t_k)$, $k = 0, 1, \ldots, n$, is said to be *inscribed in curve* γ (Fig. 16).

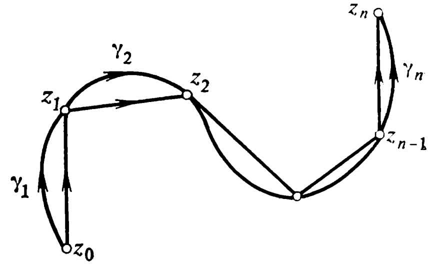


Fig. 16

Let us consider the totality of all broken lines inscribed in γ. If the set of such broken lines is bounded, γ is said to be *rectifiable* and the least upper bound of the set is called the *length* of γ.

A curve is said to be *smooth* if we can write its equation in the form $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$, where the function given by $\sigma(t)$ has a continuous and nonzero derivative, $\sigma'(t) \neq 0$, at each point belonging to $[\alpha, \beta]$; with a closed curve we must have $\sigma'(\alpha) = \sigma'(\beta)$.

A curve is said to be *piecewise smooth* if it can be partitioned into a finite number of smooth curves. A simple example of a piecewise smooth curve is a broken line.

We can write the equation of a piecewise smooth curve as $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$, where the function $\sigma(t)$ is continuous and has a piecewise continuous nonzero derivative on $[\alpha, \beta]$, i.e. $\sigma'(t) \neq 0$ on $[\alpha, \beta]$. In what follows we will write the equation of a piecewise smooth curve via such functions only.

The geometric interpretation of the derivative of a complex valued function is as follows. If a curve γ is given by the equation $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$, and at some point $t_0 \in [\alpha, \beta]$ there is a nonzero derivative $\sigma'(t) \neq 0$, then curve γ at point $z_0 = \sigma(t_0)$ has a tangent vector $\sigma'(t_0)$ (see the remark at the end of Sec. 3.1). Consequently, a piecewise smooth curve has a tangent at each of its points except, perhaps, at a finite number of them, where the tangents from the right and left have limiting positions. Such exceptional points are called *corner points of the curve.*

The reader knows from the course of mathematical analysis that a piecewise smooth curve γ: $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$, is rectifiable and its length $l(\gamma)$ is given by the formula

$$l(\gamma) = \int_{\alpha}^{\beta} |\sigma'(t)|\, dt,$$

since $|\sigma'(t)|\, dt = dl$ is the element of length of curve γ. In what follows we will consider only piecewise smooth curves.

Now let us introduce the notion of an unlimited curve. Suppose that a complex valued function $z = \sigma(t)$ is given on a ray $t \geqslant \alpha$ and $\sigma(+\infty) = \infty$, i.e. $\sigma(t) \to \infty$ and $t \to +\infty$. Then we say that we have defined an *unlimited curve*

$$z = \sigma(t), \quad \alpha \leqslant t < \infty, \tag{3.9}$$

while Eq. (3.9) is called the *parametric equation* of this curve. The unlimited curve given by (3.9) is said to be *piecewise smooth* if for each finite $\beta > \alpha$ the curve $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$, is piecewise smooth.

Similarly, we can define unlimited curves when parameter t spans the semiaxis $-\infty < t \leqslant \alpha$ or the entire real axis.

We can also write the equation of an unlimited curve (3.9) in the following form: $z = \sigma_1(\tau)$, $\alpha_1 \leqslant \tau < \beta_1$, where $\sigma_1(\tau) \to \infty$ as $\tau \to \beta_1$, with β_1 a finite number. However, for the sake of definiteness we will write the equation of such a curve only in form (3.9).

3.3 Domains A set D of points of the extended complex plane is said to be a *domain* if it is

(a) *open*, i.e. each point belonging to D has a neighborhood that belongs to D, and

(b) *connected*, i.e. any two points belonging to D can be connected by a curve (perhaps unlimited) whose all points belong to D.

A *boundary point* of D is a point in any neighborhood of which there are points belonging to D and points not belonging to D. The set of the boundary points of a domain is the *boundary* of this domain. Joining a domain D and its boundary points results in the *closure* of D denoted by $\overline{D}$.

In what follows we will only consider domains whose boundaries consist of a finite number of piecewise smooth curves and isolated points. We will also assume that all boundary curves of D are oriented in such a way that when a point travels along a boundary curve in the direction of this orientation, the domain D is to the left. Here are some examples.

Example 9. The boundary of the domain $0 < |z - a| < \varepsilon$, $\varepsilon > 0$, consists of point $z = a$ and the circle $|z - a| = \varepsilon$ oriented counterclockwise and traversed only once (Fig. 17). We will call this domain the (*open*) *circle* $|z - a| < \varepsilon$ *with point a deleted*, or the *punctured neighborhood of point a*. □

Example 10. We will depict the domain $|z| < 1$, $0 < \arg z < 2\pi$, as shown in Fig. 18 and call it the (open) circle $|z| < 1$ with a cut along the segment $[0, 1]$. The boundary Γ of this domain

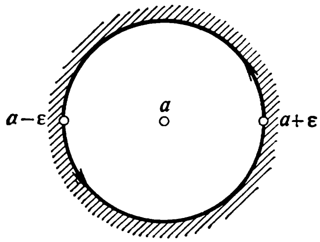


Fig. 17

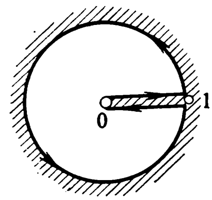


Fig. 18

consists of the following parts: the segment $[0, 1]$ traversed from point $z = 1$ to point $z = 0$ (the lower bank of the cut), the segment $[0, 1]$ traversed from point $z = 0$ to point $z = 1$ (the upper bank of the cut), and the circle $|z| = 1$ traversed one time counterclockwise. Note that to each point of the half-open interval $(0, 1]$ there correspond two different points of the boundary Γ. □

Example 11. The boundary Γ of the domain $1 < |z| < 2$ consists of two curves, i.e. $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the circle $|z| = 2$ oriented counterclockwise, and Γ_2 the circle $|z| = 1$ oriented clockwise (Fig. 19). □

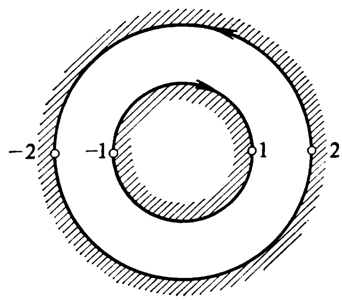


Fig. 19

A domain D is said to be *bounded* if there exists a circle K: $|z| < R$ such that $D \subset K$. Examples of bounded domains are shown in Figs. 17-19.

Example 12. The following domains are unbounded (Fig. 20):

(a) $|z| > 1$;

(b) the upper half-plane $\operatorname{Im} z > 0$ with a cut along the segment $[0, i]$;

(c) the strip $|\operatorname{Im} z| < 1$;

(d) the semistrip $|\operatorname{Im} z| < 1$, $\operatorname{Re} z > 0$ with a cut along the segment $[1, 2]$. □

A domain D in the complex plane is said to be *simply connected* if every closed curve lying in D can be continuously deformed into a point without going outside D. A continuous deformation of a curve can be given a simple geometric interpretation (Fig. 21) sufficient for our purposes, while a strict analytical definition of such a process is also possible (see Sec. 3.4).

Examples of simply connected domains are given in Figs. 18, 20b, and 20c, while the domains in Figs. 17, 19, and 20d are multiply connected.

For the extended complex plane the definition of a simply connected domain is the same as for the nonextended, the only difference

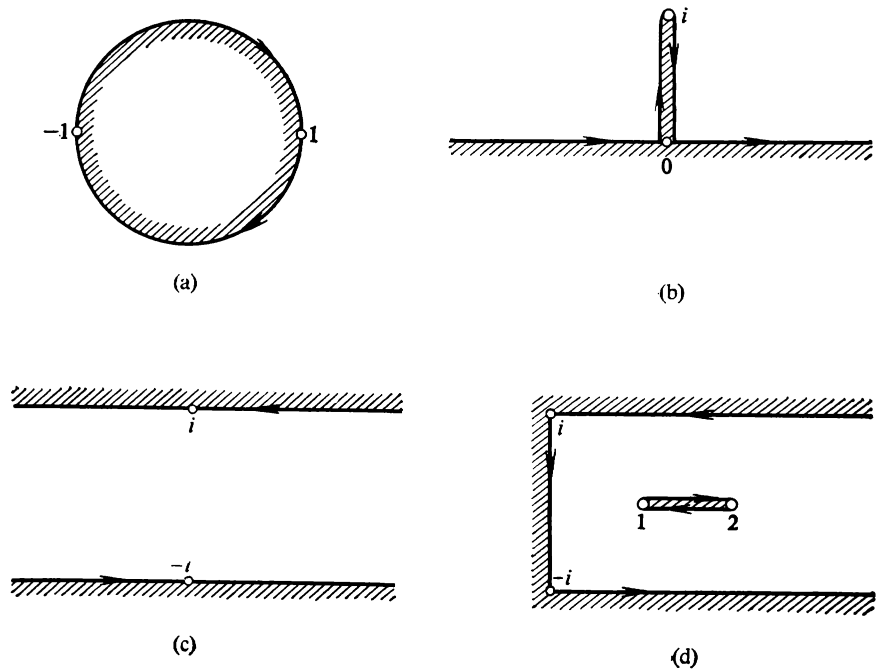


Fig. 20

being that a continuous deformation of a closed curve to point $z = \infty$ must be considered on Riemann's sphere.

Example 13. The following domains in the extended complex plane are simply connected:

(a) $|z| > 1$ (Fig. 20a);

(b) the entire extended complex plane;

(c) $z \neq a$ (the extended complex plane with point a deleted). □

Example 14. The following domains are multiply connected:

(a) $z \neq 1$, i (the extended complex plane with points 1 and i deleted);

(b) the extended complex plane with cuts along the segments $[0, 1]$ and $[i, 2i]$;

(c) $1 < |z| < \infty$. □

Domains with piecewise smooth boundaries possess the following property: the boundary of a simply connected domain in the extended complex plane consists of only one closed curve (perhaps unlimited) or only one point or not a single point (the case of the entire extended complex plane).

At this point we give without proof a theorem concerning simple closed curves.

The Jordan curve theorem *A simple closed curve (Jordan curve) separates the extended complex plane into two simply connected domains.*

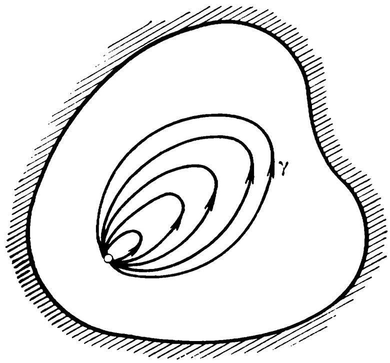


Fig. 21

For a limited simple closed curve the names of these two domains are the *interior* of the curve (the domain that does not contain the point at infinity) and the *exterior* of the curve. We will say that a simple closed curve γ is *oriented in the positive sense* if a point traversing γ in the direction of this orientation leaves the interior of γ to the left.

It is clear that a domain D in the complex plane is simply connected if and only if the interior of each simple closed curve lying in D belongs entirely to D. One can imagine a simply connected domain as a sheet of paper of arbitrary shape with, perhaps, cuts at the sides but no holes inside.

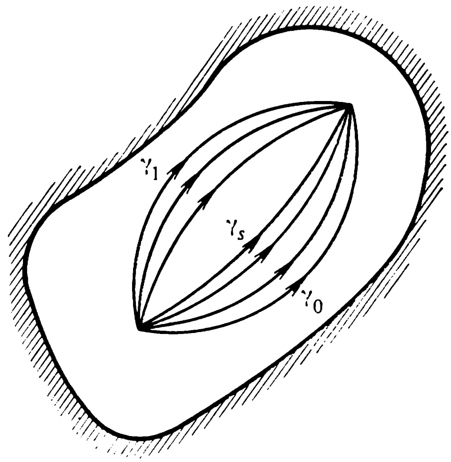


Fig. 22

3.4 Homotopic curves A continuous deformation of a curve can be interpreted geometrically (Figs. 21 and 22). Here we will give a definition of such a deformation in a strict analytical manner.

Suppose that two curves, γ_0: $z = \sigma_0(t)$, $0 \leqslant t \leqslant 1$, and γ_1: $z = \sigma_1(t)$, $0 \leqslant t \leqslant 1$, lie in a domain D and have a common beginning at point $a = \sigma_0(0) = \sigma_1(0)$ and a common end at point $b = \sigma_0(1) = \sigma_1(0)$. We will say that curve γ_0 can be continuously deformed into curve γ_1, with the process confined to D, if there exists a function $\sigma(t, s)$ that is continuous in the square $0 \leqslant t \leqslant 1$, $0 \leqslant s \leqslant 1$, and satisfies the following conditions:

(1) for each fixed $s \in [0, 1]$ the curve γ_s: $z = \sigma(t, s)$, $0 \leqslant t \leqslant 1$, lies in D;

(2) $\sigma(t, 0) \equiv \sigma_0(t)$ and $\sigma(t, 1) \equiv \sigma_1(t)$, $0 \leqslant t \leqslant 1$;

(3) $\sigma(0, s) \equiv a$ and $\sigma(1, s) \equiv b$, $0 \leqslant s \leqslant 1$.

In particular, if curve γ_0 is closed $(a = b)$ and $\sigma(t, 1) \equiv a$, $0 \leqslant t \leqslant 1$, we will say that curve γ_0 can be continuously deformed into a point, with the process confined to D.

Curves γ_2 and γ_1 are called *homotopic in* D (notation: $\gamma_0 \approx \gamma_1$ in D) if we can continuously deform curve γ_0 into curve γ_1 without leaving D (Fig. 22). A closed curve γ in D is said to be *homotopic to zero in* D (notation: $\gamma \approx 0$ in D) if this curve can be continuously deformed into a point with the process confined to D (Fig. 21).

The following properties hold:

(a) In a simply connected domain, any two curves with a common beginning and a common end are homotopic to each other, while any closed curve is homotopic to zero.

(b) Suppose that the curves $\gamma = \gamma_1\gamma_2$ and $\tilde{\gamma} = \tilde{\gamma}_1\tilde{\gamma}_2$ lie in D. Then, if $\gamma_1 \approx \tilde{\gamma}_1$ in D and $\gamma_2 \approx \tilde{\gamma}_2$ in D, we have $\gamma \approx \tilde{\gamma}$ in D.

(c) The curve $\gamma\gamma^{-1}$ is homotopic to zero in any domain that contains curve γ.

4 Continuous Functions of a Complex Variable

Suppose that on a set E in the complex plane z we have defined a complex valued function $w = f(z)$, i.e. to each point $z = x + iy \in E$ there corresponds a complex number $w = u + iv$. We can write this function in the form $f(z) = u(x, y) + iv(x, y)$, where $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$. Thus, we can think of a complex valued function of a complex number as a pair of real valued functions of two real variables.

4.1 The limit of a function Let a be a limit point of E, i.e. every neighborhood of point a contains an infinite number of points of E. A function $f(z)$ is said to tend to a *limit* A as $z \to a$ over E if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for all $z \in E$ that satisfy the condition $0 < |z - a| < \delta$ the following is true:

$$|f(z) - A| < \varepsilon.$$

We then write

$$\lim_{z \to a,\ z \in E} f(z) = A,$$

or $f(z) \to A$ as $z \to a$, $z \in E$.

The above definition of the limit of a function is equivalent to the following: $\lim\limits_{z \to a,\ z \in E} f(z) = A$ if for each sequence $\{z_n\}$, $z_n \in E$,

$z_n \neq a$ $(n = 1, 2, \ldots)$, that converges to a the sequence $\{f(z_n)\}$ converges to A, i.e. $\lim\limits_{n\to\infty} f(z_n) = A$. For brevity we will often write $\lim\limits_{z\to a} f(z)$ instead of $\lim\limits_{z\to a,\, z\in E} f(z)$.

Theorem 1 of Sec. 2 implies that the existence of the limit $\lim\limits_{z\to a} f(z)$, with $f(z) = u(x, y) + iv(x, y)$ and $a = \alpha + i\beta$, is equivalent to the existence of two limits, $\lim\limits_{\substack{x\to\alpha\\ y\to\beta}} u(x, y)$ and $\lim\limits_{\substack{x\to a\\ y\to\beta}} v(x, y)$, with

$$\lim_{z\to a} f(z) = \lim_{\substack{x\to\alpha\\ y\to\beta}} u(x, y) + i \lim_{\substack{x\to\alpha\\ y\to\beta}} v(x, y).$$

Limits of functions of a complex variable have the same properties as limits of functions of a real variable, namely, if

$$\lim_{z\to a} f(z) = A \quad \text{and} \quad \lim_{z\to a} g(z) = B,$$

then

$$\lim_{z\to a} [f(z) \pm g(z)] = A \pm B, \quad \lim_{z\to a} [f(z)\, g(z)] = AB,$$

$$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{A}{B} \quad (B \neq 0).$$

In what follows we will often use such symbols as $\sim$, o, and O. The table below gives an explanation of these symbols. Here the functions $f(z)$ and $g(z)$ are defined on set E, and a is a limit point of E.

Formula	Explanation
$f(z) \sim g(z) \quad (z\to a,\ z\in E)$	$\lim\limits_{z\to a,\ z\in E} \dfrac{f(z)}{g(z)} = 1$
$f(z) = o(g(z)) \quad (z\to a,\ z\in E)$	$\lim\limits_{z\to a,\ z\in E} \dfrac{f(z)}{g(z)} = 0$
$f(z) = O(g(z)) \quad (z\in E)$	The ratio $f(z)/g(z)$ is bounded on E, i.e. $\lvert f(z)/g(z)\rvert \leqslant M$, $z\in E$
$f(z) = O(g(z)) \quad (z\to a,\ z\in E)$	The ratio $f(z)/g(z)$ is bounded in the intersection of a neighborhood of point a with E.

The notation $f(z) = o(g(z))$ $(z \to a,\ z \in E)$ means that the function $f(z)$ is an infinitesimal compared to $g(z)$ with $z \to a$, $z \in E$. In particular, the notation $f(z) = o(1)$ $(z \to a,\ z \in E)$ means that $f(z)$ is an infinitesimal with $z \to a$, $z \in E$.

Similarly, the notation $f(z) = O(g(z))$ $(z \to a,\ z \in E)$ means that the function $f(z)$ is bounded in relation to $g(z)$ with $z \to a$, $z \in E$. In particular, the notation $f(z) = O(1)$ $(z \to a,\ z \in E)$ means that $f(z)$ is bounded with $z \to a$, $z \in E$.

Formulas of the type $f(z) \sim g(z)$ $(z \to a,\ z \in E)$ are called *asymptotic formulas*, while formulas of the type $f(z) = o(g(z))$ $(z \to a,\ z \in E)$ and $f(z) = O(g(z))$ $(z \to a,\ z \in E)$ are called *asymptotic estimates*.

Clearly the properties associated with $\sim$, o, and O for functions of a complex variable are the same as for functions of a real variable. Often for brevity we will write $(z \to a)$ instead of $(z \to a,\ z \in E)$.

Example 1. Let m and n be integers $(m > n)$. Then $z^m = o(z^n)$ $(z \to 0)$, $z^n = o(z^m)$ $(z \to \infty)$, and $z^n = O(z^m)$ $(z \in E,\ E\colon |z| \geqslant 1)$. □

Example 2. Let $P_n(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_0^n$ and $Q_m(z) = b_0 z^m + b_1 z^{m-1} + \ldots + b_m$, with $a_0 \neq 0$ and $b_0 \neq 0$. Then $P_n(z) \sim a_0 z^n$ $(z \to \infty)$ and $Q_m(z) \sim b_0 z^m$ $(z \to \infty)$. Here, if $m > n$, then $P_n(z)/Q_m(z) = o(1)$ $(z \to \infty)$, and if $m = n$, then $P_n(z)/Q_m(z) \sim a_0/b_0$ $(z \to \infty)$. □

4.2 Continuity of a function on a set Let a function $f(z)$ be defined on a set E and let a belong to E. We say that $f(z)$ is *continuous at point* a if for every positive ε there exists a positive δ such that

$$|f(z) - f(a)| < \varepsilon$$

for all $z \in E$ that satisfy the condition that $|z - a| < \delta$.

If point a is a limit point of E, then continuity of $f(z)$ at point a means that

$$\lim_{z \to a} f(z) = f(a).$$

This definition is equivalent to the following: a function $f(z) = u(x, y) + iv(z, y)$ is said to be continuous at a point $a = \alpha + i\beta$ if both $u(x, y)$ and $v(x, y)$ are continuous at point (α, β).

A function $f(z)$ is said to be *continuous on set* E if it is continuous at each point of this set.

Clearly, the sum, difference, and product of continuous functions of a complex variable are continuous, while the quotient of two continuous functions, say $f(z)$ and $g(z)$, is continuous at points where the denominator is nonzero.

A composite function in which the constituent functions are continuous is also continuous, namely, if $f(z)$ is continuous at point a

and $F(\zeta)$ is continuous at point $\zeta = f(a)$, the function $F(f(z))$ is continuous at point a, too.

Example 3. The functions z, $\operatorname{Re} z$, $\operatorname{Im} z$, z, and $|\bar{z}|$ are continuous in the entire complex plane.

Example 4. The polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n$ with complex valued coefficients is a continuous function in the entire complex plane. □

Example 5. The rational function $R(z) = P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials, is continuous at all points in the complex plane where $Q(z) \neq 0$. □

Let us introduce the following definition: a function $f(z)$ defined on E is said to be *uniformly continuous on E* if for every positive ε there exists a positive δ such that $|f(z_1) - f(z_2)| < \varepsilon$ for any two points z_1 and z_2 belonging to E that satisfy the condition $|z_1 - z_2| < \delta$.

Since the uniform continuity on E of a function $f(z) = u(x, y) + iv(x, y)$ is equivalent to the uniform continuity on E of the two functions $u(x, y)$ and $v(x, y)$, we can conclude (using the results obtained in mathematical analysis) that a function that is continuous on a closed bounded set is uniformly continuous on this set. (Note that a set is said to be closed if all its limit points belong to it.)

In our exposition we will often discuss functions that are continuous in a domain and in the closure of this domain. The following statements hold true:

(1) A function that is continuous in a domain D is uniformly continuous in a domain D_1 such that $\bar{D}_1 \subset D$.

(2) If a function is uniformly contiuuous in a bounded domain D, it can be defined at the boundary points of D in such a way that the result will be a function continuous in $\bar{D}$.

4.3 Sequences and series A sequence $\{f_n(z)\}$ is said to be *uniformly convergent on a set E* to a function $f(z)$ if for every positive ε there is a positive integer N such that

$$|f_n(z) - f(z)| < \varepsilon$$

is valid for all $n > N$ and all $z \in E$.

The series $\sum\limits_{k=1}^{\infty} g_k(z)$ is said to be *uniformly convergent on set E* if a sequence of its partial sums $S_n(z) = \sum\limits_{k=1}^{\infty} g_k(z)$ is uniformly convergent on E.

The following propositions are valid

(1) Cauchy's condition for uniform convergence of a sequence *A sequence $\{f_n(z)\}$ is uniformly convergent on set E if and only if for*

every positive ε *there is a positive integer* N *such that*

$$|f_n(z) - f_m(z)| < \varepsilon$$

for all $n > N$ *and* $m > M$ *and all* $z \in E$.

(2) Cauchy's condition for uniform convergence of a series *A series* $\sum_{k=1}^{\infty} g_k(z)$ *is uniformly convergent on set* E *if and only if for every positive* ε *there is a positive integer* N *such that*

$$\left| \sum_{k=n}^{m} g_k(z) \right| < \varepsilon$$

for all $n > N$ *and* $m \geqslant n > N$ *and all* $z \in E$.

(3) Weierstrass's condition for uniform convergence of a series *If the terms of a series* $\sum_{k=1}^{\infty} g_k(z)$ *satisfy the estimate* $|g_k(z)| \leqslant c_k$ *for all* $z \in E$, $k = 1, 2, \ldots$, *and the number series* $\sum_{k=1}^{\infty} c_k$ *is convergent, then the series in question is uniformly convergent on* E.

(4) Let $f(z) = \lim\limits_{n\to\infty} f_n(z)$, where the $f_n(z)$, $n = 1, 2, \ldots$, are continuous on E. If the sequence $\{f_n(z)\}$ is uniformly convergent on E, the function $f(z)$ is also continuous on E.

(5) Let the $g_k(z)$, $k = 1, 2, \ldots$, be continuous on set E. If the series $\sum_{k=1}^{\infty} g_k(z)$ is uniformly convergent on E, its sum $S(z) = \sum_{k=1}^{\infty} g_k(z)$ is also continuous on E.

4.4 Continuity of a function on a curve We start by taking a curve γ: $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$, and a complex valued function $w = \psi(t)$ defined on the segment $\alpha \leqslant t \leqslant \beta$. This function can be considered as a function defined on curve γ, namely, to each point $z_t = \sigma(t)$ on γ we have assigned a complex number $w = \psi(t)$. Thus, a function is defined on a curve γ if we define the following pair of functions:

$$z = \sigma(t), \quad w = \psi(t), \quad \alpha \leqslant t \leqslant \beta. \tag{4.1}$$

If γ is a simple open curve, Eqs. (4.1) define a function $w = f(z)$ that is single-valued on the set M of points $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$, of the complex plane in such a way that $w = f(\sigma(t)) \equiv \psi(t)$, $\alpha \leqslant t \leqslant \beta$.

In general, Eqs. (4.1) also define a function $w = f(z)$ of points in the complex plane, but, perhaps, a many-valued function, namely, if the curve γ has a self-intersection $\sigma(t_1) = \sigma(t_2)$, $t_1 \neq t_2$, then at point $z = \sigma(t_1) = \sigma(t_2)$ the function $f(z)$ may have two different values. However, in this case too for brevity we will write $w = f(z) = f(\sigma(t))$, $\alpha \leqslant t \leqslant \beta$, instead of Eqs. (4.1).

Let us introduce the following definition: a function $w = \psi(t)$, $\alpha \leqslant t \leqslant \beta$, is said to be continuous on a curve γ: $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$, if it is continuous on the segment $\alpha \leqslant t \leqslant \beta$; if curve γ is closed, we must have $\psi(\alpha) = \psi(\beta)$.

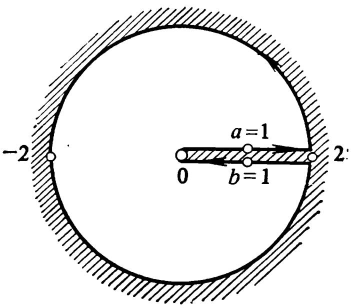


Fig. 23

The following propositions are valid:

(1) If a function is continuous in a domain D, it is continuous on each curve that lies in D.

(2) If a function is defined in a domain D and is continuous on each curve that lies in D, it is continuous in D.

4.5 Continuity of a function in a domain up to the boundary Suppose that a and b are interior or boundary points of a domain D. We define the *distance between points a and b across D as*

$$\rho_D(a, b) = \inf_{\gamma} l(\gamma),$$

where $l(\gamma)$ is the length of curve γ, and the lower bound is taken over all curves γ that connect points a and b and lie in D.

Clearly, $\rho_D(a, b) \geqslant |a - b|$ and $\rho_D(a, b) = |a - b|$ if the segment $[a, b]$ belongs to D.

Remark. It is essential that if a and b are different points on the boundary of D, the distance $\rho_D(a, b)$ is positive even if a and b coincide as points in the complex plane.

Example 6. Let D be the circle $|z| < 2$ with a cut along the segment $[0, 2]$ (Fig. 23). Then $\rho_D(-i, i) = 2$ and $\rho_D(1 - i, 1 + i) = 2\sqrt{2}$. If $a = 1$ is a point on the upper bank of the cut and $b = 1$ is a point on the lower bank (Fig. 23), then $\rho_D(a, b) = 2$, $\rho_D(a, 0) = 1$, and $\rho_D(a, 1 - i) = 1 + \sqrt{2}$. □

Let us consider a bounded domain D whose boundary Γ consists of a finite number of closed curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$. Suppose that a function $f(z)$ is defined in D and on each boundary curve Γ_k, $k = 1, 2, \ldots, n$.

Definition. A function $f(z)$ is said to be *continuous in a domain D up to its boundary* Γ if for each point a that belongs to D or Γ we have

$$\lim_{\rho_D(z, a) \to 0} f(z) = f(a).$$

Note that if the boundary point a is not a self-intersection point on the boundary curve or if a is an interior point of D, then

$$\lim_{\rho_D(z,\, a)\to 0} f(z) = \lim_{z\to a,\ z\in D} f(z).$$

Example 7. In the notation of Example 6 we have

$$\lim_{\rho_D(z,\, 2i)\to 0} f(z) = \lim_{z\to 2i,\ z\in D} f(z),$$

$$\lim_{\rho_D(z,\, 0)\to 0} f(z) = \lim_{z\to 0,\ z\in D} f(z). \ \square$$

Thus, if the boundary Γ of domain D consists of simple closed curves, the continuity of a function in D up to Γ is equivalent to the continuity of this function on $\bar{D}$. But if the boundary of D is not a simple curve, the continuity of a function in D up to Γ does not generally imply continuity of this function on $\bar{D}$.

Example 8. In the domain D of Example 6 (Fig. 23) we consider the function $f(z) = \sqrt{z} = r^{1/2}e^{i\varphi/2}$, with $z = re^{i\varphi}$ and $0 < \varphi < 2\pi$, defined at each boundary point a of D by $f(a) = \lim\limits_{\rho_D(z,\, a)\to 0} f(z)$. The function $f(z)$ is then continuous in D up to the boundary. In particular, if point $z = x > 0$ belongs to the upper bank of the cut, then

$$\lim_{\rho_D(z,\, x)\to 0} f(z) = \lim_{z\to x,\ \mathrm{Im}\, z>0} f(z) = f(x+i0) = \sqrt{x}.$$

Similarly, for a point $z = x > 0$ belonging to the lower bank we have $f(x - i0) = -\sqrt{x}$. Hence, $f(z)$ in this example is not continuous in $\bar{D}$, i.e. we cannot "paste" the function along the cut in such a way that it remains continuous. $\square$

4.6 The exponential, trigonometric, and hyperbolic functions We start with the *exponential function*. The function e^z of a complex variable $z = x + iy$ is defined thus:

$$e^z = e^{x+iy} = e^x (\cos y + i \sin y).$$

Hence,

$$\mathrm{Re}\, e^z = e^x \cos y \quad \text{and} \quad \mathrm{Im}\, e^z = e^x \sin y.$$

This definition leads to the following properties of the exponential function e^z:

(1) For any two complex numbers z_1 and z_2 we always have

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

(2) The function e^z is periodic with a period $2\pi i$:

$$e^{z+2\pi i} = e^z.$$

(3) The function e^z is continuous in the entire complex plane.
(4) For any complex number $z = x + iy$ we have

$$|e^z| = e^x, \quad \arg e^z = y.$$

(5) The function e^z assumes all values except zero, i.e. the equation $e^z = A$ is solvable for any nonzero complex number A. If $\alpha = \arg A$, all solutions of the equation $e^z = A$ are given by the formula

$$z = \ln |A| + i(\alpha + 2k\pi), \quad k = 0, \pm 1, \pm 2, \ldots. \tag{4.2}$$

In particular, if $e^z = 1$, we have $z = 2k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$.

Remark. If $e^z = A$, the complex number z is said to be the logarithm of the complex number $A \neq 0$ and is denoted by $\ln A$. Equation (4.2) implies that

$$\ln A = \ln |A| + i \arg A.$$

In particular, $\ln 1 = 2k\pi i$, $\ln(-1) = (2k + 1)\pi i$, and $\ln i = (2k + 1/2)\pi i$ (with k an integer).

Trigonometric functions. The functions $\sin z$ and $\cos z$ of a complex variable z are defined thus:

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}), \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}). \tag{4.3}$$

This definition leads to the following properties of $\sin z$ and $\cos z$:

(1) The functions $\sin z$ and $\cos z$ are continuous in the entire complex plane.

(2) The functions $\sin z$ and $\cos z$ assume all values, i.e. the equations $\sin z = A$ and $\cos z = B$ are solvable for any complex numbers A and B.

(3) All the formulas of elementary trigonometry, valid for real numbers x, are valid for all complex numbers z. For instance,

$$\sin^2 z + \cos^2 z = 1,$$
$$\sin 2z = 2 \sin z \cos z,$$
$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$$
$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

(We will prove the first formula in Sec. 15 (Example 2) in another way.)

In particular,

(a) $$\sin(z + 2\pi) = \sin z, \quad \cos(z + 2\pi) = \cos z,$$

i.e. the functions $\sin z$ and $\cos z$ are periodic functions with a period 2π, and

(b) $$\sin(-z) = -\sin z, \quad \cos(-z) = \cos z,$$

i.e. $\sin z$ is an odd function and $\cos z$ an even function.

(4) For all complex numbers $z = x + iy$ we have

$$\frac{1}{2}|e^y - e^{-y}| \leqslant |\sin z| \leqslant \frac{1}{2}(e^y + e^{-y}), \tag{4.4}$$

$$\frac{1}{2}|e^y - e^{-y}| \leqslant |\cos z| \leqslant \frac{1}{2}(e^y + e^{-y}). \tag{4.5}$$

Let us prove the validity of (4.4). Combining Eqs. (4.3) with the triangle inequality (see Eq. (1.15)), we find that

$$\frac{1}{2}\Big|\, |e^{iz}| - |e^{-iz}| \,\Big| \leqslant |\sin z| \leqslant \frac{1}{2}(|e^{iz}| + |e^{-iz}|).$$

This together with the fact that

$$|e^{iz}| = |e^{-y}e^{ix}| = e^{-y}|e^{ix}| = e^{-y}, \quad |e^{-iz}| = e^y$$

leads to (4.4). The validity of (4.5) can be proved in a similar manner.

From (4.4) and (4.5) it follows that as $y \to \infty$, the following asymptotic formulas are valid (uniformly in x, where $z = x + iy$):

$$|\sin z| \sim \frac{1}{2} e^{|y|}, \quad |\cos z| \sim \frac{1}{2} e^{|y|}.$$

Consequently, the functions $\sin z$ and $\cos z$ are unbounded in the entire complex plane (this also follows from Property 2).

(5) The following formulas are valid:

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$
$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

These formulas (or (4.4) and (4.5)) imply, for one, that the equations $\sin z = 0$ and $\cos z = 0$ have solutions only when $y = 0$, i.e. only on the real axis. Consequently, all solutions of the equation $\sin z = 0$ are given by the formula $z = k\pi$, $k = 0, \pm 1, \pm 2, \ldots$, while all solutions of the equation $\cos z = 0$ are given by the formula $z = \pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

The functions $\tan z$ and $\cot z$ are defined as

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}.$$

Properties 1 and 5 imply that the function $\tan z$ is continuous at $z \neq \pi/2 + k\pi$, while the function $\cot z$ is continuous at $z \neq k\pi$, with $k = 0, \pm 1, \pm 2, \ldots$.

Hyperbolic functions. The functions $\sinh z$ and $\cosh z$ are defined as

$$\sinh z = \frac{1}{2}(e^z - e^{-z}), \quad \cosh z = \frac{1}{2}(e^z + e^{-z}). \tag{4.6}$$

Combining (4.3) and (4.6), we can see that $\sinh z = -i \sin(iz)$ and $\cosh z = \cos(iz)$. Thus, the properties of $\sinh z$ and $\cosh z$ follow directly from the properties of $\sin z$ and $\cos z$. Note, in particular, that both $\sinh z$ and $\cosh z$ are continuous functions in the entire complex plane; all solutions of the equation $\sinh z = 0$ are given by the formula $z = k\pi i$, $k = 0, \pm 1, \pm 2, \ldots$, while all solutions of $\cosh z = 0$ are given by $z = (\pi/2 + k\pi) i$, $k = 0, \pm 1, \pm 2, \ldots$.

The functions $\tanh z$ and $\coth z$ are defined by

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}.$$

The function $\tanh z$ is continuous at $z \neq (\pi/2 + k\pi) i$, and the function $\coth z$ is continuous at $z \neq k\pi i$, with $k = 0, \pm 1, \pm 2, \ldots$.

Note that the formulas for trigonometric and hyperbolic functions valid for real x's, are valid for complex z's.

5 Integrating Functions of a Complex Variable

5.1 The definition of an integral We start with a complex valued function $f(z)$ defined on a curve γ. Let us partition γ into arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$ by the points $z_0, z_1, \ldots, z_n$ taken in the order that they appear on γ, where z_0 is the beginning and z_n the end of the curve γ (Fig. 24). Let l_k ($k = 1, 2, \ldots, n$) be the length of the arc γ_k (where z_{k-1} is the beginning and z_k the end of the kth arc) and $l = \max\limits_{1 \leqslant k \leqslant n} l_k$. On each arc γ_k we take a point $\zeta_k \in \gamma_k$ and build the Riemann sum

$$\sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}). \tag{5.1}$$

If as $l \to 0$ the integral sums (5.1) tend to a finite limit that does not depend on the choice of points z_k and ζ_k, the limit is said to be the *integral of $f(z)$ along curve γ*:

$$\int_{\gamma} f(z)\, dz = \lim_{l \to 0} \sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}). \tag{5.2}$$

Let us write $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ and introduce the notation

$$z_k = x_k + iy_k, \quad x_k - x_{k-1} = \Delta x_k, \quad y_k - y_{k-1} = \Delta y_k,$$
$$\zeta_k = \xi_k + i\eta_k.$$

Then

$$\sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} (u_k \Delta x_k - v_k \Delta y_k) + i \sum_{k=1}^{n} (v_k \Delta x_k + u_k \Delta y_k),$$

where $u_k = u(\xi_k, \eta_k)$ and $v_k = v(\xi_k, \eta_k)$. Passing to the limit with $l \to 0$, we obtain

$$\int_\gamma f(z)\,dz = \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy. \qquad (5.3)$$

Consequently, the existence of the integral $\int_\gamma f(z)\,dz$ is equivalent to the existence of two line integrals of real valued functions,

$$\int_\gamma u\,dx - v\,dy \quad \text{and} \quad \int_\gamma v\,dx + u\,dy.$$

If the curve γ is given by the equation $z = \sigma(t) = \xi(t) + i\eta(t)$,

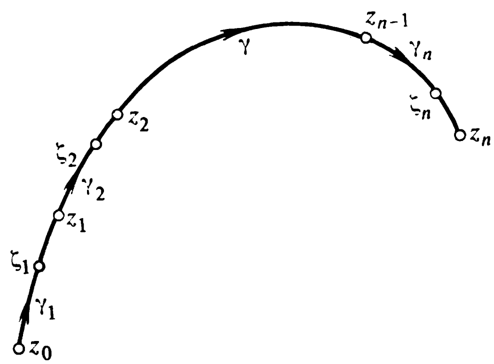


Fig. 24

$\alpha \leqslant t \leqslant \beta$, then in (5.3) $dx = \xi_1'(t)\,dt$ and $dy = \eta'(t)\,dt$ and, hence,

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta (u\xi' - v\eta')\,dt + i\int_\alpha^\beta (v\xi' + u\eta')\,dt$$

$$= \int_\alpha^\beta (u + iv)(\xi' + i\eta')\,dt = \int_\alpha^\beta f(\sigma(t))\,\sigma'(t)\,dt. \qquad (5.4)$$

Example 1. Suppose $f(z) \equiv 1$ and a and b are, respectively, the beginning and end of curve γ. Then the Riemann sum (5.1) is

$$\sum_{k=1}^{n} (z_k - z_{k-1}) = z_1 - z_0 + z_2 - z_1 + \ldots + z_n - z_{n-1} = z_n - z_0 = b - a,$$

whence $\int_\gamma dz = b - a$. Thus, this integral depends only on the beginning and end of γ and *does not depend on the path of integration.*

In this case we can write $\int_a^b dz$ instead of $\int_\gamma dz$. In particular, if $a = b$, then $\int_\gamma dz = 0$, i.e. $\int_\gamma dz$ along any closed curve is zero. □

5.2 Properties of integrals Equation (5.3) implies that a function that is continuous on a curve is integrable along the curve. The properties of line integrals lead to the following relationships:

(1)
$$\int_\gamma [af(z) + bg(z)]\, dz = a \int_\gamma f(z)\, dz + b \int_\gamma g(z)\, dz, \tag{5.5}$$

where a and b are any complex numbers (the linearity of integrals).

(2)
$$\int_\gamma f(z)\, dz = -\int_{\gamma^{-1}} f(z)\, dz, \tag{5.6}$$

i.e. the reverse of orientation of the curve changes the sign of the integral.

(3)
$$\int_{\gamma_1\gamma_2} f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz. \tag{5.7}$$

Example 2. We assume that $f(z) = z$ and γ is a curve with the beginning at point a and the end at point b. Since $f(z) = z$ is continuous on γ, the integral $\int_\gamma z\, dz$ is finite and the limit (5.2) does not depend on the choice of points z_k and ζ_k. We put $\zeta_k = z_{k-1}$. Then $\int_\gamma z\, dz = \lim_{l\to 0} S$, where $S = \sum_{k=1}^{n} z_{k-1}(z_k - z_{k-1})$. Next, if we put $\zeta_k = z_k$, we obtain $\int_\gamma z\, dz = \lim_{l\to 0} \widetilde{S}$, where $\widetilde{S} = \sum_{k=1}^{n} z_k(z_k - z_{k-1})$. Hence,

$$\int_\gamma z\, dz = \frac{1}{2} \lim_{l\to 0} (S + \widetilde{S}),$$

$$S + \widetilde{S} = \sum_{k=1}^{n} (z_k^2 - z_{k-1}^2)$$
$$= z_1^2 - z_0^2 + z_2^2 - z_1^2 + \ldots + z_n^2 - z_{n-1}^2 = z_n^2 - z_0^2 = b^2 - a^2,$$

which yield

$$\int_\gamma z\, dz = \frac{1}{2}(b^2 - a^2).$$

Thus, the value of $\int_\gamma z\,dz$ does not depend on the path of integration. In particular, $\int_\gamma z\,dz$ along any closed curve is zero. □

Example 3. Let us evaluate $I_n = \int_{C_\rho} (z-a)^n\,dz$, where n is an integer, and C_ρ the circle $|z-a| = \rho$, $\rho > 0$, oriented counterclockwise.

We write the equation of circle C_ρ in the form $z = a + \rho e^{it}$, $0 \leqslant t \leqslant 2\pi$. Then $dz = i\rho e^{it}\,dt$, and Eq. (5.4) yields

$$I_n = i\rho^{n+1} \int_0^{2\pi} e^{it(n+1)}\,dt,$$

whence for $n = -1$ we have $I_{-1} = 2\pi i$, while for $n \neq 1$ the Newton-Leibniz formula (see Sec. 3.1) yields

$$I_n = \frac{\rho^{n+1}}{n+1} e^{it(n+1)} \Big|_{t=0}^{t=2\pi} = 0.$$

Thus,

$$\int_{|z-a|=\rho} (z-a)^n\,dz = \begin{cases} 0, & n = 0,\ 1,\ \pm 2,\ \pm 3,\ \ldots, \\ 2\pi i, & n = -1. \ \square \end{cases} \tag{5.8}$$

The following property holds: if a series $f(z) = \sum_{k=1}^{n} f_n(z)$ consisting of functions $f_n(z)$ $(n = 1, 2, \ldots)$ that are continuous on a curve γ is uniformly convergent on γ, it can be integrated term-by-term, i.e.

$$\int_\gamma f(z)\,dz = \sum_{n=1}^{\infty} \int_\gamma f_n(z)\,dz$$

This follows from (5.3) and the theorem on term-by-term integration of a uniformly convergent series consisting of real terms.

5.3 Estimates of integrals

Lemma 1 *Suppose we have a function $f(z)$ continuous on a curve γ. Then the following estimate is valid:*

$$\left| \int_\gamma f(z)\,dz \right| \leqslant \int_\gamma |f(z)|\,|dz|, \tag{5.9}$$

where $|dz| = [(dx)^2 + (dy)^2]^{1/2} = ds$ is the element of length of curve γ.

Proof. We have

$$\left|\sum_{k=1}^{n} f(\zeta_k)(z_k - z_{k-1})\right| \leqslant \sum_{k=1}^{n} |f(\zeta_k)|\,|z_k - z_{k-1}|,$$

from which, going over to the limit, we arrive at estimate (5.9).

Remark. The estimate (5.9) can also be obtained from (5.4) combined with the inequality (3.6).

Corollary *Inequality* (5.9) *leads to the following estimate*:

$$\left|\int_{\gamma} f(z)\,dz\right| \leqslant Ml(\gamma), \tag{5.10}$$

where $M = \max\limits_{z\in\gamma} |f(z)|$, *and* $l(\gamma)$ *is the length of* γ.

Lemma 2 *Suppose that we have a function* $f(z)$ *continuous in a domain* D *and a curve* γ *lying in* D. *Then the integral of* $f(z)$ *along* γ *can be approximated as closely as desired by the integral of* $f(z)$ *along a broken line lying in* D, *i.e. for every positive* ε *there is a broken line* C *in* D *such that*

$$\left|\int_{\gamma} f(z)\,dz - \int_{C} f(z)\,dz\right| < \varepsilon. \tag{5.11}$$

Proof. We will take a domain D_1 such that $\bar{D}_1 \subset D$ and curve γ lies inside D_1 (the proof that such a domain exists will be given below in Lemma 4). By Property 1 of Sec. 4.2, the function $f(z)$ is uniformly continuous in D_1, i.e. for a fixed and positive ε there is a positive δ such that for all $z \in D_1$, $\zeta \in D_1$, $|z - \zeta| < \delta$ the following inequality holds true:

$$|f(z) - f(\zeta)| < \frac{\varepsilon}{3l}, \tag{5.12}$$

where l is the length of curve γ.

Let us partition curve γ into arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$ by the ordered points $z_0, z_1, \ldots, z_n$ in such a way that the length l_k of arc γ_k is smaller than δ and the circles $|z - z_k| < \delta$ $(k = 1, 2, \ldots, n)$ belong to D_1 (this can be done in the same manner as in Lemma 4 which follows). Let C be a broken line with vertices at $z_0, z_1, \ldots, z_n$, and c_k the segment $[z_{k-1}, z_k]$, with $\lambda_k = |z_k - z_{k-1}|$ the length of c_k $(k = 1, 2, \ldots, n)$.

Since l_k is less than δ, we conclude that $\lambda_k \leqslant l_k < \delta$, i.e. arc γ_k and segment c_k lie in the circle $|z - z_k| < \delta$ $(k = 1, 2, \ldots, n)$. Hence, in view of (5.12), for all $z \in \gamma_k$ and $z \in c_k$ the following estimate is valid:

$$|f(z) - f(z_k)| < \frac{\varepsilon}{3l}. \tag{5.13}$$

Let us now prove (5.11). We have

$$\int_{\gamma} f(z)\,dz-\int_{C} f(z)\,dz=\sum_{k=1}^{n}\left[\int_{\gamma_k} f(z)\,dz-\int_{c_k} f(z)\,dz\right]$$

$$=\sum_{k=1}^{n}\left\{\left[\int_{\gamma_k} f(z)\,dz-\int_{\gamma_k} f(z_k)\,dz\right]-\left[\int_{c_k} f(z)\,dz-\int_{\gamma_k} f(z_k)\,dz\right]\right\};$$

whence, using the fact that

$$\int_{\gamma_k} f(z_k)\,dz=\int_{c_k} f(z_k)\,dz=f(z_k)(z_k-z_{k-1})$$

(see Example 1), we obtain

$$\left|\int_{\gamma} f(z)\,dz-\int_{C} f(z)\,dz\right|$$

$$\leqslant\sum_{k=1}^{n}\left\{\left|\int_{\gamma_k}[f(z)-f(z_k)]\,dz\right|+\left|\int_{c_k}[f(z)-f(z_k)]\,dz\right|\right\}. \qquad (5.14)$$

Applying to each term on the right-hand side of (5.14) the estimates (5.10) and (5.13), we finally obtain

$$\left|\int_{\gamma} f(z)\,dz-\int_{C} f(z)\,dz\right|\leqslant\sum_{k=1}^{n}\left(\frac{\varepsilon}{3l}\,l_k+\frac{\varepsilon}{3l}\,\lambda_k\right)\leqslant\frac{\varepsilon}{3l}\times 2l<\varepsilon.$$

Lemma 3 *Let D be a simply connected bounded domain and Γ its boundary. If a function $f(z)$ is continuous in D up to Γ, the integral of $f(z)$ along Γ can be approximated as closely as desired by the integral of $f(z)$ along a closed broken line lying in D.*

The proof of Lemma 3 is beyond the scope of our book.

Let us take a multiply connected domain. Let the boundary Γ of a bounded domain D consist of the curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$, i.e. $\Gamma=\bigcup_{k=1}^{n}\Gamma_k$. If a function $f(z)$ is continuous in D up to Γ, the integral of $f(z)$ along Γ can be found from the following expression:

$$\int_{\Gamma} f(z)\,dz=\sum_{k=1}^{n}\int_{\Gamma_k} f(z)\,dz. \qquad (5.15)$$

Lemma 3 leads to the following

Corollary *If a function $f(z)$ is continuous in a domain D up to the boundary, the integral of $f(z)$ along the boundary can be approximated as close as desired by the sum of integrals of $f(z)$ along closed broken lines lying in D.*

Lemma 4 *A curve γ lying in a domain D can be covered by a finite set of circles belonging to D.*

Proof. Suppose Γ is the boundary of D, and $\rho(\gamma, \Gamma) = \inf_{z\in\gamma,\, \zeta\in\Gamma} |z - \zeta|$ is the distance between curve γ and the boundary Γ. From mathematical analysis the reader should know that $\rho(\gamma, \Gamma) > 0$.

We partition the curve γ into arcs $\gamma_1, \gamma_2, \ldots, \gamma_n$ by a sequence of points $z_0, z_1, \ldots, z_n$, where z_0 is the beginning and z_n the end of γ, in such a way that the length l_j of arc γ_j is less than $\rho(\gamma, \Gamma)/4$:

$$l_j < \frac{1}{4}\rho(\gamma, \Gamma), \quad j = 1, 2, \ldots, n.$$

We wish to show that γ can be covered by the following sequence of circles:

$$K_j\colon |z - z_j| < \frac{1}{2}\rho(\gamma, \Gamma), \quad j = 0, 1, \ldots, n. \tag{5.16}$$

Indeed, if $z \in \gamma_j$, then

$$|z - z_j| \leqslant l_j < \frac{1}{4}\rho(\gamma, \Gamma) < \frac{1}{2}\rho(\gamma, \Gamma), \tag{5.17}$$

i.e. the arc γ_j lies within the circle K_j ($j = 1, 2, \ldots, n$).

Corollary *Let D_1 be the union of all the circles given in* (5.16), *i.e.* $D_1 = \bigcup_{j=0}^{n} K_j$. *Then the curve γ lies in D_1 and $\bar{D}_1 \subset D$,* with $\rho(\gamma, \Gamma_1) \geqslant \rho(\gamma, \Gamma)/4$, *where Γ_1 is the boundary of D_1.*

6 The Function arg z

The function $\ln z$ was defined in Sec. 4 via the formula $\ln z = \ln |z| + i \arg z$. In Chap. IV we will show that all the elementary many-valued functions can be expressed in terms of the logarithmic function. For this reason we must thoroughly investigate the many-valued function $\arg z$.

The properties of the function $\arg z$ and especially the properties of the variation of this function along a curve (see Sec. 6.2) will be widely used starting from Sec. 21. Some of these properties, however, we will need in Sec. 13.

6.1 Polar coordinates As is known, the Cartesian and polar coordinates of a point z ($z = x + iy$) in the complex plane are related by the formulas

$$x = r\cos\varphi, \quad y = r\sin\varphi. \tag{6.1}$$

If the polar coordinates (r, φ) of a point are given, the Cartesian coordinates (x, y) are determined in a *unique manner* from Eqs. (6.1). If the Cartesian coordinates (x, y) are known, (6.1) uniquely determine $r = (x^2 + y^2)^{1/2}$, but φ is not determined uniquely, namely, the equations

$$\cos\varphi = \frac{x}{\sqrt{x^2+y^2}}, \quad \sin\varphi = \frac{y}{\sqrt{x^2+y^2}} \tag{6.2}$$

determine $\varphi = \arg z$ only to within a constant term, $2k\pi$, where k is an integer.

This fact (the nonuniqueness of the relation between (x, y) and (r, φ)) is not important when we are dealing with single-valued

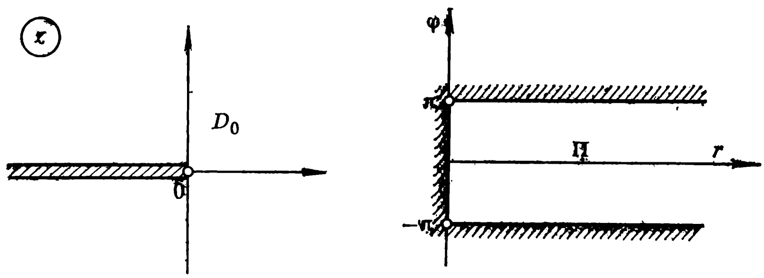


Fig. 25

functions. But when we are studying many-valued functions (such as $\ln z$) it becomes important.

Note that Eqs. (6.1) establish a one-to-one relationship between the entire complex z plane and the set $\{0 < r < \infty,\ -\pi < \varphi \leqslant \leqslant \pi\} \cup \{r = 0,\ \varphi = 0\}$ in the (r, φ) plane. This set is not a domain, i.e. an open connected set, while the complex z plane is. One-to-one correspondence can be achieved if, say, we consider the semistrip Π: $\{0 < r < \infty,\ -\pi < \varphi < \pi\}$. In the complex z plane the semistrip corresponds to the domain D_0: the plane cut along the semiaxis $(0, \infty]$ (Fig. 25).

In domain D_0 we will reckon the polar angle φ from the semiaxis $x > 0$ (for $x > 0$ we assume that $\varphi = 0$). Then φ will vary from $-\pi$ to π, and to each point $z \in D_0$ there will correspond only one value of φ equal to $\varphi(z)$. This means that in D_0 we have specified a single-valued and continuous function $\varphi(z)$. This function can be differentiated an infinite number of times in the variables x and y in D_0, since the mapping (6.1) of the semistrip Π on D_0 can be differentiated an infinite number of times, is a one-to-one mapping, and its Jacobian $J = r$ does not vanish.

The function $\varphi(z)$ is determined uniquely by Eqs. (6.2) since $-\pi < \varphi < \pi$. At every point $z \in D_0$ the value of $\varphi(z)$ corresponds to one of the values of the many-valued function $\arg z$ (see Sec. 1.4). For this reason the function $\varphi(z)$ is said to be a (*single-valued*) *con-*

tinuous branch of arg z, which for brevity we will denote by $\varphi =$ arg z.

The domain D_0 contains an infinite number of continuous branches $\varphi_k(z)$ of arg z. All are described by the single formula

$$\varphi_k(z) = \varphi(z) + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots .$$

We can express, via (6.2), the function $\varphi = \arg z$ defined above in terms of inverse trigonometric functions:

$$\arg z = \arctan(y/x) \quad \text{if} \quad x > 0,$$
$$\arg z = \pi + \arg\tan(y/x) \quad \text{if} \quad x < 0 \quad \text{and} \quad y > 0,$$
$$\arg z = -\pi + \arctan(y/x) \quad \text{if} \quad x < 0 \quad \text{and} \quad y < 0.$$

However, in the case of an arbitrary domain there is no simple

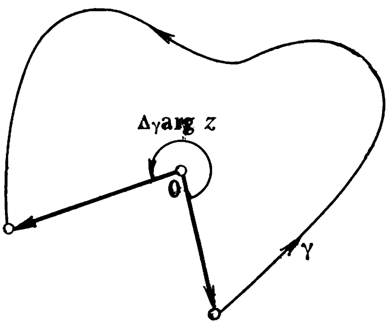


Fig. 26

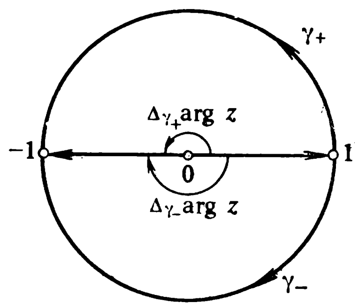


Fig. 27

formula that relates a branch of arg z and inverse trigonometric functions, since these functions vary either from $-\pi/2$ to $\pi/2$ or from 0 to π, while the function arg z may vary within any limits. A more convenient representation of arg z is the integral representation (see Eq. (6.15) below).

6.2 Variation of the argument along a curve Suppose that we have a curve γ that does not pass through the point $z = 0$. We will call the angle of rotation of vector z as point z moves along curve γ from the beginning to the end of γ the *variation of* arg z *along curve* γ and will denote it by $\Delta_\gamma \arg z$ (Fig. 26).

Example 1. (a) If γ is the line segment with the beginning at point $1 - i$ and end at point $1 + i$, then $\Delta_\gamma \arg z = \pi/2$.

(b) If γ_+ is the semicircle $|z| = 1$, $\operatorname{Im} z \geqslant 0$ oriented counterclockwise, then $\Delta_{\gamma_+} \arg z = \pi$ (Fig. 27).

(c) If γ_- is the semicircle $|z| = 1$, $\operatorname{Im} z \leqslant 0$ oriented clockwise, then $\Delta_{\gamma_-} \arg z = -\pi$ (Fig. 27). □

Let us derive a formula for determining $\Delta_\gamma \arg z$. From (6.1) we have

$$dx = \cos\varphi\, dr - r\sin\varphi\, d\varphi, \quad dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi, \tag{6.3}$$

whence $r\,d\varphi = -\sin\varphi\, dx + \cos\varphi\, dy$. Consequently,

$$d\varphi = d\arg z = \frac{-y\,dx + x\,dy}{x^2+y^2}. \tag{6.4}$$

Now let us investigate the integral $\int\limits_\gamma d\arg z$. It is equal to the difference of the values of arg z at the end and beginning of γ, i.e. the variation of arg z along the curve, or $\Delta_\gamma \arg z$. Consequently,

$$\Delta_\gamma \arg z = \int\limits_\gamma \frac{-y\,dx + x\,dy}{x^2+y^2}. \tag{6.5}$$

We can write this formula as

$$\Delta_\gamma \arg z = \operatorname{Im} \int\limits_\gamma \frac{d\zeta}{\zeta}, \tag{6.6}$$

since

$$\operatorname{Im}\frac{dz}{z} = \operatorname{Im}\frac{dx + i\,dy}{x+iy} = \frac{-y\,dx + x\,dy}{x^2+y^2},$$

and the integration variable in (6.6) can be denoted by any letter.

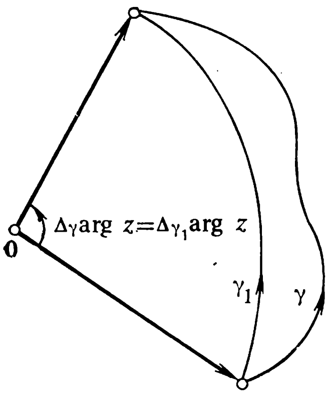


Fig. 28

Let us study the properties of the variation of arg z.

(1) Suppose that a curve γ can be continuously deformed into a curve γ_1 that does not pass through point $z = 0$ (i.e. the curves γ and γ_1 are homotopic in the domain $0 < |z| < \infty$) (Fig. 28). Then we have

$$\Delta_\gamma \arg z = \Delta_{\gamma_1} \arg z. \tag{6.7}$$

Proof. Consider the integral

$$\int\limits_\gamma P\,dx + Q\,dy, \tag{6.8}$$

where the functions $P(x, y)$, $Q(x, y)$, $\partial P/\partial y$, and $\partial Q/\partial x$ are continuous in domain D and curve γ lies in D. In courses of mathematical analysis (see Kudryavtsev [1]) the reader will find the proof of the following

Theorem *If a domain D is simply connected, then the integral* (6.8) *along any closed curve γ lying in D is zero if and only if*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{6.9}$$

at every point of D.

The theorem leads to the following

Corollary *If in a domain D (perhaps multiply connected) condition (6.9) is met and curve γ can be continuously deformed into curve γ_1, with the process confined to D (i.e. the curves γ and γ_1 are homotopic in D), then*

$$\int_{\gamma} P\,dx + Q\,dy = \int_{\gamma_1} P\,dx + Q\,dy. \tag{6.10}$$

In (6.10) we put

$$P(x, y) = \frac{-y}{x^2+y^2}, \quad Q(x, y) = \frac{x}{x^2+y^2}.$$

We can now directly verify that these functions satisfy condition (6.9) in the domain $0 < |z| < \infty$. Thus, from (6.5) and (6.10) we obtain (6.7).

Equation (6.7) also follows from the geometric interpretation of the variation of arg z along a curve (see Fig. 28).

Property 1 leads, among other things, to the following:

(2) Suppose that a closed curve γ does not pass through point $z = 0$ and can be continuously deformed into a point without crossing point $z = 0$ (i.e. the curve is homotopic to zero in the domain $0 < |z| < \infty$). Then

$$\Delta_\gamma \arg z = 0. \tag{6.11}$$

Note that Eq. (6.7) does not hold for all curves γ and γ_1 with a common beginning and a common end (cf. Examples 1b and 1c). In the same way (6.11) does not hold for all closed curves γ.

Example 2. If γ is the circle $|z| = 1$ oriented counterclockwise and traversed once, then $\Delta_\gamma \arg z = 2\pi$. □

Here are other properties of the variation of the argument.

(3) If a curve γ does not pass through point $z = 0$, then

$$\Delta_\gamma \arg z = -\Delta_{\gamma^{-1}} \arg z. \tag{6.12}$$

(4) If a curve $\gamma = \gamma_1\gamma_2$ does not pass through point $z = 0$, then

$$\Delta_{\gamma_1\gamma_2} \arg z = \Delta_{\gamma_1} \arg z + \Delta_{\gamma_2} \arg z. \tag{6.13}$$

The last two properties follows from Eq. (6.6) and the properties of integrals (see Eqs. (5.6) and (5.7)).

6.3 Continuous branches of arg z Suppose D is a simply connected domain not containing points $z = 0$ and $z = \infty$. We fix a point $z_0 \in D$ and find arg z_0, one of the values of the argument of z_0. We put

$$\arg z = \arg z_0 + \Delta_\gamma \arg z, \tag{6.14}$$

where the curve γ with the beginning at point z_0 and the end at point z lies in D.

By Property 1 of Sec. 6.2, the variation of the argument, $\Delta_\gamma \arg z$, does not depend on the shape of curve γ, since in a simply connected domain any curves with a common beginning and a common end can be continuously deformed into each other, with the process confined

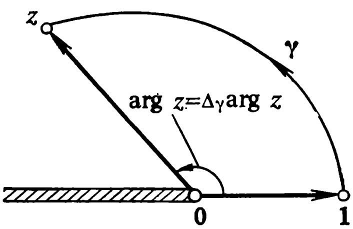
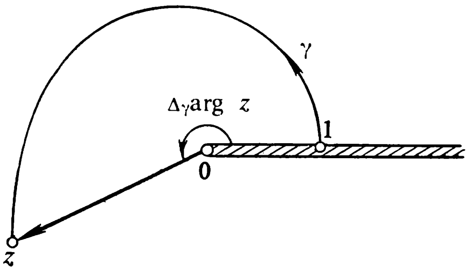


Fig. 29



ig. 30

to D (see Sec. 3.4). Consequently, the function given by (6.14) is single-valued in D. It is also continuous in D, since we can write

$$\arg z = \arg z_0 + \int\limits_{z_0}^{z} \frac{-y\,dx + x\,dy}{x^2+y^2}. \tag{6.15}$$

Thus the function given by (6.14) is a *single-valued continuous branch of the many-valued function* arg z *in* D.

There is obviously an infinite number of such branches:

$$(\arg z)_k = \arg z_0 + \Delta_\gamma \arg z + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots, \tag{6.16}$$

i.e. the many-valued function arg z in D splits into single-valued continuous branches given by (6.16). This implies that a continuous branch of arg z in D is determined solely by the value of this function at a single point $z_0 \in D$.

Example 3. Let D_0 be the entire complex plane with a cut along the ray $(-\infty, 0]$. We put $z_0 = 1$ and $\arg 1 = 0$. Then

$$\arg z = \Delta_\gamma \arg z, \tag{6.17}$$

where curve γ starts at $z_0 = 1$, ends at z, and lies in D_0 (Fig. 29). Obviously, the function given by (6.17) coincides with the function $\varphi(z)$ considered in Sec. 6.1. In particular, we have

$$\begin{aligned} \arg x &= 0 && \text{if} \quad x > 0, \\ \arg (iy) &= \pi/2 && \text{if} \quad y > 0, \\ \arg (iy) &= -\pi/2 && \text{if} \quad y < 0, \quad \text{etc.} \ \square \end{aligned}$$

In Eq. (6.14) the point z_0 can be a boundary point of D.

Example 4. Let D_1 be the entire complex plane with a cut along the ray $[0, +\infty)$, $z_0 = 1$ a point on the upper bank of the cut, and

$\arg z_0 = \arg 1 = 0$. Then

$$\arg z = \Delta_\gamma \arg z, \tag{6.18}$$

where curve γ starts at $z_0 = 1$, ends at z, and lies in D (Fig. 30). In particular, we have

$$\begin{aligned} \arg(iy) &= \pi/2 \quad &&\text{if} \quad y > 0, \\ \arg x &= \pi \quad &&\text{if} \quad x < 0, \\ \arg(iy) &= 3\pi/2 \quad &&\text{if} \quad y < 0, \quad \text{etc.} \end{aligned}$$

(see Example 3).

Let us find the values of the function (6.18) on the upper and lower

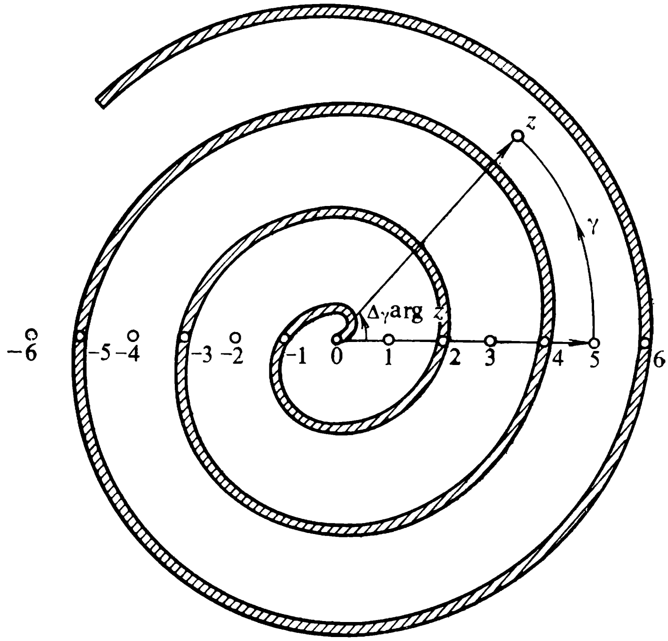


Fig. 31

banks of the cut. At $x > 0$ we have $\arg(x + i0) = \lim\limits_{y \to +0} \arg(x + iy) = 0$; similarly $\arg(x - i0) = 2\pi$. Hence, the function given by (6.18) cannot be "pasted" along the ray $(0, +\infty)$ in such a way that it remains constant. (The function (6.17) cannot either be pasted continuously along the ray $(-\infty, 0)$.) This, among other things, means that in the domain $0 < |z| < \infty$ we cannot isolate a continuous branch of $\arg z$. □

Equation (6.14) shows that the function given by (6.14) is single-valued in D if and only if the variation of the argument, $\Delta_\gamma \arg z$, does not depend on the choice of curve γ, i.e. along any closed curve lying in D the variation of the argument is zero. In other words, domain D must not contain simple closed curves with point $z = 0$ in their interiors, i.e. we must ensure that there is no possibility of circling point $z = 0$ (the same holds for point $z = \infty$). The entire complex plane with a cut along an unlimited curve that connects points $z = 0$ and $z = \infty$ is such a domain. In it and in any of its subdomains the many-valued function $\arg z$ can be separated into single-valued continuous branches.

Note that the function given by (6.17) varies between $-\pi$ and π, namely, $-\pi < \arg z < \pi$, $z \in D_0$, while the function given by (6.18) varies between 0 and 2π, namely, $0 < \arg z < 2\pi$, $z \in D_1$. But in the case of an arbitrary domain a continuous branch of $\arg z$ can vary within any limits.

Example 5. Let D_2 be the entire complex plane with a cut along the curve $z = (1/\pi)\, e^{it}$, $0 \leqslant t < \infty$. We put $\arg 5 = 2\pi$. Then

$$\arg z = 2\pi + \Delta_\gamma \arg z,$$

where curve γ starts at point $z_0 = 5$, ends at z, and lies in D_2 (Fig. 31). In particular, calculating the values of the variation of the argument, we find: $\arg(-6) = 3\pi$, $\arg 7 = 4\pi$, $\arg(-4) = \pi$, $\arg 3 = 0$, $\arg(-2) = -\pi$, $\arg 1 = -2\pi$, etc. □

Chapter II

Regular Functions

7 Differentiable Functions. The Cauchy-Riemann Equations

7.1 The Derivative Let a (complex valued) function $f(z)$ be defined in a neighborhood of a point z_0. If the quotient $[f(z_0 + \Delta z) - f(z_0)]/\Delta z$ tends to a finite limit as $\Delta z \to 0$, then we say that this limit is the *derivative* of $f(z)$ at point z_0 and denote it by $f'(z_0)$ and call function $f(z)$ *differentiable at point* z_0. Thus,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}. \tag{7.1}$$

A function is said to be *differentiable in a domain* if it is differentiable at every point of the domain.

We put $\Delta f = f(z_0 + \Delta z) - f(z_0)$. Then we can write Eq. (1) in the following form:

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = f'(z_0). \tag{7.2}$$

This means that for each positive ε there exists a positive $\delta = \delta(\varepsilon)$ such that

$$\left| \frac{\Delta f}{\Delta z} - f'(z_0) \right| < \varepsilon$$

when $0 < |\Delta z| < \delta$. From (7.2) it follows that

$$\Delta f = f'(z_0)\,\Delta z + o(\Delta z) \quad (\Delta z \to 0).$$

Conversely, if the increment Δf of a function $f(z)$ can be written as

$$\Delta f = A\,\Delta z + o(\Delta z), \tag{7.3}$$

with A a complex valued constant independent of Δz, then $f(z)$ is differentiable at point z_0, and $A = f'(z_0)$.

Thus, Eq. (7.3) is both necessary and sufficient for $f(z)$ to be differentiable at z_0. One result that follows from Eq. (7.3) is that a function differentiable at point z_0 is continuous at this point.

Example 1. The function $f(z) = z^n$ ($n \geqslant 1$ is an integer) is differentiable in the entire complex plane since

$$\lim_{\Delta z \to 0} \frac{(z+\Delta z)^n - z^n}{\Delta z} = \lim_{\Delta z \to 0} \frac{nz^{n-1}\Delta z + o(\Delta z)}{\Delta z} = nz^{n-1}. \tag{7.4}$$

Therefore,

$$(z^n)' = nz^{n-1}. \ \square \tag{7.5}$$

From the definition of the derivative and the properties of limits it follows that the rules of differentiation of real valued functions of a real variable known from courses of mathematical analysis remain valid for functions of a complex variable.

(1) If two functions, say $f(z)$ and $g(z)$, are differentiable at a point z, then their sum, product, and quotient (with a nonzero denominator) are also differentiable at this point, and

$$\begin{gathered}(f \pm g)' = f' \pm g', \quad (cf)' = cf' \quad (c = \text{const}),\\ (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.\end{gathered} \tag{7.6}$$

(2) If a function $f(z)$ is differentiable at a point z and another function $F(w)$ is differentiable at the point $w = f(z)$, then the function $\Phi(z) = F[f(z)]$ is differentiable at point z, with

$$\Phi'(z) = F'(w) f'(z). \tag{7.7}$$

Example 2. Formulas (7.5) and (7.6) imply that

(a) the function $f(z) = z^m$, with m a negative integer, is differentiable in the entire complex plane except at point $z = 0$, and $(z^m)' = mz^{m-1}$; in particular,

$$(z^{-1}) = \left(\frac{1}{z}\right)' = -\frac{1}{z^2};$$

(b) the polynomial $P_n(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$ is differentiable in the entire complex plane, and

$$P_n'(z) = na_0 z^{n-1} + (n-1) a_1 z^{n-2} + \ldots + 2a_{n-2} z + a_{n-1};$$

(c) the rational function $R(z) = P_n(z)/Q_m(z)$ has a derivative at all points where $Q_m(z) \neq 0$, and the formula for $R'(z)$ is the same as with real x's. $\square$

In the definition of the derivative there is a requirement that the limit in (7.1) must not depend on the way in which Δz tends to zero. This imposes restrictions on differentiable functions of a complex variable much more stringent than in the case of differentiable functions of a real variable. In Sec. 12 we will prove that a function that is differentiable in a domain has derivatives of any order in this domain.

In Sec. 4 we noted that continuity of a function of a complex variable, say $f(z) = u(x, y) + iv(x, y)$, at a point $z = x + iy$ is equivalent to continuity of functions u and v at point (x, y). However, there is no similar statement for differentiability. Precisely, the condition that $f(z) = u + iv$ is differentiable imposes additional conditions on the partial derivatives of u and v.

7.2 The Cauchy-Riemann equations

Theorem 1 *A function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$ if and only if*

(1) *both $u(x, y)$ and $v(x, y)$ are differentiable at point (x, y);*

(2) *at point (x, y) the partial derivatives of u and v satisfy the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{7.8}$$

For the derivative $f'(z)$ the following formula is true:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \tag{7.9}$$

Proof. Necessity. Suppose $f(z)$ is differentiable at point z. Then in view of Eq. (7.3) we have

$$\Delta f = f'(z)\,\Delta z + \varepsilon(\rho), \tag{7.10}$$

where $\varepsilon(\rho) = o(\rho)$ as $\rho \to 0$. Here $\rho \equiv |\Delta z| = [(\Delta x)^2 + (\Delta y)^2]^{1/2}$. Since $\varepsilon(\rho)$ is complex valued, we can write $\varepsilon(\rho) = \varepsilon_1(\rho) + i\varepsilon_2(\rho)$, where $\varepsilon_1(\rho)$ and $\varepsilon_2(\rho)$ are real valued. Since $\varepsilon(\rho)/\rho \to 0$ as $\rho \to 0$, we find that $\varepsilon_1(\rho)/\rho \to 0$ and $\varepsilon_2(\rho)/\rho \to 0$ as $\rho \to 0$; whence

$$\varepsilon_1(\rho) = o(\rho), \quad \varepsilon_2(\rho) = o(\rho) \quad (\rho \to 0). \tag{7.11}$$

We introduce the notation $\Delta f = \Delta u + i\,\Delta v$ and $f'(z) = A + iB$. Substituting this into Eq. (7.10), we obtain

$$\Delta u + i\,\Delta v = (A + iB)(\Delta x + i\,\Delta y) + \varepsilon_1 + i\varepsilon_2. \tag{7.12}$$

Collecting the real and imaginary parts on both sides, we obtain

$$\Delta u = A\,\Delta x - B\,\Delta y + \varepsilon_1, \quad \Delta v = B\,\Delta x + A\,\Delta y + \varepsilon_2. \tag{7.13}$$

This proves that the functions u and v are differentiable at point (x, y).

From Eqs. (7.13) we find that

$$A = \frac{\partial u}{\partial x}, \quad -B = \frac{\partial u}{\partial y}, \quad B = \frac{\partial v}{\partial x}, \quad A = \frac{\partial v}{\partial y},$$

from which the Cauchy-Riemann equations and Eq. (7.9) follow immediately since $f'(z) = A + iB$.

Sufficiency. Let us assume that the functions $u(x, y)$ and $v(x, y)$ are differentiable at point (x, y) and conditions (7.8) are met. Then Eqs. (7.13) hold, with $\varepsilon_1 = o(\rho)$ and $\varepsilon_2 = o(\rho)$. Multiplying the second equation in (7.13) by i and adding the product to the first equation, we obtain

$$\Delta u + i\,\Delta v = A\,\Delta x - B\,\Delta y + i(B\,\Delta x + A\,\Delta y) + \varepsilon_1 + i\varepsilon_2,$$

or

$$\Delta f = (A + iB)(\Delta x + i\,\Delta y) + \varepsilon_1 + i\varepsilon_2,$$

or

$$\Delta f = (A + iB)\,\Delta z + \varepsilon(\rho),$$

where $\varepsilon(\rho) = o(\rho)$. From this, in view of (7.3), follows the differentiability of $f(z)$ at point z. The proof of the theorem is complete.

Example 3. (a) The function $e^z = e^x \cos y + ie^x \sin y$ is differentiable in the entire complex plane since

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}.$$

Equation (7.9) then yields

$$(e^z)' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y = e^z,$$

i.e.

$$(e^z)' = e^z. \tag{7.14}$$

(b) The functions $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ are differentiable in the entire complex plane, and their derivatives are given by the following formulas:

$$(\sin z)' = \cos z, \qquad (\cos z)' = -\sin z, \tag{7.15}$$

$$(\sinh z)' = \cosh z, \quad (\cosh z)' = \sinh z. \tag{7.16}$$

(c) Let us consider the function $\bar{z}^2 = x^2 - y^2 - i2xy$. We have $\partial u/\partial x = 2x$, $\partial u/\partial y = -2y$, $\partial v/\partial x = -2y$, and $\partial v/\partial y = -2x$. Conditions (7.8) are met only at point $x = y = 0$. This means that $\bar{z}^2$ is differentiable only at point $z = 0$. ☐

We write $z = re^{i\varphi}$. Then $f(z) = u(r, \varphi) + iv(r, \varphi)$, and the Cauchy-Riemann equations in polar coordinates are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \varphi}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \varphi}. \tag{7.17}$$

Hence,

$$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{z}\left(\frac{\partial v}{\partial \varphi} - i\frac{\partial u}{\partial \varphi}\right). \tag{7.18}$$

Example 4. Let D be the complex z plane with a cut along the positive real semiaxis.

(a) The function $\sqrt{z} = \sqrt{r}e^{i\varphi/2}$, where $z = re^{i\varphi}$, $0 < \varphi < 2\pi$, satisfies conditions (7.17), which means that $\sqrt{z}$ is differentiable in D. Equations (7.18) yield $(\sqrt{z})' = (2\sqrt{r}e^{i\varphi/2})^{-1}$, i.e

$$(\sqrt{z})' = \frac{1}{2\sqrt{z}}. \tag{7.19}$$

(b) The function $\ln z = \ln r + i\varphi$ ($z = re^{i\varphi}$, $0 < \varphi < 2\pi$) satisfies conditions (7.17), and

$$(\ln z)' = \frac{1}{z}. \quad \square \tag{7.20}$$

7.3 Conjugate harmonic functions Suppose a complex valued function $f(z) = u + iv$ is differentiable in a domain D and, in addition, let the functions u and v have continuous partial derivatives up to second order inclusive. Then, if we differentiate the first equation in (7.8) with respect to x and the second with respect to y, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y\,\partial x}.$$

Adding these equations and bearing in mind that the mixed derivatives are equal because of their continuity, we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{7.21}$$

Similarly we obtain

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

A real valued function of two real variables that in a domain has second-order continuous partial derivatives and satisfies Eq. (7.21) is said to be *harmonic* in this domain, while Eq. (7.21) is known as *Laplace differential equation.*

Earlier we noted that a function that is differentiable in a domain has, in the same domain, derivatives of all orders. Hence, the function possesses continuous partial derivatives of any finite order. For this reason the real and imaginary parts of a function $f(z) = u + iv$ that is differentiable in a domain are harmonic in this domain.

A pair of harmonic functions $u(x, y)$ and $v(x, y)$ related through the Cauchy-Riemann equations is said to be *conjugate harmonic.* Thus, the real and imaginary parts of a function that is differentiable in a certain domain constitute in this domain a pair of conjugate harmonic functions.

Conversely, if a pair of conjugate harmonic functions $u(x, y)$ and $v(x, y)$ are given in a domain D, then, in view of Theorem 1, the function $f(z) = u + iv$ is differentiable in D. We have thus arrived at the following

Theorem 2 *A function $f(z) = u + iv$ is differentiable in a domain D if and only if the functions $u(x, y)$ and $v(x, y)$ constitute a pair of conjugate harmonic functions in the same domain.*

If in a simply connected domain we know one of the functions, u or v, the other can be found.

Theorem 3 *For each function $u(x, y)$ that is harmonic in a simply connected domain D there can be found the conjugate harmonic function, which is defined to within an arbitrary constant term.*

Proof. Since by hypothesis $u(x, y)$ is a harmonic function in a simply connected domain D, we have

$$\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right),$$

which means that $-(\partial u/\partial y)\,dx + (\partial u/\partial x)\,dy$ is the total differential of a single-valued function $v(x, y)$ defined to within an arbitrary constant term C through the following formula (see Sec. 6.2):

$$v(x, y)=\int_{(x_0,\, y_0)}^{(x,\, y)} -\frac{\partial u}{\partial y}\,dx+\frac{\partial u}{\partial x}\,dy+C, \tag{7.22}$$

where $(x_0, y_0) \in D$ and $(x, y) \in D$ (the value of the integral in (7.22) does not depend on the curve that connects points (x_0, y_0) and (x, y) but solely on point (x, y) if point (x_0, y_0) is fixed).

Formula (7.22) yields

$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x},$$

which implies that $v(x, y)$ is a harmonic function in D and conjugate to $u(x, y)$, i.e. u and v constitute a pair of conjugate harmonic functions.

Theorems 2 and 3 imply that if a harmonic function $u(x, y)$ is defined in a simply connected domain D, then we can find, to within a constant term, a function $f(z) = u + iv$ that is differentiable in D. In other words, we can reconstruct a differentiable function from its real or imaginary part in a simply connected domain. If D is multiply connected, then the function v defined via (7.22) as well as the function $f(z) = u + iv$ may prove to be not single-valued.

Note that in reconstructing the function $v(x, y)$ from the function $u(x, y)$ (or vice versa) it often proves more convenient to use the Cauchy-Riemann equations directly instead of formula (7.22) (see the example below).

Example 5. Find the differentiable function $f(z)$ if

$$\operatorname{Re} f(z) = u(x, y) = y^3 - 3x^2y.$$

The function $u = y^3 - 3x^2y$ is harmonic in the entire complex plane. We have

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=-6xy,$$

whence

$$v = -3xy^2 + g(x). \tag{7.23}$$

This yields

$$\frac{\partial v}{\partial x}=-3y^2+g'(x). \tag{7.24}$$

On the other hand, in view of (7.8) we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -3y^2 + 3x^2. \qquad (7.25)$$

If we now compare (7.24) with (7.25), we will see that $g'(x) = 3x^2$, whence $g(x) = x^3 + C$, where C is a real constant. Equation (7.23) then yields $v = -3xy^2 + x^3 + C$. The sought-for function is

$$f(z) = u + iv = y^3 - 3x^2y + i(x^3 - 3xy^2) + iC = i(z^3 + C),$$

and this function is differentiable in the entire complex plane. □

7.4 The concept of a regular function We will now introduce a concept basic to the theory of functions of a complex variable, the concept of a regular function.

Definition 1. Let us assume that we are dealing with a function $f(z)$ that is defined in a neighborhood of the point $z = a$ $(a \neq \infty)$ and can be expanded in a power series

$$f(z) - \sum_{n=0}^{\infty} c_n (z-a)^n \qquad (7.26)$$

that is convergent in the neighborhood of point $z = a$ (i.e. in the circle $|z - a| < \rho$, with ρ positive). Then $f(z)$ is said to be *regula at point* $z = a$.

A function is said to be *regular in a domain* if it is regular at every point of this domain.

Theorem 4 *If a function $f(z)$ is regular at a point $z = a$, then it is differentiable at this point.*

Proof. By hypothesis, the power series in (7.26) is convergent in a neighborhood of point a, which implies that $f(a) = c_0$. We consider the quotient

$$\frac{f(z) - f(a)}{z-a} = \frac{f(z) - c_0}{z-a} = \sum_{n=1}^{\infty} c_n (z-a)^{n-1}. \qquad (7.27)$$

Since the series in (7.27) is uniformly convergent in the circle $|z - a| \leqslant \rho_1 < \rho$, its sum is continuous in this circle, and in the right-hand side of (7.27) we can pass to the limit term-by-term (a fuller exposition of the theory of power series will be given in Sec. 11) as $z \to a$, with the limit being equal to c_1. Therefore, the left-hand side tends to a limit, too, as $z \to a$, i.e. we have $f'(z) = c_1$.

Remark 1. In Sec. 12 we will show that a function that is differentiable in a domain is regular in the same domain.

Example 6. The function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ is regular at point $z = 0$ (the series is convergent in the circle $|z| < 1$). □

Definition 2. Let a function $f(z)$ be defined in a neighborhood of a point at infinity and suppose we can expand it in a power series

$$f(z) = \sum_{n=0}^{\infty} \frac{c^n}{z^n}, \tag{7.28}$$

that is convergent in a neighborhood of point $z = \infty$ (i.e. in a domain $|z| > R$). Then $f(z)$ is said to be *regular at the point at infinity.*

Remark 2. Definition 2 implies that a function $f(z)$ is regular at point $z = \infty$ if and only if the function $g(\zeta) = f(1/\zeta)$ is regular at point $\zeta = 0$.

Example 7. The function $f(z) = z/(z - 1)$ is regular at point $z = \infty$ since the function $g(\zeta) = f(1/\zeta) = 1/(1 - \zeta)$ is regular at point $\zeta = 0$. □

8 The Geometric Interpretation of the Derivative

8.1 The concept of univalence In Sec. 4 we introduced the concept of a function of one complex variable. This concept can be given the following geometric interpretation.

Suppose we have defined a function $w = f(z)$ on a set E in the

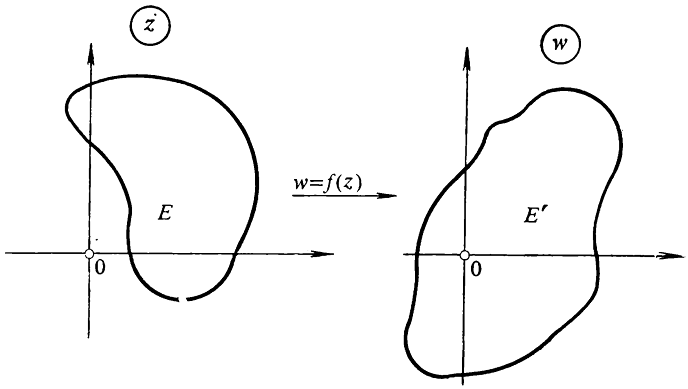


Fig. 32

complex z plane. We denote by E' the set (range) of the function's values in the complex w plane (see Fig. 32). We have thus defined a *mapping* of set E onto set E'. Point $w \in E'$ is called an *image* of point $z \in E$ and point z the *preimage* of point w in the mapping $w = f(z)$.

It may happen that some points of E' have not one but several preimages, i.e. the mapping $w = f(z)$ may be not a one-to-one mapping. If the mapping $w = f(z)$ is one-to-one, the function $f(z)$ is said to be *univalent*. Here is a more detailed definition of univalence.

Definition 1. The function $w = f(z)$ is said to be *univalent on a set* E if at different points of E it assumes different values.

The mapping $w = f(z)$ performed by a univalent function is one to-one and is called a *univalent mapping*.

Obviously, a function $w = f(z)$ is univalent on a set E if for any two points z_1 and z_2 of this set the fact that $f(z_1) = f(z_2)$ is true if

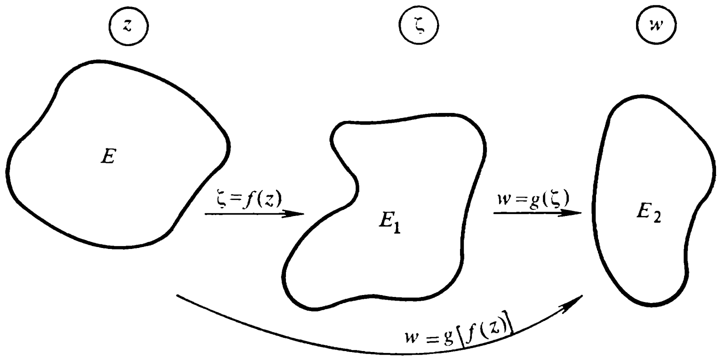


Fig. 33

and only if $z_1 = z_2$. In other words, the function $w = f(z)$ is univalent on a set E if this set does not have a single pair of points z_1 and z_2 such that $f(z_1) = f(z_2)$.

The definition of univalence implies that if a function is univalent on a set E and if $E_1 \subset E$, this function is univalent on E_1, too.

The result of several univalent mappings performed in a sequence is a univalent mapping, i.e. if a function $\zeta = f(z)$ is univalent on a set E $(E \to E_1)$ and another function $w = g(\zeta)$ is univalent on E_1 $(E_1 \to E_2)$, the function $w = g[f(z)]$ is also univalent on E $(E \to E_2)$ (see Fig. 33).

If the mapping $w = f(z)$: $E \to E'$ is univalent, each point $w \in E'$ has corresponding to it one and only one point $z \in E$ such that $f(z) = w$. This is also a definition on E' of a function $z = h(w)$ that is the *inverse* of $f(z)$. Obviously, we can write

$$f[h(w)] \equiv w, \quad w \in E'; \quad h[f(z)] \equiv z, \quad z \in E.$$

A function $w = f(z)$ defined on a set E and mapping E on E' is univalent on E if and only if the inverse function $z = h(w)$ is single-valued on E'.

8.2 Examples of univalent mappings

Example 1. The linear function

$$w = f(z) = az + b, \tag{8.1}$$

where a and b are complex valued constants ($a \neq 0$), is univalent in the entire complex plane since the inverse function

$$z = h(w) = \frac{1}{a} w - \frac{b}{a} \tag{8.2}$$

is single-valued.

The function given by (8.1) maps the extended complex z plane onto the extended complex w plane in a one-to-one manner. The

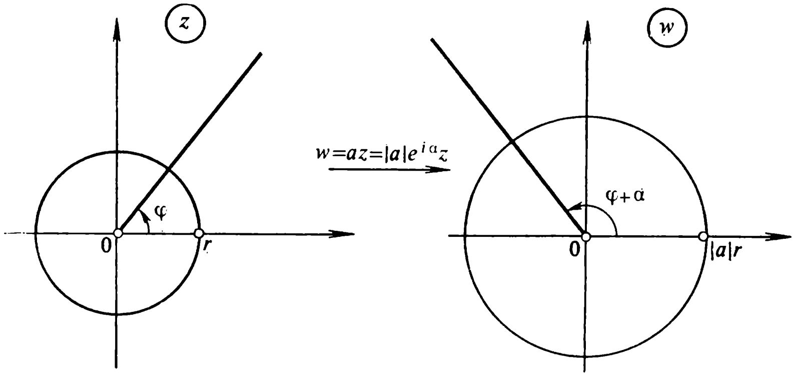


Fig. 34

point $z = \infty$ in the process of this mapping is mapped into point $w = \infty$, while the relationship between the finite points in the complex z and w planes is given by formulas (8.1) and (8.2).

Let us study the case where $b = 0$. Then

$$w = az, \tag{8.3}$$

whence

$$|w| = |a|\,|z|, \quad \arg w = \arg a + \arg z. \tag{8.4}$$

Equations (8.4) imply that the mapping (8.3) corresponds to a stretching or contraction of the entire complex z plane by a factor $|a|$, with the center of similitude at the origin of coordinates, together with a rotation of the plane as a whole about point $z = 0$ through the angle $\alpha = \arg a$. Under the mapping (8.3) the ray $\arg z = \varphi$ is mapped into the ray $\arg w = \varphi + \alpha$ and the circle $|z| = r$ into the circle $|w| = |a|r$ (see Fig. 34). Under such a mapping the circle $|z| < R$ is mapped into the circle $|w| < |a|R$.

If $|a| = 1$, i.e. $a = e^{i\alpha}$, the mapping (8.3) corresponds to a rotation of the entire complex z plane through the angle α. For instance, the mapping $w = iz$ is the rotation through the angle $\pi/2$, while the mapping $w = -z$ is the rotation through the angle π.

The mapping (8.1) is a combination of the following mappings:

$$\zeta = |a|\,z, \quad \tau = \zeta e^{i \arg a}, \quad w = \tau + b.$$

Hence, the mapping $w = az + b$ corresponds to the following sequence of mappings:

(a) stretching or contraction by a factor $|a|$ of the complex z plane (with the center of similitude at point $z = 0$);

(b) rotation of the entire complex ζ plane through the angle $\alpha = \arg a$ about point $\zeta = 0$;

(c) translation of the complex τ plane through the displacement vector b. □

Example 2. The function

$$w = \frac{1}{z} \tag{8.5}$$

maps in a one-to-one manner the extended complex z plane onto the extended complex w plane (the inverse function $z = 1/w$ is single-

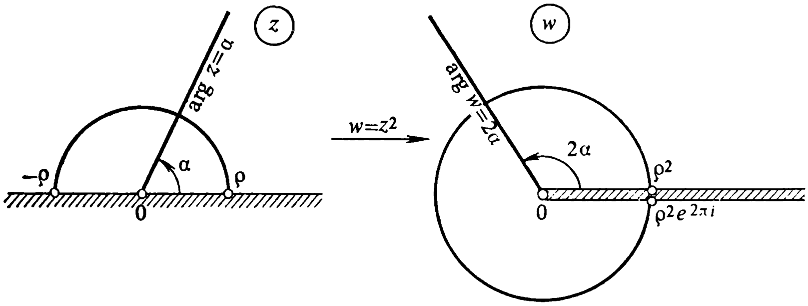


Fig. 35

valued). Point $w = \infty$ corresponds to point $z = 0$, while point $w = 0$ corresponds to point $z = \infty$. The ray $\arg z = \varphi$ is mapped by (8.5) into the ray $\arg w = -\varphi$, the circle $|z| = r$ into the circle $|w| = 1/r$, and the circle $|z| > R$ onto the domain $|w| > 1/R$. □

Example 3. Let us consider the function

$$w = z^2. \tag{8.6}$$

If $z_1^2 = z_2^2$, then either $z_1 = z_2$ or

$$z_1 = -z_2. \tag{8.7}$$

Two points related through (8.7) are symmetric with respect to the origin of coordinates. Hence, the function $w = z^2$ is univalent in a domain D if and only if this domain has not a single pair of points symmetric with respect to point $z = 0$. For instance, the function (8.6) is univalent in the upper half-plane $\operatorname{Im} z > 0$.

Let us take the ray $\arg z = \alpha$, with $0 < \alpha < \pi$, which lies in the upper half-plane (Fig. 35). The mapping (8.6) maps this ray into the ray $\arg w = 2\alpha$. We rotate the ray $\arg z = \alpha$ by continuously increasing α from 0 to π. Then the ray $\arg w = 2\alpha$, which is the image of arg $z = \alpha$, will rotate counterclockwise. If the ray in the z plane rotates through the upper half-plane, its image will rotate through the entire complex w plane. In the process the rays arg $z = 0$ and arg $z = \pi$, which constitute the boundary of the domain Im $z > 0$, will be mapped into the rays arg $w = 0$ and arg $w = 2\pi$,

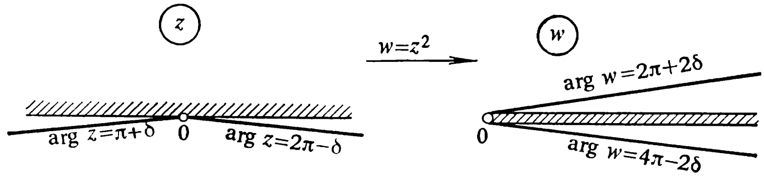


Fig. 36

respectively. Geometrically these last two rays coincide with the positive real semiaxis in the complex w plane. To ensure that the mapping (8.6) is one-to-one not only inside the domain Im $z > 0$ but on its boundary as well, we will cut the complex w plane along the positive real semiaxis and assume the ray arg $z = 0$ (i.e. the positive real semiaxis in the complex z plane) is mapped onto the upper bank of the cut, while the ray arg $z = \pi$ is mapped onto the lower bank.

Thus, the function $w = z^2$ is univalent in the upper half-plane and maps this domain onto the complex w plane with a cut along the positive real semiaxis (Fig. 35). We note that the function $w = z^2$ maps the semicircle $z = \rho e^{i\theta}$, $0 \leqslant \theta \leqslant \pi$, into the "open" circle $|w| = \rho^2$ (points $w_1 = \rho^2$ and $w_2 = \rho^2 e^{2\pi i}$, which are the images of the points $z_1 = \rho$ and $z_2 = \rho e^{i\pi} = -\rho$, coincide but lie on different banks of the above-mentioned cut).

The function $w = z^2$ is also univalent in the lower half-plane and maps the domain Im $z < 0$ onto the complex w plane with a cut along the positive real semiaxis (Fig. 36). Under this mapping the rays arg $z = \pi$ and arg $z = 2\pi$, which constitute the boundary of the domain Im $z < 0$, transform into the upper and lower banks of the cut, respectively. Indeed, the ray arg $z = \pi + \delta$ (with δ a small positive quantity), which is adjacent to the ray arg $z = \pi$, is mapped into the ray arg $w = 2\pi + 2\delta$, which lies above the upper bank of the cut. Similarly, the ray arg $z = 2\pi - \delta$ is mapped into the ray arg $w = 4\pi - 2\delta$, which lies below the lower bank.

Note that the function $w = z^2$ maps the right half-plane, Re $z > 0$, and left half-plane, Re $z < 0$, into the complex w plane with a cut along the negative real semiaxis (Fig. 37). □

Example 4. Let us take the mapping

$$w = e^z. \tag{8.8}$$

We wish to find the condition that a domain D must satisfy so that this mapping is univalent (in this domain). If $e^{z_1} = e^{z_2}$, i.e. $e^{z_1 - z_2} = 1$, then (see Sec. 4.6)

$$z_1 - z_2 = 2k\pi i \quad (k = 0, \pm 1, \pm 2, \ldots). \tag{8.9}$$

Hence, the mapping (8.8) is univalent if and only if the domain D it maps has not a single pair of different points satisfying condition (8.9). For instance, the mapping $w = e^z$ is univalent in the horizontal strip $a < \operatorname{Im} z < b, 0 < b - a \leqslant 2\pi$.

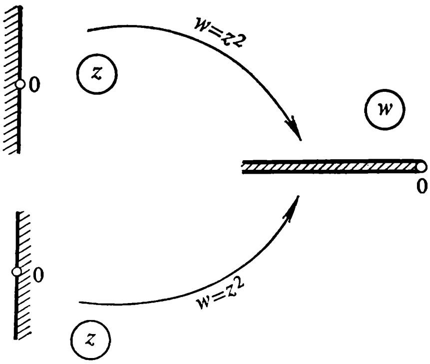


Fig. 37

Let us take the strip D_1: $0 < \operatorname{Im} z < 2\pi$ (Fig. 38). The function (8.8) maps the straight line $z = x + iC$, with C fixed and lying between zero and 2π and $-\infty < x < +\infty$, a line parallel to the real axis and lying within the strip D_1, into the straight line $w = e^{x+iC} = e^x e^{iC}$, i.e. into the ray $\arg w = C$. We will move the straight line $z = x + iC$ parallel to the real axis and continuously increase C from 0 to 2π. Then the ray $\arg w = C$, which is the image of the straight line $z = x + iC$, rotates counterclockwise and describes the entire complex w plane. The straight lines $z = x$

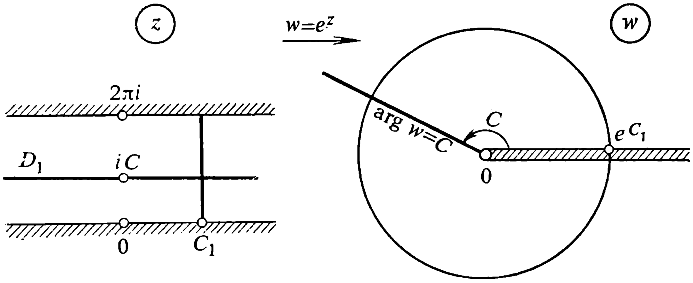


Fig. 38

($-\infty < x < \infty$) and $z = x + i2\pi$, which constitute the boundary of the strip D_1, will be mapped onto the rays $\arg w = 0$ and $\arg w = 2\pi$, respectively.

Thus, the function $w = e^z$, which is univalent within the strip $0 < \operatorname{Im} z < 2\pi$, maps the strip onto the complex w plane with a cut along the ray $[0, +\infty)$ in such a way that the lower edge of the strip is mapped into the upper bank of the cut, while the upper edge of the strip is mapped into the lower bank.

Note that the function (8.8) maps the segment $z = C_1 + iy$ (C_1 fixed and $0 \leqslant y \leqslant 2\pi$) lying in D_1 parallel to the imaginary axis into the "open" circle $w = e^{C_1}e^{iy}$ $(0 \leqslant y \leqslant 2\pi)$ with radius e^{C_1} (point $w_1 = e^{C_1}$ on the upper bank of the cut corresponds to point

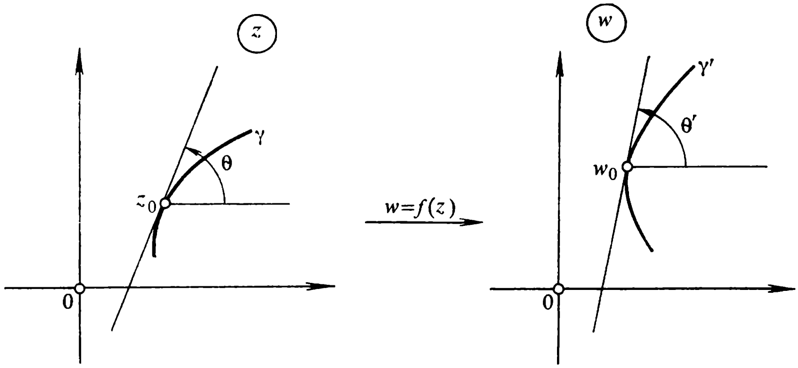


Fig. 39

$z_1 = C_1$, while the point $w_2 = e^{C_1}e^{2\pi i}$, which geometrically coincides with point w_1 and lies on the lower bank, corresponds to point $z_2 = C_1 + 2\pi i$).

By reasoning along similar lines we can show that the strip D_2: $2\pi < \text{Im } z < 4\pi$ is mapped by the function $w = e^z$ onto the complex w plane with a cut along the ray $[0, +\infty)$ in such a way that the lower edge of the strip D_2 is mapped into the upper bank of the cut and the upper edge into the lower bank. Similarly, it can be found that the function $w = e^z$ is univalent in each strip D_k: $2(k-1)\pi < \text{Im } z < 2k\pi$ (with k an integer) and maps each strip onto the complex w plane with a cut along the ray $[0, +\infty)$. □

8.3 The concept of conformal mapping

(1) Preservation of angles between curves. Let a function $w = f(z)$ be differentiable in a neighborhood of a point z_0 and suppose that $f'(z_0) \neq 0$. Take a smooth curve γ: $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$ (Fig. 39) passing through the point $z_0 = \sigma(t_0)$, $t_0 \in (\alpha, \beta)$. Suppose that θ is the angle between the tangent to curve γ at point z_0 and the positive dir ction on the real axis (the tangent is assumed to be directed in the same direction as the curve). Then $\theta = \arg \sigma'(t_0)$.

Now suppose that γ' is the image of γ created by the mapping $w = f(z)$, i.e. γ': $w = w(t) = f[\sigma(t)]$, $\alpha \leqslant t \leqslant \beta$, and w_0 is the image of z_0, i.e. $w_0 = f[\sigma(t_0)] = f(z_0)$. According to the rule of differentiation of a composite function,

$$w'(t_0) = f'(z_0)\,\sigma'(t_0). \tag{8.10}$$

Since by hypothesis $f'(z_0) \neq 0$ and $\sigma'(t_0) \neq 0$ (see Sec. 3), we conclude that $w'(t_0) \neq 0$, i.e. curve γ' has a tangent at point w_0. Suppose that $\arg w'(t_0) = \theta'$. Then (8.10) yields

$$\theta' = \arg w'(t_0) = \arg f'(z_0) + \arg \sigma'(t_0),$$

i.e.

$$\theta' = \theta + \arg f'(z_0). \tag{8.11}$$

The quantity $\alpha = \theta' - \theta$ is called the *angle of rotation of curve* γ at point z_0 under the mapping $w = f(z)$. From (8.11) it follows that if $f'(z_0) \neq 0$, the angle of rotation at point z_0 does not depend on the type of curve and is equal to $\arg f'(z_0)$, i.e. all curves passing through point z_0 are rotated under the mapping $w = f(z)$ ($f'(z_0) \neq 0$) through the same angle, equal to the argument of the derivative at point z_0.

Thus, the mapping $w = f(z)$, where $f(z)$ is a function that is differentiable in a neighborhood of point z_0 and whose derivative at this point is not zero, preserves the angles between curves that pass through point z_0 not only in magnitude but also in sense (Fig. 40).

Example 5. Find the angle of rotation α under the mapping $w = f(z)$ at point z_0.

(a) $f(z) = \dfrac{z - z_0}{z - \bar{z}_0}$, with $\operatorname{Im} z_0 = y_0 > 0$. Then

$$f'(z) = \frac{z_0 - \bar{z}_0}{(z - \bar{z}_0)^2}, \quad f'(z_0) = \frac{1}{2i \operatorname{Im} z_0} = -\frac{i}{2y_0},$$

$$\alpha = \arg f'(z_0) = -\frac{\pi}{2}.$$

(b) $f(z) = \dfrac{z - z_0}{1 - z\bar{z}_0}$, with $|z_0| < 1$. Then

$$f'(z) = \frac{1 - z_0\bar{z}_0}{(1 - z\bar{z}_0)^2}, \quad f'(z_0) = \frac{1}{1 - |z_0|^2} > 0.$$

$$\alpha = \arg f'(z_0) = 0. \ \square$$

(2) Constancy of stretching. Let a function $w = f(z)$ be differentiable in a neighborhood of a point z_0 and suppose $f'(z_0) \neq 0$. We consider a point z on curve γ lying close to point z_0 (Fig. 41). We introduce the notation $\Delta z = z - z_0$ and $\Delta w = f(z) - f(z_0) = w - w_0$. The definition of the derivative yields

$$\frac{\Delta w}{\Delta z} = f'(z_0) + \varepsilon(\Delta z), \quad \text{where} \quad \varepsilon(\Delta z) \to 0 \quad \text{as} \quad \Delta z \to 0,$$

whence

$$\lim_{\Delta z \to 0} \left| \frac{\Delta w}{\Delta z} \right| = |f'(z_0)|,$$

or

$$|\Delta w| = |f'(z_0)|\,|\Delta z| + o(|\Delta z|), \tag{8.12}$$

Suppose that $|z - z_0| = |\Delta z| = \rho$, with ρ small. Then from (8.12) we find that the circle $|z - z_0| = \rho$ is mapped under the

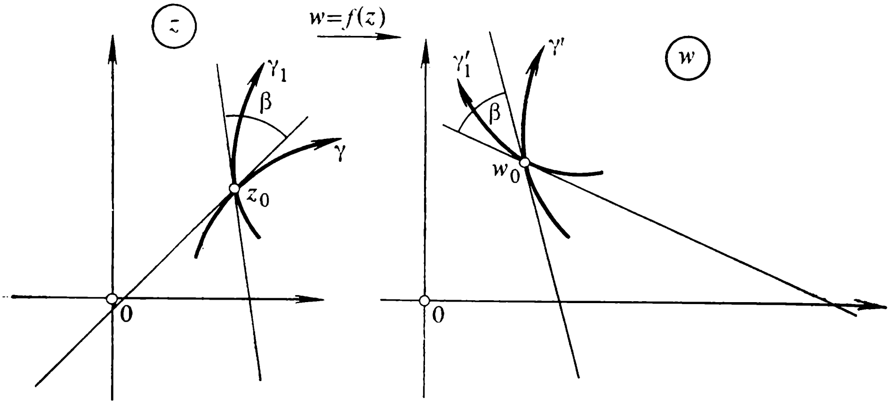


Fig. 40

mapping $w = f(z)$ into a curve that differs little from the circle

$$|w - w_0| = \rho\,|f'(z_0)|.$$

In other words, the function $w = f(z)$ stretches the circle $|\Delta z| < \rho$,

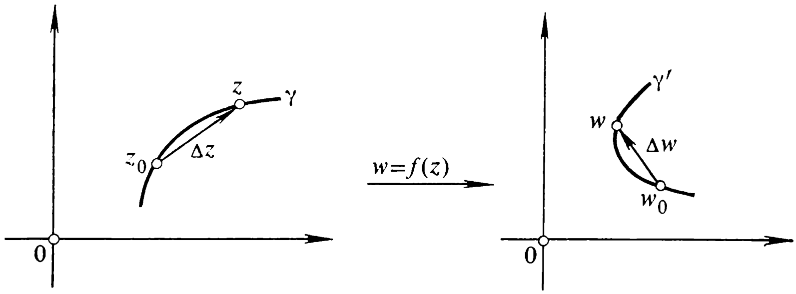


Fig. 41

to within infinitesimals of an order higher than Δz, by a factor $|f'(z_0)|$.

The quantity $\lim\limits_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = k$ is called the *linear stretching* of curve γ at point z_0 under the mapping $w = f(z)$. Hence, the linear stretching at point z_0 does not depend on the type of curve or its orientation and is equal to $|f'(z_0)|$.

(3) The definition of conformal mapping. Suppose a function $f(z)$ is defined in a neighborhood of a point z_0.

Definition 2. The mapping $w = f(z)$ is said to be *conformal at point* z_0 if it preserves the angle between any two curves passing through this point and if the linear stretching at this point is constant.

The above results indicate that if $f(z)$ is differentiable in a neighborhood of point z_0 (is regular at point z_0) and $f'(z_0) \neq 0$, the mapping $w = f(z)$ is conformal at point z_0.

Remark 1. The condition that the derivative of $f(z)$ at z_0 be nonzero means that the Jacobian of the mapping $w = f(z)$ at point z_0 is nonzero, too. Indeed, the mapping $w = f(z) = u + iv$ is equivalent to the mapping

$$u = u(x, y), \quad v = v(x, y). \tag{8.13}$$

The Jacobian of (8.13) is

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}.$$

Employing the Cauchy-Riemann equation (7.8), we obtain

$$J = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2.$$

Since $f'(z) = (\partial u/\partial x) + i(\partial v/\partial x)$, we find that

$$J = |f'(z)|^2. \tag{8.14}$$

Thus, $J(z_0) \neq 0$ if $f'(z_0) \neq 0$.

Definition 3. Let a function $f(z)$ be univalent in a domain D and let the mapping $w = f(z)$ be conformal at each point of D. Then we simply say that the mapping $w = f(z)$ is *conformal*.

Definitions 1 and 2 together with the properties of the derivatives imply that if a function $f(z)$ (i) is differentiable in a domain D, (ii) is univalent in D, and (iii) has a nonzero derivative in D, then the mapping $w = f(z)$ is conformal.

Note that condition (iii) follows from conditions (i) and (ii) (see Sec. 31).

Examples of conformal mappings were given in Sec. 8.2. The linear mapping $w = az + b$ $(a \neq 0)$ is conformal in the entire complex plane. The function $w = z^2$ performs a conformal mapping of the upper half-plane $\operatorname{Im} z > 0$ onto the complex plane with a cut along the ray $[0, +\infty)$. Finally, the mapping $w = e^z$ is conformal in the strip $0 < \operatorname{Im} z < 2\pi$.

We will study conformal mappings in greater detail in Chap. IV.

Remark 2. If a function $f(z)$ is regular at a point z_0 but $f'(z_0) = 0$, the mapping $w = f(z)$ is not conformal at z_0. Let us clarify this point using the function $f(z) = z^2$ as an example. At point $z_0 = 0$ the derivative of z^2 vanishes ($f_0'(z) = 2z$). Take two rays that start at point $z = 0$, say $\arg z = \alpha$ and $\arg z = \beta$. The function $w = z^2$ maps these rays into two rays, $\arg w = 2\alpha$ and $\arg w = 2\beta$. The initial rays form an angle of $\beta - \alpha$, while their images form an angle of $2(\beta - \alpha)$. We see that at point $z = 0$ all angles are doubled, i.e. the mapping $w = z^2$ at point $z = 0$ is not conformal.

(4) The area of the image of a domain and the length of the image of a curve. Suppose we are dealing with a function $w = f(z)$ that conformally maps a domain D onto another domain D'. Then the Jacobian of this mapping is $J = |f'(z)|^2$ and the area of D' is

$$S(D') = \iint_{D'} du\, dv = \iint_{D} |J|\, dx\, dy = \iint_{D} |f'(z)|^2\, dx\, dy.$$

Suppose γ is a curve lying in D and γ' is its image under the mapping $w = f(z)$. Then the length of γ' is

$$l(\gamma') = \int_{\gamma'} |dw| = \int_{\gamma} |f'(z)|\, |dz|.$$

Remark 3. Here is the geometric interpretation of (8.14). As shown in courses of mathematical analysis, the value of $|J|$, where J is the Jacobian of the mapping (8.13), is the coefficient of stretching of areas under mapping (8.13), i.e. the mapping $w = f(z) = u + iv$. As shown earlier, the linear stretching under the mapping $w = f(z)$ does not depend on direction and is equal to $|f'(z_0)|$. Hence, the coefficient of stretching of areas is $|f'(z_0)|^2$.

9 Cauchy's Integral Theorem

In this section we will prove Cauchy's integral theorem, which is one of the most important results of the theory of functions of a complex variable.

9.1 Cauchy's integral theorem for the case of a continuous derivative

Theorem 1 *Suppose that a function $f(z)$ is differentiable in a simply connected domain D and its derivative is continuous in D. Then the value of the integral of $f(z)$ along each closed curve γ lying inside D is zero:*

$$\int_{\gamma} f(z)\, dz = 0. \tag{9.1}$$

Proof. If $f(z) = u(x, y) + iv(x, y)$, then Eq. (5.3) yields

$$\int_{\gamma} f(z)\, dz = J_1 + iJ_2,$$

where

$$J_1 = \int_{\gamma} u\, dx - v\, dy, \quad J_2 = \int_{\gamma} v\, dx + u\, dy.$$

Since $f(z)$ has a continuous derivative in D, the first-order partial derivatives of the unctions u and v are continuous in D and the Cauchy-Riemann equations are satisfied, i.e.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}. \tag{9.2}$$

In view of the theorem of Sec. 6.2 we see that Eqs. (9.2) yield $J_1 = J_2 = 0$. Thus, $\int_\gamma f(z)\, dz = J_1 + iJ_2 = 0$.

9.2 Cauchy's integral theorem (the general case) Theorem 2 (Cauchy's integral theorem) *Let a function $f(z)$ be differentiable in a simply connected domain D. Then the value of the integral of $f(z)$ along every closed curve γ lying in D is zero:*

$$\int_\gamma f(z)\, dz = 0. \tag{9.3}$$

Proof. We will give the proof of Cauchy's integral theorem proposed by E.J.B. Goursat.

(1) We start by examining the case where the closed curve γ is the contour of a triangle lying in D. We will give a reduction ad absurdum proof. Suppose the theorem does not hold. Then there exists a triangle (the contour of this triangle and the triangle proper is denoted by Δ) such that

$$\left|\int_\Delta f(z)\, dz\right| = \alpha > 0. \tag{9.4}$$

By connecting the midpoints of the sides of Δ by straight lines (Fig. 42) we partition Δ into four triangles $\Delta^{(k)}$ ($k = 1, 2, 3, 4$). Then we note that

$$\sum_{k=1}^{4} \int_{\Delta^{(k)}} f(z)\, dz = \int_\Delta f(z)\, dv. \tag{9.5}$$

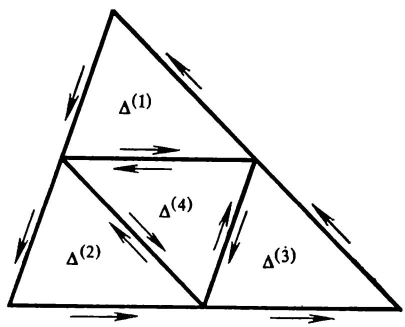


Fig. 42

Indeed, the left-hand side of (9.5) is the sum of the integral along the contour of Δ and the integrals are taken two times (in opposite directions) along each side of $\Delta^{(4)}$ (these latter integrals cancel out).

Equations (9.4) and (9.5) imply that at least for one of the integrals on the left-hand side of (9.5) (the corresponding triangle is denoted by Δ_1) the following estimate is true:

$$\left|\int_{\Delta_1} f(z)\, dz\right| \geqslant \frac{\alpha}{4}, \tag{9.6}$$

since otherwise

$$\alpha = \left| \int_{\Delta} f(z)\, dz \right| \leqslant \sum_{k=1}^{4} \left| \int_{\Delta(k)} f(z)\, dz \right| < 4\, \frac{\alpha}{4} = \alpha,$$

i.e. $\alpha < \alpha$, which is impossible.

Next, partitioning the triangle Δ_1 in the above-mentioned manner into four triangles and following the same line of reasoning, we arrive at a triangle Δ_2 such that

$$\left| \int_{\Delta_2} f(z)\, dz \right| \geqslant \frac{\alpha}{4^2}.$$

Continuing this process, we obtain a sequence of triangles $\{\Delta_n\}$ such that each Δ_n contains Δ_{n+1} $(n = 1, 2, \ldots)$ and

$$J_n = \left| \int_{\Delta n} f(z)\, dz \right| \geqslant \frac{\alpha}{4^n}. \qquad (9.7)$$

This gives the lower estimate for J_n. Let us find the upper estimate. Suppose that P is the perimeter of the initial triangle. Then the perimeter P_n of Δ_n will be $P/2^n$; hence, $P_n \to 0$ as $n \to \infty$. Thus the sequence of triangles $\{\Delta_n\}$ forms a nest: each triangle Δ_n contains all the subsequent triangles $\Delta_{n+1}, \Delta_{n+2}, \ldots$, and the perimeter of Δ_n tends to zero as $n \to \infty$. From this it follows that there is only one point z_0 lying in the interior or on the boundary of Δ and belonging to all the triangles $\Delta_1, \Delta_2, \ldots$. By hypothesis, z_0 belongs to D. Since the function $f(z)$ is differentiable at point z_0, we find that

$$f(z) = f(z_0) + f_0'(z - z_0) + o(z - z_0),$$

whence

$$\int_{\Delta n} f(z)\, dz = f(z_0) \int_{\Delta n} dz + f'(z_0) \int_{\Delta n} z\, dz$$
$$- z_0 f'(z_0) \int_{\Delta n} dz + \int_{\Delta n} o(z - z_0)\, dz. \qquad (9.8)$$

Since $\int_{\Delta n} dz = 0$ and $\int_{\Delta n} z\, dz = 0$ (see Examples 1 and 2 in Sec. 5), from (9.8) we obtain

$$\int_{\Delta n} f(z)\, dz = \int_{\Delta n} o(z - z_0)\, dz. \qquad (9.9)$$

The definition of $o(z - z_0)$ implies that for every positive ε there is a positive $\delta = \delta(\varepsilon)$ such that for all z: $|z - z_0| < \delta$ we have

$$|o(z - z_0)| < \varepsilon\, |z - z_0|. \qquad (9.10)$$

We select n so large that Δ_n would lie in the circle $|z - z_0| < \delta$. Then from Eqs. (9.9) and (9.10) we have

$$J_n = \left|\int_{\Delta n} f(z)\,dz\right| \leqslant \varepsilon \int_{\Delta n} |z - z_0|\,|dz| < \varepsilon P_n \int_{\Delta n} |dz| = \varepsilon P_n^2 = \varepsilon \frac{P^2}{4^n},$$

i.e.

$$J_n < \varepsilon \frac{P^2}{4^n}. \tag{9.11}$$

Comparing (9.7) and (9.11), we obtain $\alpha/4^n < \varepsilon P^2/4^n$, i.e. $\alpha < \varepsilon P^2$, which at $\alpha > 0$ is impossible since we can select the positive ε as small as desired. Hence $\alpha = 0$, i.e. Eq. (9.3) holds for all triangles that lie inside D.

(2) Now suppose that γ is an arbitrary polygon lying in D.

If the polygon is convex, it can be partitioned into triangles via diagonals that start at a single vertex. If we write $J = \int_\gamma f(z)\,dz$ as a sum of integrals along the boundaries of the triangles into which the polygon is partitioned, we find that $J = 0$.

Next, since an arbitrary polygon can always be partitioned into a finite number of convex polygons, we can always write

$$\int_\gamma f(z)\,dz = 0.$$

(3) Finally, suppose γ is an arbitrary closed curve in D. By Lemma 2 of Sec. 5 we can always approximate $\int_\gamma f(z)\,dz$ as accurately as desired by an integral along a closed broken line lying in D, i.e. for every positive ε there is a closed broken line such that

$$\left|\int_\gamma f(z)\,dz - \int_L f(z)\,dz\right| < \varepsilon.$$

As proved earlier, $\int_L f(z)\,dz = 0$, and, hence, the above inequality takes the form $\left|\int_\gamma f(z)\,dz\right| < \varepsilon$, which in view of the arbitrariness of $\varepsilon > 0$ implies that $\int_\gamma f(z)\,dz = 0$.

9.3 Corollaries and remarks related to Cauchy's integral theorem

Remark 1. The function $f(z) = 1/z$ is differentiable in the annulus $0 < |z| < 2$, but $\int_{|z|=1} \frac{dz}{z} \neq 0$ (see Example 3 in Sec. 5). This example shows that the condition that the domain in Cauchy's integral theorem be simply connected is important.

Corollary 1 *If a function $f(z)$ is differentiable in a simply connected domain D, the value of the integral of $f(z)$ does not depend on the path of integration. Precisely, if two curves, γ and γ_1, lie in D and have a common beginning and a common end, then*

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

Thus, curve γ can be deformed inside D (not involving its beginning and end) and yet the integral remains the same.

Employing the corollary of Sec. 6.2, we arrive at a theorem that is also called Cauchy's integral theorem:

Theorem 3 *If a function $f(z)$ is differentiable in a domain D and curves γ_1 and γ_2 are homotopic in D,*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

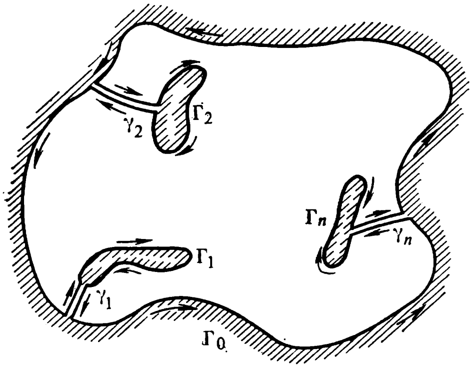


Fig. 43

The domain D may be multiply connected.

Cauchy's integral theorem also holds for the case where γ is the boundary of D. Here is the appropriate theorem:

Theorem 4 *Suppose D is a bounded, simply connected domain with a piecewise smooth boundary Γ and let $f(z)$ be a function that is differentiable in D and is continuous up to the boundary of D. Then*

$$\int_{\Gamma} f(z)\,dz = 0.$$

The proof of Theorem 4 follows from Theorem 2 and Lemma 3 of Sec. 5.

Theorem 4 is valid for multiply connected domains, too.

Corollary 2 *Suppose the boundary Γ of a multiply connected domain D consists of a closed piecewise smooth curve Γ_0 and closed piecewise smooth curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ that lie inside Γ_0 and are pairwise nonconcurrent, and suppose that $f(z)$ is differentiable in D up to the boundary of D. Then*

$$\int_{\Gamma_0} f(z)\,dz + \sum_{k=1}^{n} \int_{\Gamma_k} f(z)\,dz = 0. \tag{9.12}$$

The curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ are oriented in such a way that in traversing each of them we find that D remains to the left.

Proof. We connect the curve Γ_0 with $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ by cuts $\gamma_1, \gamma_2, \ldots, \gamma_n$ (Fig. 43) in such a way that the resulting domain $\tilde{D}$ is simply connected. The boundary $\tilde{\Gamma}$ of $\tilde{D}$ consists of curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ and cuts $\gamma_1, \gamma_2, \ldots, \gamma_n$. By Theorem 4, $\int\limits_{\tilde{\Gamma}} f(z)\,dz = 0$. Taking into account the fact that integration along each cut γ_k ($k = 1, 2, \ldots, n$) is performed two times (in opposite directions) and, hence, $\int\limits_{\gamma_k} f(z)\,dz = 0$, we arrive at formula (9.12).

Note a particular case of Corollary 2. Let $f(z)$ be differentiable in D, which may be multiply connected, and let γ and γ_1 be two simple closed curves with one lying inside the other and constituting the boundary of a domain $D_1 \subset D$ (Fig. 44). Then

$$\int\limits_{\gamma} f(z)\,dz = \int\limits_{\gamma_1} f(z)\,dz, \tag{9.13}$$

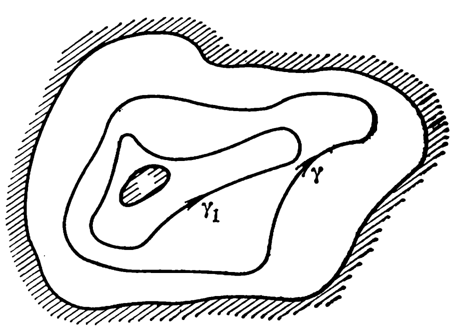


Fig. 44

where the traversal of curves γ and γ_1 is performed in one direction. The above formula implies that deformation of a closed contour in a domain where $f(z)$ is differentiable does not influence the value of $\int\limits_{\gamma} f(z)\,dz$.

9.4 The integral and the primitive Suppose that a function $f(z)$ is defined in a domain D and another function $F(z)$ is differentiable in D. If $F'(z) = f(z)$ for all $z \in D$, the function $F(z)$ is said to be a *primitive* of $f(z)$ in D.

Theorem 5 *If a function $f(z)$ is differentiable in a simply connected domain D, it has a primitive in the same domain.*

Proof. Consider the function

$$F(z) = \int\limits_{z_1}^{z} f(\zeta)\,d\zeta, \tag{9.14}$$

where the integral is evaluated along any curve lying in D. Since the value of this integral does not depend on the path of integration (Corollary 1), $F(z)$ is single-valued in D. We wish to show that

$F(z)$ is a primitive of $f(z)$, i.e

$$F'(z) = \left(\int_{z_0}^{z} f(\zeta)\, d\zeta \right)' = f(z).$$

Suppose $z + \Delta z$ is a point in D that lies in a small neighborhood of point $z \in D$. Consider the ratio

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z} \left\{ \int_{z_0}^{z+\Delta z} f(\zeta)\, d\zeta - \int_{z_0}^{z} f(\zeta)\, d\zeta \right\}$$

$$= \frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(\zeta)\, d\zeta. \qquad (9.15)$$

We have to show that $\sigma = \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)$ tends to zero as $\Delta z \to 0$.

Since $\int_{z}^{z+\Delta z} d\zeta = \Delta z$ (see Example 1 in Sec. 5), we can write

$$\frac{1}{\Delta z} \int_{z}^{z+\Delta z} f(z)\, d\zeta = f(z). \qquad (9.16)$$

Employing the fact that the values of the integrals in (9.15) and (9.16) do not depend on the path of integration, we can take the segment connecting points $z + \Delta z$ and z as the path of integration. We then have

$$\sigma = \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z} \int_{z}^{z+\Delta z} [f(\zeta) - f(z)]\, d\zeta,$$

whence

$$|\sigma| \leqslant \frac{1}{|\Delta z|} \int_{z}^{z+\Delta z} |f(\zeta) - f(z)|\, |d\zeta|. \qquad (9.17)$$

Since the function $f(z)$ is continuous at point z, for each positive ε we can find a positive $\delta = \delta(\varepsilon)$ such that at $|z - \zeta| < \delta$ we will have

$$|f(\zeta) - f(z)| < \varepsilon. \qquad (9.18)$$

Since in (9.17) ζ belongs to the segment $[z, z + \Delta z]$, we can write $|z - \zeta| \leqslant |\Delta z|$ and, hence, (9.18) holds if $|\Delta z| < \delta$. Combining (9.17) with (9.18) yields $|\sigma| < \frac{1}{|\Delta z|} \varepsilon |\Delta z|$, or $|\sigma| < \varepsilon$, if

$|\Delta z| < \delta$. Hence,

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z)-F(z)}{\Delta z} = f(z),$$

i.e. $F'(z) = f(z)$. The proof is complete.

Theorem 5 leads to the following

Corollary 3 *If a function $f(z)$ is continuous in a domain D and the value of the integral of $f(z)$ along any closed curve lying in D is equal to zero, the function* $F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$ *is a primitive of $f(z)$.*

Note that if $F(x)$ is a primitive of $f(z)$ in a domain D, then $F(z) + C$, where C is an arbitrary complex valued constant, is a primitive of $f(z)$ in D, too. The converse statement is also true, namely, we have

Theorem 6 *The totality of all the primitives of a function $f(z)$ in a domain D is given by the formula $F_1(z) + C$, where $F_1(z)$ is one of the primitives of $f(z)$, and C is an arbitrary constant.*

Proof. Let $F_1(z)$ and $F_2(z)$ be two primitives of $f(z)$ in D. Then the function $F(z) = F_2(z) - F_1(z) = u + iv$ is constant in D. Indeed, by the hypothesis $F'(z) = F_2'(z) - F_1'(z) = f(z) - f(z) = 0$ for all $z \in D$. From this it follows that (see Eqs. (7.8) and (7.9)) $\partial u/\partial x = \partial u/\partial y = \partial v/\partial x = \partial v/\partial y \equiv 0$ in D, and from a well-known theorem of mathematical analysis we obtain $F(z) \equiv \text{const}$, i.e. $F_2(z) = F_1(z) + C$, with C a complex valued constant.

Corollary 4 *Under the hypothesis of Theorem 5, any primitive $F(z)$ of $f(z)$ can be expressed thus:*

$$F(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta + C, \tag{9.19}$$

where C is a complex valued constant.

Corollary 5 *Under the hypothesis of Theorem 5, the following Newton-Leibniz formula is true:*

$$\int_{z_0}^{z_1} f(\zeta)\, d\zeta = F(z_1) - F(z_0). \tag{9.20}$$

Proof. Putting $z = z_0$ in (9.19), we find that $C = F(z_0)$. Putting $z = z_1$ in (9.19), we find that

$$F(z_1) = \int_{z_0}^{z_1} f(\zeta)\, d\zeta + C = \int_{z_0}^{z_1} f(\zeta)\, d\zeta + F(z_0),$$

from which (9.20) readily follows.

Corollary 6 *If two functions, $f(z)$ and $g(z)$, satisfy the hypothesis* *of Theorem 5, the integration-by-parts formula is valid*:

$$\int_{z_0}^{z_1} f(\zeta)\, g'(\zeta)\, d\zeta = [f(\zeta)\, g(\zeta)]_{z_0}^{z_1} - \int_{z_0}^{z_1} f'(\zeta)\, g(\zeta)\, d\zeta. \tag{9.21}$$

Proof. Integration of the identity $fg' = (fg)' - gf'$ and the formula

$$\int_{z_0}^{z_1} (fg)'\, d\zeta = f(z_1)\, g(z_1) - f(z_0)\, g(z_0) = [f(\zeta)\, g(\zeta)]_{z_0}^{z_1},$$

lead to Eq. (9.21).

Note that integrals of differentiable elementary functions of a complex variable in a simply connected domain are evaluated by the same methods and formulas as used for real valued functions. For instance,

$$\int_{z_1}^{z_2} e^{\zeta}\, d\zeta = e^{z_2} - e^{z_1}; \quad \int_{z_1}^{z_2} \zeta^n\, d\zeta = \frac{z_2^{n+1} - z_1^{n+1}}{n+1} \quad (n \text{ a nonnegative integer}).$$

Example 1. The function $f(z) = 1/z$ is differentiable in the multiply connected domain D: $0 < |z| < \infty$. Suppose $\tilde{D}$ is a simply connected domain and $\tilde{D} \subset D$. Then the function

$$F(z) = \int_1^z \frac{d\zeta}{\zeta}, \quad z \in \tilde{D},$$

where the integral is taken along any curve lying in $\tilde{D}$, is a primitive of $f(z)$, by Theorem 6, and $F'(z) = 1/z$. However, the function

$$\Phi(z) = \int_1^z \frac{d\zeta}{\zeta}, \quad z \in D,$$

is not single-valued in D because

$$\int_{|z|=1} \frac{d\zeta}{\zeta} = 2\pi i \neq 0. \ \square$$

10 Cauchy's Integral Formula

Cauchy's integral theorem leads to one of the most important formulas of the theory of functions of a complex variable, Cauchy's integral formula.

Theorem *Suppose $f(z)$ is a function that is differentiable in a simply connected domain D and let γ be a simple closed curve lying in D and oriented in the positive direction. Then, for every point z lying in the* *interior of* γ,

$$f(z)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta-z}\,d\zeta. \tag{10.1}$$

This is called *Cauchy's integral formula.*

Proof. The function $f(\zeta)/(\zeta-z)$ is differentiable with respect to ζ in D with point z deleted. Let us select a ρ in such a way that the circle $|\zeta-z|<\rho$ and its boundary C_ρ: $|\zeta-z|=\rho$ lie inside γ. Then, using Corollary 2 of Cauchy's integral theorem (see Eq. (9.13)), we obtain (Fig. 45)

$$J=\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\frac{1}{2\pi i}\int_{C_\rho}\frac{f(\zeta)}{\zeta-z}\,d\zeta$$
$$=\frac{1}{2\pi i}\int_{C_\rho}\frac{f(\zeta)-f(z)+f(z)}{\zeta-z}\,d\zeta=J_1+f(z)\frac{1}{2\pi i}\int_{C_\rho}\frac{d\zeta}{\zeta-z},$$

where $J_1=\frac{1}{2\pi i}\int_{C_\rho}\frac{f(\zeta)-f(z)}{\zeta-z}\,d\zeta$. Since $\frac{1}{2\pi i}\int_{C_\rho}\frac{d\zeta}{\zeta-z}=1$ (see Example 3 in Sec. 5), we can write

$$J=\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta-z}\,d\zeta=J_1+f(z), \tag{10.2}$$

and to prove the theorem we must only show that $J_1=0$.

In view of the continuity of $f(\zeta)$ at point z, for any positive ε there exists a positive $\delta=\delta(\varepsilon)$ such that $|f(\zeta)-f(z_1)|<\varepsilon$ for $|\zeta-z|<\delta$. Hence,

$$|J_1|\leqslant\frac{1}{2\pi}\int_{C_\rho}\frac{|f(\zeta)-f(z)|}{|\zeta-z|}\,|d\zeta|$$
$$<\frac{1}{2\pi}\frac{\varepsilon}{\rho}\int_{C_\rho}|d\zeta|=\varepsilon$$

if $\rho\leqslant\delta$. Recalling that J_1 does not depend on ρ, we obtain $J_1=0$, i.e. $J=f(z)$. The proof of the theorem is complete.

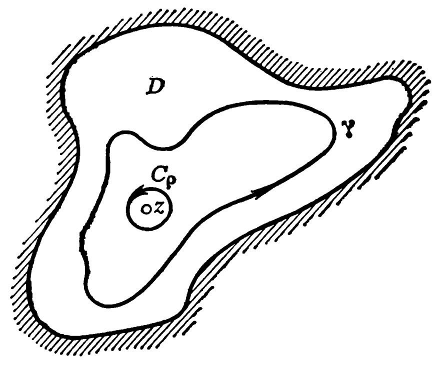


Fig. 45

Remark 1. Let D be a bounded, simply connected domain with a piecewise smooth boundary Γ, and let $f(z)$ be a function differentiable in D and continuous up to the boundary of D. Then for every point z lying in D we have

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}\,d\zeta. \tag{10.3}$$

The validity of (10.3) can be proved in the same manner as we proved the above theorem; Theorem 4 of Sec. 9 is also used.

Formula (10.3) remains valid for the case where D is a multiply connected domain. The proof of this is done in the same way as for formula (9.12).

Formula (10.3) is used to express the value of $f(z)$ inside a domain in terms of its values at the boundary of this domain.

Note a particular case of (10.3). Let $f(z)$ be differentiable in D and let γ and γ_1 be two simple closed curves (with γ_1 lying inside γ) that constitute the boundary of a domain $D_1 \subset D$ (see Fig. 44). Then for all $z \in D_1$ we have

$$f(z)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(\zeta)}{\zeta-z}\,d\zeta-\frac{1}{2\pi i}\int_{\gamma_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta. \tag{10.4}$$

Here γ and γ_1 are oriented in the positive sense.

Remark 2. If on the right-hand side of (10.3) point z belongs to the exterior of Γ, i.e. lies outside $\bar{D}$, the integrand is differentiable with respect to ζ everywhere in D, and, by Cauchy's integral theorem, the integral vanishes. Thus,

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\begin{cases} f(z), & z\in D,\\ 0, & z \text{ outside } \bar{D}.\end{cases}$$

The mean value theorem *Let a function $f(z)$ be differentiable in the circle K: $|z-z_0|<R$ and continuous in the closure $\bar{K}$. Then the mean arithmetic value of $f(z)$ on the circumference of the circle is equal to the value of $f(z)$ at the center of the circle:*

$$f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\varphi})\,d\varphi. \tag{10.5}$$

Proof. Suppose that Γ in (10.3) is the circumference of a circle of radius R centered at point z_0. Then

$$\zeta=z_0+Re^{i\varphi},\quad 0\leqslant\varphi\leqslant 2\pi,\quad d\zeta=iRe^{i\varphi}\,d\varphi,$$

$$f(z_0)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)\,d\zeta}{\zeta-z_0}=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_0+Re^{i\varphi})\,iRe^{i\varphi}}{Re^{i\varphi}}\,d\varphi$$

$$=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+Re^{i\varphi})\,d\varphi.$$

The proof of the theorem is complete.

The mean value theorem for harmonic functions *Let*

$$u(z)=u(x,y),\quad z=x+iy,$$

be harmonic in the circle K: $|z - z_0| < R$ *and continuous in the closure* $\overline{K}$. *Then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + R^{i\varphi})\, d\varphi. \qquad (10.6)$$

Proof. Suppose $f(z)$ is a function that is regular in the circle K and such that $\operatorname{Re} f(z) = u(z)$. By the mean value theorem we have

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \rho e^{i\varphi})\, d\varphi, \quad 0 < \rho < R. \qquad (10.7)$$

Separating the real part from the imaginary, we obtain

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\varphi})\, d\varphi,$$

whence, going over to the limit as $\rho \to R$, we arrive at (10.6). The proof of the theorem is complete.

11 Power Series

11.1 Domain of convergence of a power series A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n (z-a)^n, \qquad (11.1)$$

where a and c_n ($n = 0, 1, 2, \ldots$) are given complex numbers, and z is the complex valued variable. At $a = 0$ the series (11.1) takes the form

$$\sum_{n=0}^{\infty} c_n z^n. \qquad (11.2)$$

Obviously, all the properties of power series of the type (11.2) are valid for (11.1).

The *domain of convergence* of the power series (11.2) is the set of all points z at which (11.2) is convergent. Point $z = 0$ always belongs to the domain of convergence of (11.2). There are power series that converge only at $z = 0$ (see Example 3).

Example 1. The series $\sum_{n=0}^{\infty} (-1)^n z^n$ is convergent at $|z| < 1$ and divergent at $|z| \geqslant 1$. □

Example 2. The series $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$ is convergent in the the entire complex plane because for each z there is a positive integer n_0 such that at $n > n_0$ we have $|z/n| < 1/2$, i.e. $|z^n/n^n| < 1/2^n$, from which the convergence of the series at point z follows. □

Example 3. The series $\sum_{n=0}^{\infty} n^n z^n$ is convergent only at $z=0$ because $z \neq 0$, then for $n > 1/|z|$ we have $|nz| > 1$ and $|nz|^n > 1$ (the necessary condition for convergence of a series is not met). □

Theorem 1 (Abel's theorem) *If a power series of the type* (11.2) *is convergent at a point* $z_0 \neq 0$, *it is absolutely convergent in the circle* K_0: $|z| < |z_0|$, *while in any smaller circle* K_1: $|z| \leqslant R_1 < |z_0|$ *this series is uniformly convergent.*

Proof. In view of the fact that (11.2) is convergent at point z_0 we have $\lim_{n\to\infty} c_n z_0^n = 0$, and, hence, there is a positive constant M such that for all n we have $|c_n z_0^n| < M$. Let z be an arbitrary point of K_0. Then

$$|c_n z^n| = |c_n z_0^n| \left|\frac{z}{z_0}\right|^n < Mq^n, \tag{11.3}$$

where $q = |z/z_0| < 1$, and this implies that (11.2) is absolutely convergent in K_0.

If $z \in K_1$, then $|c_n z^n| \leqslant M |z/z_0|^n \leqslant Mq_1^n$, where $q_1 = R_1/|z_0| < 1$ does not depend on z, and by Weierstrass's test the series (11.2) is uniformly convergent in K_1.

Let R be the least upper bound of the distances between point $z = 0$ and points z at which (11.2) is convergent. Then at $|z| > R$ this series is divergent. Abel's theorem leads to the following

Corollary 1 *The series* (11.2) *is convergent in the circle* K: $|z| < R$, *while in any smaller circle* $|z| \leqslant R_1 < R$ *this series is uniformly convergent.*

The circle K is said to be the *circle of convergence*, and its radius R the *radius of convergence* of (11.2). At the points on the circle $|z| = R$ the series (11.2) may be either convergent or divergent. If (11.2) is convergent only at $z = 0$, its radius of convergence is zero, while if it is convergent in the entire complex plane, the radius of convergence is infinite.

The radius of convergence of (11.2) is given by the Cauchy-Hadamard formula

$$R = 1/l, \quad \text{where} \quad l = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}. \tag{11.4}$$

The proof of (11.4) can be found in Bitsadze [1].

Consider the series

$$\sum_{n=1}^{\infty} nc_n z^{n-1}, \tag{11.5}$$

which consists of the derivatives of the terms in (11.2). Since $\lim_{n\to\infty} \sqrt[n]{n} = 1$, the Cauchy-Hadamard formula (11.4) results in

Corollary 2 *The radius of convergence of the series* (11.5) *is equal to the radius of convergence of the series* (11.2).

11.2 Term-by-term differentiation of a power series

Theorem 2 *Let the radius of convergence of the power series*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{11.6}$$

be $R \neq 0$. Then this series can be differentiated any number of times inside the circle $|z| < R$. The resulting series have the same radius of convergence as (11.6).

Proof. Consider the series

$$S(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}, \tag{11.7}$$

which consists of the derivatives of the terms in (11.6). By Corollary 2, the series (11.7) is uniformly convergent in the circle K_1: $|z| \leqslant R_1 < R$ and its sum $S(z)$ is continuous in K_1. We wish to show that the function $f(z)$ is differentiable in K_1 and that

$$S(z) = f'(z). \tag{11.8}$$

Let γ be an arbitrary curve lying within K_1 and connecting points 0 and z. Then (see Sec. 9)

$$\int_0^z \zeta^k \, d\zeta = \frac{z^{k+1}}{k+1}.$$

Hence,

$$\int_0^z n c_n \zeta^{n-1} \, d\zeta = c_n z^n, \quad n = 1, 2, \ldots. \tag{11.9}$$

Integrating the uniformly convergent series (11.7) along curve γ termwise and taking into account the fact that the value of the integral $\int_0^z S(\zeta)\, d\zeta$ is independent of the path of integration, we obtain

$$\int_0^z S(\zeta)\, d\zeta = \sum_{n=0}^{\infty} \int_1^z n c_n \zeta^{n-1} \, d\zeta = \sum_{n=1}^{\infty} c_n z^n. \tag{11.10}$$

Combining (11.10) with (11.6), we obtain

$$\int_0^z S(\zeta)\, d\zeta = f(z) - c_0. \tag{11.11}$$

In view of Corollary 3 of Sec. 9, the function $\int\limits_0^z S(\zeta)\, d\zeta$ is a primitive of $S(z)$ and, hence, $S(z) = f'(z)$. Thus, the function $f(z)$ is differentiable in the circle K_1 and Eq. (11.8) is true, i.e. the series (11.6) can be differentiated termwise in K_1. But the radius R_1 of K_1 can be chosen as close to R as desired, whence (11.6) can be differentiated termwise in K.

The operation of term-by-term differentiation can obviously be applied to (11.6) any number of times. The proof of the theorem is complete.

Corollary 3 *The coefficients c_n of the power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \tag{11.12}$$

which converges in the circle K: $|z - a| < R$ $(R \neq 0)$, are given by the following formulas:

$$c_0 = f(a), \quad c_n = \frac{f^{(n)}(a)}{n!} \quad (n = 1, 2, \ldots). \tag{11.13}$$

Proof. Applying Theorem 2 to the power series (11.12), we obtain

$$f^{(n)}(z) = n!c_n + (n+1)\, c_{n+1}(z-a) + \ldots \tag{11.14}$$

for all $z \in K$. Putting $z = a$ in (11.14) and (11.12), we arrive at (11.13).

From (11.13) we can see that the expansion of a function in a power series is unique.

The power series $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(a)}{n}(z-a)^n$ is said to be a *Taylor series* of $f(z)$. Thus, every power series (11.12) within its circle of convergence is the Taylor series of the sum of (11.12).

12 Properties of Regular Functions

The definition of a regular function was given in Sec. 7.4. Here we will prove the concepts of differentiability and regularity in a domain to be equivalent and study the properties of regular functions.

12.1 The regularity of a function that is differentiable in a domain

Theorem 1 *If a function $f(z)$ is differentiable in a domain D, it is regular in this domain.*

Proof. Let $z = a$ be an arbitrary point of D. Consider the circle K: $|z - a| < \rho$, $\rho > 0$, lying in D together with its boundary γ_ρ:

$|\zeta - a| = \rho$. Suppose z is an arbitrary point in K. By virtue of Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \tag{12.1}$$

Next we expand $\frac{1}{\zeta - z}$ in a power series in $z - a$ (a geometric progression):

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - a)\left(1 - \frac{z-a}{\zeta - a}\right)} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\zeta - a)^{n+1}}. \tag{12.2}$$

If $\zeta \in \gamma_\rho$, then

$$|\zeta - a| = \rho, \quad \left|\frac{z-a}{\zeta - a}\right| = \frac{|z-a|}{\rho} < 1,$$

and, hence, the series in (12.2) converges uniformly in ζ on γ_ρ (Weierstrass's test). The series

$$\frac{f(\zeta)}{\zeta - z} = \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - a)^{n+1}} (z-a)^n, \tag{12.3}$$

which is (12.2) multiplied by $f(\zeta)$, is also uniformly convergent on γ_ρ since $f(\zeta)$ is continuous on γ_ρ and, hence, bounded on γ_ρ. Integrating the series in (12.3) termwise along γ_ρ and employing (12.1), we obtain

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \tag{12.4}$$

where

$$c_n = \frac{1}{2\pi i} \int\limits_{|\zeta - a| = \rho} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta. \tag{12.5}$$

The series (12.4) is convergent in the circle K: $|z - a| < \rho$, which means that $f(z)$ is regular at point a. But since a is an arbitrary point of D, the function $f(z)$ is regular in D. The proof of the theorem is complete.

Theorem 1 and Theorem 4 of Sec. 7 result in the following

Corollary 1 *A function $f(z)$ is regular in a domain D if and only if it is differentiable in this domain.*

Thus, in D the concepts of regularity and differentiability are equivalent. This together with the properties of differentiable functions (see Sec. 7) implies, for one, that if two functions, $f(z)$ and $g(z)$, are regular in a domain D, their sum, product, and quotient (provided the denominator is nonzero) are regular in D.

Similarly, if $f(z)$ is regular in D and the function $F(w)$ is regular in G and if the range of values of $w = f(z)$ $(z \in D)$ belongs to G, then the function $\Phi(z) = F[f(z)]$ is regular in D.

Theorem 1 has

Corollary 2 *The series* (12.4) *is sure to converge in the circle* $|z - a| < R_1$, *where* R_1 *is the distance between point* $z = a$ *and the boundary of the domain* D *in which* $f(z)$ *is differentiable.*

For this reason the radius of convergence of the power series (12.4) is no less than R_1.

Corollary 3 *If a function* $f(z)$ *is regular in a circle* K: $|z - a| < R$, *it can be represented by a Taylor series*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

convergent in the entire circle K.

Corollary 4 *If* $f(z)$ *is regular at* $z = a$, *it is regular in a neighborhood of point* a.

Proof. A regular function $f(z)$ can be represented by a convergent series (12.4) in a circle K: $|z - a| < \rho$ and, hence, is differentiable in this circle (Theorem 2 of Sec. 11). But by Theorem 1 the function $f(z)$ is regular in K. This means that if $z_0 \in K$, then

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n.$$

The resulting series is convergent in a circle $|z - z_0| < \rho_1$, $\rho_1 \geqslant d$, where d is the distance between point z_0 and the boundary of K.

Remark 1. A function that is differentiable at a point $z = a$ may not be regular at this point, since a function regular at a point $z = a$ is differentiable not only at the point $z = a$ but in a neighborhood of this point as well. For instance, the function $f(z) = \bar{z}^2$ is differentiable only at $z = 0$ (see Example 3c in Sec. 7) and therefore is not regular at this point.

12.2 The infinite differentiability of regular functions

Theorem 2 *If a function* $f(z)$ *is differentiable in a domain* D, *it is infinitely differentiable in this domain, with*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int\limits_{\gamma_\rho} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta, \quad z \in D, \tag{12.6}$$

where γ_ρ *is the boundary of the circle* $|\zeta - z| \leqslant \rho$ *lying in* D.

Proof. By Theorem 1, the function $f(z)$ is regular in D. Let $z = a \in D$. Since $f(z)$ is regular at point $z = a$, we can write

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \tag{12.7}$$

where the series in (12.7) is convergent in a circle $|z-a|<\rho$ ($\rho>0$). According to Theorem 2 of Sec. 11, the series in (12.7) can be differentiated in the circle $|z-a|<\rho$ termwise any number of times and (see (11.13))

$$c_0=f(a), \quad c_n=\frac{f^{(n)}(a)}{n!} \quad (n=1,\ 2,\ a..). \tag{12.8}$$

On the other hand,

$$c_n=\frac{f^{(n)}(a)}{n!}=\frac{1}{2\pi i}\int\limits_{\gamma_\rho}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta,$$

whence, substituting z for a, we arrive at (12.6).

This theorem implies, for one, that the derivative of a regular function is a regular function, too.

Remark 2. Formula (12.6) can formally be obtained from Cauchy's integral formula

$$f(z)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(\zeta)\,d\zeta}{\zeta-z}$$

by differentiating the left and right sides of the latter n times.

Remark 3. If the function $f(z)$ is differentiable in a neighborhood of point a, it is regular at this point (Theorem 1) and can be represented in the form of a power series that is the Taylor series of $f(z)$ (see Corollary 3 of Sec. 11). Thus, the formal Taylor series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(z-a)^n$$

of a function $f(z)$ differentiable in a neighborhood of a converges to this function in a (generally different) neighborhood of point a. A similar proposition for functions of a real variable does not hold. For instance, the function

$$f(x)=\begin{cases} e^{-1/x^2}, & x\neq 0,\\ 0, & x=0,\end{cases}$$

is differentiable everywhere and has an infinite number of derivatives at point $x=0$ equal to zero, and, hence, all the coefficients of the Taylor series of $f(x)$ are zero at $x=0$, but $f(x)\not\equiv 0$.

From Theorem 2 and Sec. 7.3 follows

Corollary 5 *A function that is harmonic in a domain is infinitely differentiable in this domain.*

12.3 Sufficient conditions for regularity Theorem 1 states that the differentiability of a function $f(z)$ in a domain D is a sufficient condition for the regularity of this function in D. Here are other sufficient conditions.

Theorem 3 (Morera's theorem) *If a function $f(z)$ is continuous in a simply connected domain D and the value of the integral of this function along any closed contour lying in D is zero, the function is regular in D.*

Proof. By virtue of Corollary 3 of Sec. 9, the function $f(z)$ has a primitive, i.e. there is a differentiable function $F(z)$ such that $F'(z) = f(z)$ for all $z \in D$. According to Theorem 1, the function $F(z)$ is regular in D and, hence, its derivative is a function regular in D, i.e. the function $f(z) = F'(z)$ is regular in D.

Theorem 4 (Weierstrass's first theorem) *Let the functions $f_n(z)$ $(n = 1, 2, \ldots)$ be regular in a domain D and let the series*

$$f(z) = \sum_{n=1}^{\infty} f_n(z) \tag{12.9}$$

be uniformly convergent in each closure $\bar{D}_1$ lying in D. Then $f(z)$ is regular in D.

Proof. Suppose z_0 is an arbitrary point of D. Consider the circle K: $|z - z_0| < \rho$ lying together with its boundary in D. By hypothesis, the series (12.9) is uniformly convergent in $\bar{K}$ and, hence, in K. Moreover, the functions $f_n(z)$ $(n = 1, 2, \ldots)$ are regular in K and, hence, continuous in K. For this reason $f(z)$ is continuous in K, since it is the sum of a uniformly convergent series whose terms are continuous functions.

Let γ be a closed contour lying in K. Integrating the uniformly convergent series (12.9) along γ termwise, we obtain

$$\int_{\gamma} f(z)\,dz = \sum_{n=1}^{\infty} \int_{\gamma} f_n(z)\,dz.$$

By Cauchy's integral theorem, $\int_{\gamma} f_n(z)\,dz = 0$ $(n = 1, 2, \ldots)$ and, hence $\int_{\gamma} f(z)\,dz = 0$. By Morera's theorem, the function $f(z)$ is regular in circle K and, for one, at point z_0. Since z_0 is an arbitrary point of D, the function $f(z)$ is regular in D. The proof of the theorem is complete.

Theorem 5 (Weierstrass's second theorem) *Under the hypothesis of Theorem 4, the series (12.9) can be differentiated term-by-term any number of times. The resulting series are uniformly convergent in closures $\bar{D}_1$ that lie inside D.*

We give only the formulation of Weierstrass's second theorem. The interested reader can find its proof in, say, Markushevich [1].

Other sufficient conditions for regularity related to integrals depending on a parameter will be given in Sec. 15.

In conclusion of this section we give a brief summary of the basic properties of regular functions. The reader may have already noted that along with the term "regular function" other equivalent terms are used in the literature, namely,

$$\{\textit{regular function}\} \equiv \{\textit{holomorphic function}\} \equiv \{\textit{single-valued analytic function}\}.$$

Here are the criteria (necessary and sufficient) for the regularity of a function $f(z)$ in a domain D:

(1) the differentiability of $f(z)$ in D;

(2) the Cauchy-Riemann equations.

Morera's theorem and Weierstrass's first theorem are sufficient conditions for the regularity of a function $f(z)$ in a domain D.

Finally, regular functions possess the following properties:

(i) the sum, difference, product, quotient (with a nonzero denominator), and composite function consisting of regular functions are regular functions, too;

(ii) a regular function is infinitely differentiable;

(iii) a regular function obeys Cauchy's integral theorem and formula;

(iv) the primitives of a function that is regular in a simply connected domain are regular.

12.4 Some methods of power series expansion Every function $f(z)$ that is regular in the circle $|z-a| < \rho$ can be expanded into a power series convergent in this circle (see Corollary 3 of Theorem 1):

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \tag{12.10}$$

where the expansion coefficients c_n are given by the formulas

$$c_n = \frac{1}{n!} f^{(n)}(a), \tag{12.11}$$

or

$$c_n = \frac{1}{2\pi i} \int\limits_{|\zeta - a| = \rho_1} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta, \quad \rho_1 < \rho. \tag{12.12}$$

This power series is the Taylor series of the function $f(z)$ in a neighborhood of point $z = a$.

By directly calculating the derivatives of the elementary functions e^z, $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ at point $z = 0$ (see Eqs. (7.14), (7.15), and (7.16)) the following expansions that are convergent in the entire complex plane can be obtained

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \tag{12.13}$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}, \tag{12.14}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \qquad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}. \tag{12.15}$$

Another example is the series

$$\frac{1}{1-z} = \sum_{n=1}^{\infty} z^n, \tag{12.16}$$

which is convergent in the circle $|z| < 1$.

Note further that the formulas (12.12) are usually not employed when calculating the expansion coefficients in (12.10). Often the expansion coefficients of a Taylor series are found through known expansions (for instance, through (12.13)-(12.16)) and by employing some special methods.

Example 1. The series

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n \quad (|z| < 1)$$

is obtained by differentiating the series (12.16). □

Example 2. To find the Taylor series in a neighborhood of point $z = 0$ of the rational function

$$f(z) = \frac{1}{(1-z^2)(z^2+4)}$$

we write

$$f(z) = \frac{1}{5}\left(\frac{1}{1-z^2} + \frac{1}{z^2+4}\right) = \frac{1}{5}\left[\frac{1}{1-z^2} + \frac{1}{4\left(1+\frac{z^2}{4}\right)}\right],$$

whence by virtue of (12.16) we obtain

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{5}\left[1 + \frac{(-1)^n}{4^{n+1}}\right] z^{2n},$$

which converges in the circle $|z| < 1$. □

Here are some methods of expansion into power series.

(1) Arithmetic operations on power series. Suppose two functions

$f(z)$ and $g(z)$, are regular in a neighborhood of a point $z = a$ and can be expanded in the following power series:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \tag{12.17}$$

$$g(z) = \sum_{n=0}^{\infty} d_n (z-a)^n, \tag{12.18}$$

where both series are convergent in the circle $|z - a| < R$. Then we can write

$$Af(z) = \sum_{n=0}^{\infty} Ac_n (z-a)^n, \quad A = \text{const}, \tag{12.19}$$

$$f(z) \pm g(z) = \sum_{n=0}^{\infty} (c_n \pm d_n)(z-a)^n, \tag{12.20}$$

$$f(z)\, g(z) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} c_k d_{n-k}\Big)(z-a)^n. \tag{12.21}$$

The series (12.19)-(12.21) are convergent in the circle $|z - a| < R$.

Example 3. To expand the function $e^z \cos z$ in a neighborhood of point $z = 0$ we multiply the series (12.13) and (12.14). However, to calculate the expansion coefficients more effectively it is expedient to employ the identity

$$e^z \cos z = e^z \left(\frac{e^{iz}+e^{-iz}}{2}\right) = \frac{1}{2}\left(e^{z(1+i)} + e^{z(1-i)}\right).$$

Since $1 + i = \sqrt{2} e^{i\pi/4}$ and $1 - i = \sqrt{2} e^{-i\pi/4}$, employing the series (12.13) yields the following expansion:

$$e^z \cos z = \sum_{n=0}^{\infty} \frac{2^{n/2} e^{i\pi n/4} + 2^{n/2} e^{-i\pi n/4}}{2n!} z^n = \sum_{n=0}^{\infty} \frac{2^{n/2}}{n!} \cos \frac{\pi n}{4} z^n,$$

which is convergent in the entire complex plane. □

(2) Method of undetermined coefficients. Let us investigate the problem of finding the expansion coefficients in a Taylor series in a neighborhood of a point $z = a$ for a function $f(z)$ that is the quotient of two regular functions ($f(z) = g(z)/h(z)$) whose Taylor series are known ($h(a) \neq 0$). If

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad g(z) = \sum_{n=0}^{\infty} a_n (z-a)^n,$$

$$h(z) = \sum_{n=0}^{\infty} b_n (z-a)^n.$$

then, equating the coefficients of the same powers of $z - a$ in the equality $f(z)\,h(z) = g(z)$, we arrive at equations of the form $c_0 b_n + c_1 b_{n-1} + \ldots + c_n b_0 = a_n$, from which we can find the coefficients c_0, c_1, c_2, etc. It is then easy to express the c_n in terms of $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ via a determinant (see Markushevich [2], pp. 212-217).

Example 4. Applying the method of undetermined coefficients to the function $z/(e^z - 1)$, we arrive at the following expansion

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n. \tag{12.22}$$

Here the B_n are the Bernoulli numbers given by the formulas $B_0 = 1$ $B_0 \binom{n+1}{0} + B_1 \binom{n+1}{1} + \ldots + B_n \binom{n+1}{n} = 0$ $(n = 1, 2, \ldots)$, where the $\binom{n+1}{k}$ $(k = 0, 1, 2, \ldots, n)$ are binomial coefficients. The series (12.22) is convergent in the circle $|z| < 2\pi$.

Using the identity

$$\cot z = \frac{\cos z}{\sin z} = i\,\frac{e^{2iz}+1}{e^{2iz}-1} = i + \frac{2i}{e^{2iz}-1}$$

and the expansion (12.22), we arrive at the expansion

$$z \cot z = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)} z^{2n} \quad (|z| < \pi). \ \square \tag{12.23}$$

(3) A series of power series Let

$$f(z) = \sum_{n=1}^{\infty} f_n(z), \tag{12.24}$$

where all the series

$$f_n(z) = \sum_{k=0}^{\infty} c_n^{(k)} (z-a)^k \quad (n = 1, 2, \ldots) \tag{12.25}$$

are convergent in a single circle K: $|z - a| < \rho$ and besides, the series (12.24) is uniformly convergent in every circle $|z - a| \leqslant \rho_1$, where $\rho_1 < \rho$. By Weierstrass's theorems,

$$c_k = \frac{f^{(k)}(a)}{k!} = \sum_{n=1}^{\infty} \frac{f_n^{(k)}(a)}{k!} = \sum_{n=1}^{\infty} c_n^{(k)},$$

whence

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k = \sum_{k=0}^{\infty} \Big(\sum_{n=1}^{\infty} c_n^{(k)}\Big) (z-a)^k. \tag{12.26}$$

(4) Substitution of a series into a series. Consider the function $f(z) = g[h(z)]$, where the function $w = h(z)$ is regular in the circle K_1: $|z - a| < R_1$ and the function $g(w)$ is regular in the circle K: $|w - b| < R$, with $h(a) = b$. Suppose

$$g(w) = \sum_{n=0}^{\infty} b_n (w-b)^n, \quad h(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

are power series of $g(w)$ and $h(z)$. Since the function $h(z)$ is regular in K_1, there is a circle K_2: $|z - a| < R_2 \leqslant R_1$ such that $|h(z) - h(a)| < R$, i.e. $|w - b| < R$. The function $f(z)$ is regular in the circle K_2 since it is a composite function of regular functions. The expansion coefficients in $f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k$ are determined via Eqs. (12.25)-(12.26), where $f_n(z) = b_n [h(z) - b]^n$, since

$$f(z) \equiv g(w) = \sum_{n=0}^{\infty} b_n (w-b)^n = \sum_{n=0}^{\infty} b_n [h(z)-b]^n. \tag{12.27}$$

The resulting series

$$f(z) = \sum_{n=0}^{\infty} b_n [h(z)-b]^n = \sum_{k=0}^{\infty} c_k (z-a)^k \tag{12.28}$$

is sure to converge in K_2: $|z - a| < R_2$, where R_2 is chosen in such a way that $|w - b| = |h(z) - b| < R$ when $|z - a| < R_2$.

(5) A re-expansion of a power series. Let us consider the following special case of substituting a series into a series. Let the series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \tag{12.29}$$

be convergent in the circle K: $|z - a| < R$ and let b be a point within the circle of convergence K. We write

$$z - a = (b - a) + (z - b). \tag{12.30}$$

Substituting (12.29) into (12.30), we obtain

$$f(z) = \sum_{n=0}^{\infty} c_n [(z-b)+(b-a)]^n. \tag{12.31}$$

If $|z - b| < \rho$, where $\rho = R - |b - a|$, then $|z - a| < R$ and, applying the method of substituting a series into another series to (12.31), we arrive at the expansion

$$f(z) = \sum_{n=0}^{\infty} d_n (z-b)^n, \tag{12.32}$$

which is convergent in the circle $|z - b| < \rho$.

12.3 The zeros of regular functions A point $z = a$ is said to be a *zero* of a regular function $f(z)$ if $f(a) = 0$.

(1) Let $a \neq \infty$ be a zero of $f(z)$. We will consider the expansion of $f(z)$ into a power series in a neighborhood of point $z = a$:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n. \tag{12.33}$$

Since point $z = a$ is a zero of $f(z)$, we can immediately write $c_0 = f(a) = 0$. Suppose that c_m in (12.33) is the first nonzero expansion coefficient ($c_0 = c_1 = \ldots = c_{m-1} = 0$, $c_m \neq 0$), i.e.

$$f(z) = c_m (z-a)^m + c_{m+1} (z-a)^{m+1} + \ldots, \quad c_m \neq 0. \tag{12.34}$$

Then the number m is said to be the *order of the zero* $z = a$ of $f(z)$ and $z = a$ the *zero of the mth order*. Since $c_k = f^{(k)}(a)/k!$ ($k = 1, 2, \ldots$), the order of the zero $z = a$ of $f(z)$ is equal to the lowest order of the derivative of this function differing from zero at point $z = a$.

Obviously, (12.34) can be rewritten in the following form:

$$f(z) = (z-a)^m [c_m + c_{m+1} (z-a) + \ldots], \tag{12.35}$$

where the series $h(z) = c_m + c_{m+1}(z-a) + \ldots$ is convergent in the same circle as the series (12.34). Hence, the function $h(z)$ is regular at point $z = a$, with $h(a) = c_m \neq 0$. Thus, if $z = a$ is an mth order zero of $f(z)$, then we can write

$$f(z) = (z-a)^m h(z), \quad h(a) \neq 0, \tag{12.36}$$

where $h(z)$ is regular at $z = a$.

Conversely, if a function $f(z)$ can be represented in the form (12.36), where the function $h(z)$ is regular at point $z = a$, then Eqs. (12.34) and (12.35) are valid, i.e. point $z = a$ is an mth order zero of $f(z)$.

(2) Let $z = \infty$ be a zero of a function $f(z)$. Since $f(z)$ is regular at point $z = \infty$ (see Sec. 7.4), we can write

$$f(z) = c_0 + \sum_{n=1}^{\infty} \frac{c_n}{z^n}. \tag{12.37}$$

By hypothesis, $c_0 = f(\infty) = 0$. Let c_m be the first nonzero expansion coefficient in (12.37), i.e. $c_1 = c_2 = \ldots = c_{m-1} = 0$ but $c_m \neq 0$ (the number m is said to be the *order of the zero* $z = \infty$ of $f(z)$ and $z = \infty$ the *zero of the mth order*). Then

$$f(z) = \sum_{n=m}^{\infty} \frac{c_n}{z^n} = \frac{1}{z^m} \left(c_m + \frac{c_{m+1}}{z} + \ldots\right), \tag{12.38}$$

whence

$$f(z) = z^{-m}\psi(z), \quad \psi(\infty) = c_m \neq 0, \tag{12.39}$$

where the function $\psi(z)$ is regular at point $z = \infty$.

Conversely, if a function $f(z)$ can be represented in the form (12.39), where $m = 1, 2, \ldots$, and $\psi(z)$ is regular at point $z = \infty$, then Eq. (12.38) is valid, i.e. point $z =$ is an mth order zero of $f(z)$. We have therefore proved

Theorem 6 *A point $a \neq \infty$ is an mth order zero of a function $f(z)$ if and only if this function can be represented in the form $f(z) = (z - a)^m h(z)$, with $h(z)$ regular at point $z = a$ and $h(a) \neq 0$.*

Similarly, the representation of a function $f(z)$ in the form $f(z) = z^{-m}\psi(z)$, where $\psi(z)$ is regular at point $z = \infty$, $\psi(\infty) \neq 0$, and $m = 1, 2, \ldots$, is necessary and sufficient for point $z = \infty$ to be an mth order zero of $f(z)$.

Corollary 6 *If a point $z = a$ is an mth order zero of a function $f(z)$, then it is a pmth order zero of the function $g(z) = [f(z)]^p$, with $p = 1, 2, \ldots$.*

Remark 4. The asymptotic behavior

$$f(z) \sim c_m (z - a)^m, \quad c_m \neq 0 \quad (z \to a), \tag{12.40}$$

which follows from (12.36), is the necessary and sufficient condition for the function $f(z)$ regular at point $a \neq \infty$ to have an mth order zero at this point.

Similarly, a function $f(z)$ regular at point $z = \infty$ has an mth order zero at this point if and only if

$$f(z) \sim \frac{A}{z^m}, \quad A \neq 0 \quad (z \to \infty). \tag{12.41}$$

The asymptotic formulas (12.40) and (12.41) can be taken as definitions of the order of a zero at points $z = a$ $(a \neq \infty)$ and $z = \infty$, respectively.

Example 5. The function $f(z) = \sin(1/z)$ has first order zeros at the points $z_k = 1/k\pi$ $(k = \pm 1, \pm 2, \ldots)$ and at point $z = \infty$. □

Example 6. The function $f(z) = (e^z + 1)^3$ has third order zeros at points $z_k = (2k + 1)\pi i$, $k = 0, \pm 1, \pm 2, \ldots$. □

Example 7. The function

$$f(z) = \frac{(z^3+1)^6}{(z^2+4)^{11}} e^{1/z}$$

has sixth order zeros at the points $z_k = e^{(2k+1)\pi i/3}$, $k = 0, 1, 2$; point $z = \infty$ is a forth order zero of this function:

$$f(z) \sim \frac{z^{18}}{z^{22}} e^{1/z} \sim \frac{1}{z^4} \quad (z \to \infty). \ □$$

We will now prove the following theorem on the zeros of a regular function:

Theorem 7 *Let a function $f(z)$ be regular at a point $z = a$ and $f(a) = 0$. Then either $f(z) \equiv 0$ in a neighborhood of a point a or there is a neighborhood of point a where there are no zeros of $f(z)$ different from a.*

Proof. Two cases are possible here. (i) All the expansion coefficients in (12.33) are zeros, which means that $f(z) \equiv 0$ in a neighborhood of point $z = a$. (ii) There is a positive integer m such that $c_0 = c_1 = \ldots = c_{m-1} = 0$ but $c_m \neq 0$.

In the second case, point z is an mth order zero of $f(z)$ and, hence, by Theorem 6, $f(z) = (z - a)^m h(z)$, where $h(z)$ is a function regular at point a, and $h(a) \neq 0$. In view of the continuity of $h(z)$ and from the condition $h(a) \neq 0$ we infer that $h(z) \neq 0$ in a neighborhood of point a. Thus, there is a neighborhood of point a where there are no other zeros of $f(z)$ except a. Consequently, the zeros of a regular function are always isolated.

13 The Inverse Function

13.1 The inverse function theorem The term "inverse function" was introduced in Sec. 8.1. Here we will give a more detailed definition of an inverse function.

Let the function $w = f(z)$ be defined on a set E and let E' be the

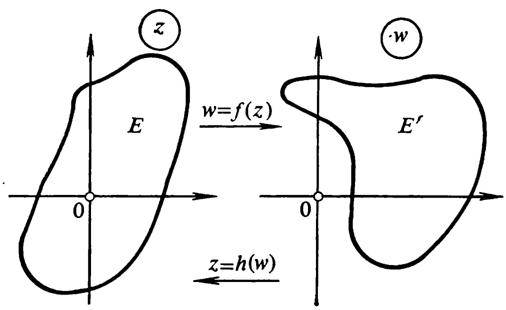


Fig. 46

set of values of this function (Fig. 46). Then for each value $w \in E'$ there is one or several values $z \in E$ such that $f(z) = w$, i.e. for each $w \in E'$ the equation

$$f(z) = w \tag{13.1}$$

has one or several solutions $z \in E$. These solutions determine on E' a function $z = h(w)$ called the *inverse* of the function $w = f(z)$.

Thus, to find a function that is the inverse of $w = f(z)$ we must find all the solutions of Eq. (13.1) for each value $w \in E'$. The definition of the inverse function implies that on E' we have the identity

$$f[h(w)] \equiv w.$$

Here are sufficient conditions for the regularity of the inverse function.

The inverse function theorem *Let a function $w = f(z)$ be regular at point z_0 and let $f'(z_0) \neq 0$. Then*

(1) *there is a circle K: $|z - z_0| < \rho$ and a circle K': $|w - w_0| < \rho'$, $w_0 = f(z_0)$, such that for each $w \in K'$ Eq. (13.1) has a single solution $z = h(w)$, where $z \in K$;*

(2) *the function $z = h(w)$, which is the inverse of $w = f(z)$, is regular at point w_0;*

(3) *in a neighborhood of point w_0 the following formula is valid:*

$$h'(w) = \frac{1}{f'(z)} = \frac{1}{f'[h(w)]}. \tag{13.2}$$

Proof. Putting $z = x + iy$ and $w = u + iv$, we substitute for Eq. (13.1) the following set of equations

$$\left.\begin{aligned} u(x, y) &= u, \\ v(x, y) &= v. \end{aligned}\right\} \tag{13.3}$$

The Jacobian $J(x, y) = J(z)$ of the mapping (13.3), which by Eq. (8.14) is equal to $|f'(z)|^2$, is nonzero at point $z_0 = x_0 + iy_0$ by hypothesis and, hence, is nonzero in a small neighborhood of this point.

By a well-known theorem of mathematical analysis (e.g. see Kudryavtsev [1]), in a neighborhood of the point $w_0 = u_0 + iv_0$ there is a one-to-one continuous mapping $x = x(u, v)$, $y = y(u, v)$ that is the inverse to the mapping (13.3). This means that there is a circle K': $|w - w_0| < \rho'$ such that for each $w \in K'$ Eq. (13.1) has only one solution

$$z = x(u, v) + iy(u, v) = h(w)$$

such that $z \in K$ and $z = h(w)$ is a continuous function.

We still have to prove that $h(w)$ is regular at point w_0. Let $w \in K'$ and $w + \Delta w \in K'$. Consider the quotient $\Delta z/\Delta w$, with $\Delta w \neq 0$ and $\Delta z = h(w + \Delta w) - h(w)$. Note that $\Delta w \neq 0$ implies $\Delta z \neq 0$, since the function $w = f(z)$ maps a small neighborhood of point z_0 onto a small neighborhood of point w_0 in a one-to-one manner.

Consider the identity

$$\frac{\Delta z}{\Delta w} = \frac{1}{\frac{\Delta w}{\Delta z}}. \tag{13.4}$$

Send Δw to 0. Then because of continuity of $h(w)$ we have $\Delta z \to 0$. On the right-hand side of Eq. (13.4) we go over to the limit as $\Delta z \to 0$.

Such a limit exists and does not depend on the way Δz tends to 0, since $w = f(z)$ is differentiable in a neighborhood of point z_0. Its value is $1/f'(z)$. Hence, the left-hand side of Eq. (13.4) also has a limit as $\Delta w \to 0$ and Eq. (13.2) is valid. The proof of the theorem is complete.

13.2 The function $\sqrt{z}$ Let us consider the function $w = z^2$. To find the inverse function, we must solve the equation

$$z^2 = w \tag{13.5}$$

for z. This equation, as shown in Sec. 1.5, has two solutions for a $w \neq 0$. If one of these solutions is denoted by $\sqrt{w}$, the other solution is $-\sqrt{w}$.

Thus, the function $z = h(w)$, which is the inverse of $w = z^2$, is double-valued. Note that the function $w = z^2$ is defined on the entire complex z plane and the range of its values is the entire complex w plane.

It is natural to ask about the existence and the way in which we can build a single-valued continuous function such that its value at each point of a domain D coincides with one of the values of the double-valued function that is the inverse of $w = z^2$.

As usual, we will denote the independent variable by z and the dependent variable by w. Suppose D_0 is the complex z plane with a cut along the real positive semiaxis. We write the variable z in the exponential form and consider in D_0 the function

$$w = f_1(z) = \sqrt{r}\, e^{i\varphi/2}, \quad 0 < \varphi < 2\pi. \tag{13.6}$$

The function $f_1(z)$ is single-valued and continuous in D_0 and satisfies the condition $f_1^2(z) = z$, i.e. it is solution of the equation

$$w^2 = z. \tag{13.7}$$

The range of values of the function $w = f_1(z)$ is the upper half-plane (Fig. 47). This follows from the definition of function (13.6) and also from the fact that under the mapping (13.7), which is the inverse of the mapping (13.6), the upper half-plane is mapped into the complex z plane with a cut along the real positive semiaxis (see Example 3 in Sec. 8).

Thus, the function $w = f_1(z)$ is single-valued and continuous in D_0, the plane with the cut along $[0, +\infty]$, and maps this domain onto the upper half-plane.

Similarly, the function

$$w = f_2(z) = -\sqrt{r}\, e^{i\varphi/2}, \quad 0 < \varphi < 2\pi,$$

is single-valued and continuous in D_0, satisfies the condition $f_2^2(z) = z$ and maps the domain D_0 on the lower half-plane (Fig. 47).

We will say that $f_1(z)$ and $f_2(z)$ constitute in D_0 two continuous branches of the double-valued function $\sqrt{z}$. These functions are differentiable and, hence, regular in D_0, by the inverse function theorem. According to (13.2),

$$f'_k(z)=\frac{1}{2f_k(z)}, \quad k=1, 2. \tag{13.8}$$

The differentiability of the functions $f_1(z)$ and $f_2(z)$ and the validity of (13.8) can be established directly by applying the Cauchy-Riemann equations in polar coordinates (see Example 4 in Sec. 7). The

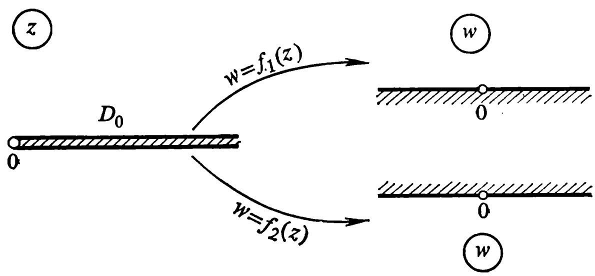


Fig. 47

functions $f_1(z)$ and $f_2(z)$ are said to be the regular branches of the double-valued function $\sqrt{z}$ in D_0. Often both branches are denoted by the same symbol, $\sqrt{z}$. To establish which of the two branches of the double-valued function $\sqrt{z}$ is considered, we must (a) fix the value of the function at an inner point of D_0 or (b) specify the value of the function at a boundary point (at the cut); in the latter case we must specify on which of the two banks of the cut, the upper or the lower, the point is taken.

For instance, if we are studying the regular branch of the function $\sqrt{z}$ on which $w_0 = i$ at point $z_0 = -1$, then we are speaking of the function $w = f_1(z)$, which maps D_0 onto the upper half-plane. But if we know that -1 is mapped into $-i$, then we are speaking of the function $w = f_2(z)$, which maps D_0 onto the lower half-plane.

Similarly, the regular branch of the function $\sqrt{z}$ on which $w_0 = 1$ at the point $z_0 = 1 + i0$ lying on the upper bank of the cut along $[0, +\infty)$ represents the function $w = f_1(z)$. But if $1 + i0$ mapped into -1, the branch represents $w = f_2(z)$.

Let us consider a domain $\tilde{D}$ that is the complex z plane with a cut along the negative real semiaxis (Fig. 48). Obviously, two regular branches of $\sqrt{z}$ can be isolated in $\tilde{D}$, namely,

$$w=\tilde{f}_1(z)=\sqrt{r}\, e^{i\varphi/2}, \quad w=\tilde{f}_2(z)=-\sqrt{r}\, e^{i\varphi/2} \quad (z=re^{i\varphi}, -\pi<\varphi<\pi).$$

The function $w = \tilde{f}_1(z)$ maps $\tilde{D}$ onto the right half-plane, Re $w > 0$, while the function $w = \tilde{f}_2(z)$ maps $\tilde{D}$ onto the left half-plane, Re $w > 0$; cf. Fig. 37 in Sec. 8.2.

Thus, if we take the complex z plane cut either along the real positive semiaxis or along the negative real semiaxis, in such a domain the double-valued function $\sqrt{z}$ splits into two regular branches. It is easy to see that $\sqrt{z}$ splits into two regular branches in any plane with a cut along the ray arg $z = \alpha$. We denote this domain by D_α, so that $\tilde{D} = D_\pi$. The investigation we have just undertaken shows that

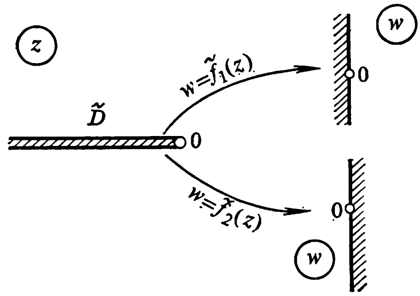


Fig. 48

(1) the double-valued function $\sqrt{z}$ splits into two regular branches in a domain D_α, i.e. the complex z plane with a cut along the ray arg $z = \alpha$ connecting the points $z = 0$ and $z = \infty$;

(2) at a point z_0 of D_α the values of the function are w_0 on one branch and $-w_0$ on the other;

(3) the range of values that the function admits on the two branches depends essentially on the domain D_α in which these branches split;

(4) a regular branch of $\sqrt{z}$ in D_α is fixed if the image of an inner point of D_α is specified or if the image of a boundary point is specified (in the latter case it must be stated on which of the two banks of the cut the point is taken).

13.3 The function ln *z* The concept of the logarithm of a complex number was introduced in Sec. 4.6. It is natural to think of the logarithmic function as the inverse of the exponential function. We must solve the equation

$$e^w = z \tag{13.9}$$

for w. Let $z = re^{i\varphi}$ and $w = u + iv$. Then Eq. (13.9) yields $u = \ln r$ and $v = \varphi + 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$. Hence,

$$w = \ln z = \ln |z| + i(\arg z + 2k\pi), \tag{13.10}$$

where arg z is a fixed value of the argument of z, and k is an integer. Thus, Eq. (13.9) has at $z \neq 0$ an infinite number of solutions, which are determined by (13.10), i.e. the logarithmic function assumes at each point z $(z \neq 0)$ an infinite number of values. The real part of this function ($\ln |z|$) is determined uniquely.

Let us now study the question of how to isolate in a domain D a single-valued continuous branch of the logarithmic function, i.e. a continuous function whose value at each point of D conincides

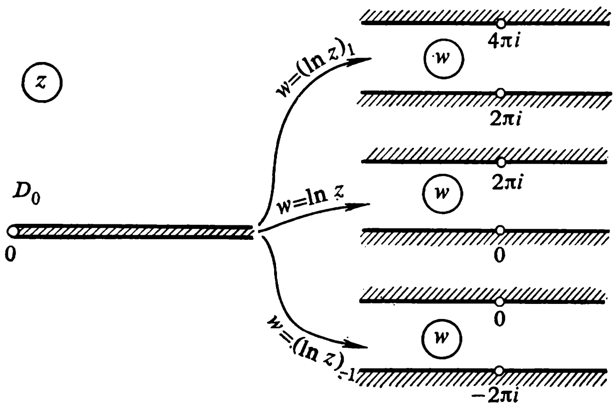


Fig. 49

with one of the values of the multiple-valued function ln z. Obviously, it is possible to isolate a single-valued continuous branch of ln z in D if we can isolate in this domain a single-valued continuous branch for arg z.

As in the case of the function $\sqrt{z}$, we will take the domain D_0, which is the complex z plane with a cut along $[0, +\infty)$, as D. In D_0 the function allows separation into single-valued continuous branches. Let $\varphi = \arg z$ (the notation is the same as for the multiple-valued function arg z) be the continuous branch of the argument in D_0 on which

$$0 < \varphi < 2\pi. \tag{13.11}$$

Retaining the former designation for the logarithmic function, we put

$$w = \ln z = \ln |z| + i \arg z, \tag{13.12}$$

where $\varphi = \arg z$ satisfies (13.11).

The function $w = \ln z$ satisfies Eq. (13.9). From (13.11) and (13.12) we can see that this function is single-valued and continuous in D_0. The function $w = \ln z$ maps D_0 onto the strip $0 < \operatorname{Im} w < 2\pi$ in a one-to-one manner (Fig. 49).

If we use the inverse function theorem or Example 4 in Sec. 7, we find that the function ln z defined via (13.11)-(13.12) is regular in D_0. This function is called the *regular branch in D_0 of the multiple-valued function* ln z, and its derivative is calculated by the formula

$$(\ln z)' = \frac{1}{z}.$$

Note that in D_0 there is an infinite number of single-valued continuous branches of the argument, and all of them have the form

$$(\arg z)_k = \arg z + 2k\pi \quad (k \text{ an integer}), \tag{13.13}$$

where $\varphi = \arg z$ is the branch we have just considered, i.e. the one that obeys (13.11). Putting $k = 1$ in (13.13), we obtain the branch of the argument $(\arg z)_1$ and the corresponding regular branch of $\ln z$:

$$w = (\ln z)_1 = \ln |z| + i \arg z + 2\pi i = \ln z + 2\pi i. \tag{13.14}$$

The function $(\ln z)_1$ maps D_0 onto the strip $2\pi < \operatorname{Im} w < 4\pi$ (see Fig. 49).

Similarly, if we take $k = -1$, we have the function

$$w = (\ln z)_{-1} = \ln z - 2\pi i, \tag{13.15}$$

which maps D_0 onto the strip $-2\pi < \operatorname{Im} z < 0$ in a one-to-one manner (see Fig. 49).

Both $(\ln z)_1$ and $(\ln z)_{-1}$ are regular branches of $\ln z$ in D_0.

To isolate a regular branch of $\ln z$ in D_0 it is sufficient to isolate the corresponding continuous branch of $\arg z$ (formula (13.13); the latter is uniquely determined by the value of $\arg z$ at an inner point of D_0 or on the boundary of D_0, i.e. on the upper or lower bank of the cut along $(0, +\infty)$. In particular, the regular branch of $\ln z$ that assumes real values on the upper bank is given by Eqs. (13.11)-(13.12).

In conclusion we note that regular branches of $\ln z$ can be isolated in other domains as well. In Chap. IV we will consider this aspect in greater detail.

14 The Uniqueness Theorem

14.1 The uniqueness theorem

Theorem 1 (the uniqueness theorem) *Let $f(z)$ be regular in D and let $f(z_n) = 0$, $n = 1, 2, \ldots$, where $\{z_n\}$ is a sequence of different points, $z_n \in D$, $n = 1, 2, \ldots$, such that $\lim\limits_{n\to\infty} z_n = a$, $a \in D$. Then $f(z) \equiv 0$ in D.*

Proof. Let us expand $f(z)$ in a Taylor series in powers of $z - a$:

$$f(z) = \sum_{k=0}^{\infty} c_k (z-a)^k, \tag{14.1}$$

and let us show that all the expansion coefficients are zero. We assume the contrary is true. Then, by Theorem 7 of Sec. 12, there is a neighborhood U of point z such that $f(z) \neq 0$ at $z \in U$, $z \neq a$. But this contradicts the hypothesis of the theorem. Hence, all the c_n

are zeros. The series (14.1) converges in the circle K: $|z - a| < \rho_0$, where ρ_0 is the distance between point a and the boundary of D. Thus, $f(z) \equiv 0$ in K.

Now let b be an arbitrary point in D. We wish to show that $f(b) = 0$. We connect points a and b by a broken line lying entirely in D. Let ρ be the distance between γ and the boundary Γ of D; since γ is a finite broken line lying in D, ρ must be positive. Next we construct the circles $K_0, K_1, \ldots, K_n$ centered, respectively, at points $z_0 = a, z_1, \ldots, z_n = b$ on γ, each circle of radius ρ; points z_j are selected in such a way that $|z_j - z_{j-1}| < \rho/2$ at $j = 1, 2, \ldots, n$. Then all the circles lie in D, and the center of each circle K_{j+1} lies inside the circle K_j, with $j = 0, 1, \ldots, n-1$ (Fig. 50).

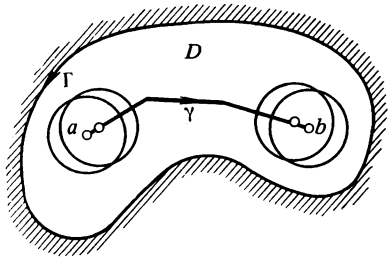


Fig. 50

Since $\rho_0 \geqslant \rho$, cirle K contains circle K_0; hence, $f(z) \equiv 0$ in K_0. We expand $f(z)$ in a Taylor series in powers of $z - z_1$. Circle K_1 lies in D, whence this series converges in K_1. Since the center z_1 of K_1 lies in the circle K_0, then $f(z) \equiv 0$ in a neighborhood of z_1 and we can show, by a similar method, that $f(z) \equiv 0$ in K_1. Continuing this reasoning in the same manner, we can show that the function $f(z)$ is identically zero in all circles K_j, so that $f(b) = 0$. The proof of the theorem is complete.

Corollary 1 *Let a function $f(z)$ be regular in a domain D, and $f(z) \equiv 0$ on a set E contained in D, which set has a limit point $a \in D$. Then $f(z) \equiv 0$ in D.*

Proof. By the definition of a limit point, there is a sequence of different points $\{z_n\}$, $n = 1, 2, \ldots$, such that $z_n \in E$ and $\lim\limits_{n\to\infty} z_n = a$. Since $f(z_n) = 0$ for all values of n and since the points z_n lie in D, we can say that $f(z) \equiv 0$ in D, by the uniqueness theorem.

Corollary 2 *Let two functions, $f(z)$ and $g(z)$, be regular in a domain D and coincide on a set E contained in D, which set has a limit point $a \in D$. Then $f(z) \equiv g(z)$ in D.*

Proof. The function $h(z) \equiv f(z) - g(z)$ is regular in D and $h(z) \equiv 0$ at $z \in E$, so that $h(z) \equiv 0$ in D, by Corollary 1. Whence $f(z) \equiv g(z)$, $z \in D$.

14.2 Some additional remarks

Remark 1. Consider the function $f(z) = \sin(1/z)$. Then $f(z_n) = 0$ at $z_n = 1/\pi n$, $n = \pm 1, \pm 2, \ldots$, and $\lim\limits_{n\to\infty} z_n = 0$, but nevertheless $f(z) \not\equiv 0$. This example does not contradict the uniqueness theorem, since the limit point a of the sequence $\{z_n\}$ is not a point at which the function $\sin(1/z)$ is regular.

Remark 2. The uniqueness theorem and Corollaries 1 and 2 are also valid when D is the extended complex plane.

More often we will use the weaker variant of the uniqueness theorem:

Corollary 3 *Let a function $f(z)$ be regular in a domain D and $f(z) \equiv 0$ on a curve γ lying in D or in a circle $K \subset D$. Then $f(z) \equiv 0$ in D.*

The uniqueness theorem is one of the most important properties of regular functions and only shows how strongly the properties of differentiable functions of a complex variable differ from those of differentiable functions of a real variable. For instance, suppose that on a segment I of the real axis a function $f(x)$ of the real variable x is continuously differentiable or twice continuously differentiable or n times continuously differentiable or continuously differentiable an infinite number of times. Let $I_1 \subset I$ be a smaller segment and $g(x)$ a function defined on I_1 and possessing the same properties, i.e. differentiable the same number of times, as $f(x)$. Then there is an infinite number of functions $f(x)$ with the above-noted properties at $x \in I$ and coinciding with $g(x)$ at $x \in I_1$.

15 Analytic Continuation

15.1 The definition and the main properties

Definition 1. Suppose the following conditions are met:

(1) a function $f(z)$ is defined on a set E;

(2) a function $F(z)$ is regular in a domain D that contains E;

(3) $F(z) \equiv f(z)$ at $z \in E$.

Then the function $F(z)$ is called the *analytic continuation of $f(z)$ (from set E into domain D).*

An important property of the analytic continuation is its uniqueness.

Theorem 1 (the analytic continuation principle) *Suppose set E has a limit point a belonging to domain D. Then the analytic continuation from set E into domain D is unique.*

Proof. Suppose the function $f(z)$ defined on E has two analytic continuations, $F_1(z)$ and $F_2(z)$, into D. Since $F_1(z) \equiv F_2(z)$ at $z \in E$, we have, by the uniqueness theorem of Sec. 14, $F_1(z) \equiv F_2(z)$ in D.

In particular, if E is a curve lying in D or a subdomain of D, then there is no more than one analytic continuation of $f(z)$ into D.

Example 1. Find the analytic continuation of the function

$$f(z) = \sum_{n=0}^{\infty} z^n.$$

This series is convergent and is regular in the circle K: $|z| < 1$. We have $f(z) = 1/(1 - z)$, $|z| < 1$.

The function $F(z) = 1/(1 - z)$ is regular in the extended complex z plane with point $z = 1$ deleted (we call this domain the D domain) and $F(z) \equiv f(z)$ at $|z| < 1$. Hence, the function $F(z)$ is the (unique) analytic continuation of $f(z)$ from K into D. □

15.2 The analytic continuation of the exponential, trigonometric, and hyperbolic functions The functions e^z, $\sin z$, and $\cos z$ were introduced in Sec. 4. We wish to show that these functions can be defined as the analytic continuations of e^x, $\sin x$, and $\cos x$, respectively. Let us assume by, definition, that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The series on the right-hand side converges for all z's, whence the sum of the series is regular for all z's (Sec. 12). The function e^z for real values of z, i.e. $z = x$, coincides with the function e^x known from mathematical analysis. Hence, the function e^z is the analytic continuation of e^x from the real axis into the entire complex plane.

We introduce the following

Definition 2. A function that is regular in the entire complex plane is called an *entire function.*

For example, polynomials and e^z are entire functions.

Theorem 2 *If $f(z)$ and $g(z)$ are entire functions, then $f(z) \pm g(z)$, $f(z)\,g(z)$, and $f(g(z))$ are entire functions, too.*

The *proof* of this theorem follows from Definition 2 and the properties of regular functions (Sec. 12).

Next we introduce the functions $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ through the sums of the following power series:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}, \qquad \cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!},$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \qquad \cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.$$

Since all these series converge at all z's, the functions $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ are entire functions. Moreover, these functions are the analytic continuations of the functions $\sin x$, $\cos x$, $\sinh x$, and $\cosh x$ from the real axis into the complex z plane.

Now we can introduce the functions $\tan z$, $\cot z$, $\tanh z$, and $\coth z$. By definition,

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z},$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z}.$$

The function $\tan z$ is regular for all $z \neq \infty$ for which $\cos z \neq 0$, i.e. at $z \neq \pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$. The function $\cot z$ is regular at $z \neq k\pi$, the function $\tan z$ at $z \neq i(\pi/2 + k\pi)$, and the function $\coth z$ at $z \neq ik\pi$, with $k = 0, \pm 1, \pm 2, \ldots$.

From courses of elementary trigonometry and mathematical analysis we know of certain addition theorems and a number of identities involving trigonometric and hyperbolic functions of a real independent variable. It can easily be shown that these formulas remain valid for a complex valued independent variable as well.

Example 2. Let us show that

$$\sin^2 z + \cos^2 z = 1$$

for all values of z.

By Theorem 2, the function $f(z) = \sin^2 z + \cos^2 z - 1$ is an entire function, since $\sin z$ and $\cos z$ are entire functions. Employing the fact that $f(x) \equiv 0$ for real x, we obtain, by the uniqueness theorem, that $f(z) \equiv 0$ for all values of z. □

Example 3. Let us show that

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \tag{15.1}$$

for all complex z_1 and z_2.

For real z_1 and z_2 this formula is valid. Let z_2 be real and fixed. Then the left- and right-hand sides of Eq. (15.1) are entire functions of the variable z_1. For real z_1 these entire functions coincide; hence, by the uniqueness theorem, these functions coincide for complex valued z_1. Thus, Eq. (15.1) is valid for all complex z_1 and for all real z_2.

Now let us assume that z_1 is complex valued and fixed. Then the left- and right-hand sides of Eq. (15.1) are entire functions of z_2. Since they coincide for all real z_2, they coincide for all complex valued z_2. We have therefore proved that Eq. (15.1) is valid for all complex valued z_1 and z_2. □

16 Integrals Depending on a Parameter

16.1 The regularity of integrals depending on a parameter Let us consider the integral

$$F(z) = \int_\gamma f(\zeta, z)\, d\zeta \tag{16.1}$$

Theorem 1. *Let the following conditions be met:*

(1) *γ is a finite piecewise smooth curve;*

(2) *the function $f(\zeta, z)$ is continuous in (ζ, z) at $\zeta \in \gamma$ and $z \in D$, where D is a domain in the complex z plane;*

(3) *for each fixed* $\zeta \in \gamma$ *the function* $f(\zeta, z)$ *is regular in* z *in* D. *Then the integral* (16.1) *is a regular function in* D.

Proof. By virtue of Conditions 1 and 2, the function $F(z)$ is continuous in D. Let us select an arbitrary point $a \in D$ and build a circle K that contains point a and lies inside D. Next we apply Morera's theorem. Let γ' be a closed curve lying in K. Then we can write

$$\int_{\gamma'} F(z)\,dz = \int_{\gamma'} \left(\int_{\gamma} f(\zeta, z)\,d\zeta \right) dz = \int_{\gamma} \left(\int_{\gamma'} f(\zeta, z)\,dz \right) d\zeta = 0, \qquad (16.2)$$

since we can interchange the order of integration and the integral along γ' vanishes (Cauchy's integral theorem). By Morera's theorem, the function $F(z)$ is regular in K; hence, $F(z)$ is regular in D. The proof of the theorem is complete.

Corollary 1 *Let* γ *be an unlimited piecewise smooth curve and let Conditions* 2 *and* 3 *and the following condition be met*:

(4) *the integral* (16.1) *is uniformly convergent in* $z \in D'$, *where* D' *is any closed subdomain of* D.

Then $F(z)$ *is regular in* D.

Corollary 2 *Let Conditions* 1 *and* 3 *be met while* $f(\zeta, z)$ *may have singularities at the ends of curve* γ. *If* $f(\zeta, z)$ *is continuous in* (ζ, z) *at* $z \in D$ *and* $\zeta \in \gamma$ *when* ζ *does not belong to the ends of* γ *and Condition* 4 *is met, then* $F(z)$ *is regular in* D.

The *proof* of Corollaries 1 and 2 is the same as that of Theorem 1; the integrals in (16.2) can be interchanged due to the uniform convergence of the integral in (16.1).

Theorem 2 *Suppose the conditions of Theorem 1 are satisfied. Then*

$$F'(z) = \int_{\gamma} \frac{\partial f(\zeta, z)}{\partial z}\,d\zeta, \quad z \in D. \qquad (16.3)$$

Proof. Let K be the circle $|z-a| \leqslant r$ lying in D and let γ' be its boundary. Then for $|z-a| < r$ we have

$$F'(z) = \frac{1}{2\pi i} \int_{\gamma'} \frac{F(t)}{(t-z)^2}\,dt = \frac{1}{2\pi i} \int_{\gamma'} \frac{1}{(t-z)^2} \left(\int_{\gamma} f(\zeta, t)\,d\zeta \right) dt$$

$$= \int_{\gamma} \left(\frac{1}{2\pi i} \int_{\gamma'} \frac{f(\zeta, t)}{(t-z)^2}\,dt \right) d\zeta = \int_{\gamma} \frac{\partial f(\zeta, z)}{\partial z}\,d\zeta.$$

The order of integration can be interchanged due to the continuity of the integrand and the finiteness of the curves γ and γ'.

Remark. Theorem 2 remains valid when the conditions of Corollary 1 or 2 are met, and the integral in (16.3) converges uniformly in $z \in D'$, where D' is any closed subdomain of D.

16.2 The analytic properties of integral transformations The most widely-used integral transformations in mathematical physics are the Laplace, Fourier, and Mellin transformations.

Let a function $f(t)$ be defined on the semiaxis $t \geqslant 0$. Its *Laplace transform* is defined thus:

$$F(z) = \int_0^\infty e^{-zt} f(t)\, dt. \tag{16.4}$$

Theorem 3 *Let the function $f(t)$ be continuous for $t \geqslant 0$. Suppose it satisfies the estimate*

$$|f(t)| \leqslant Ce^{\alpha t}, \quad t \geqslant 0. \tag{16.5}$$

Then its Laplace transform $F(z)$ is a function regular in the half-plane $\operatorname{Re} z > \alpha$.

Proof. We will use Corollary 1 of Theorem 1. Conditions 2 and 3 of Theorem 1 are met. Let $\delta > 0$ and $\operatorname{Re} z \geqslant \alpha + \delta$, $t \geqslant 0$. Then

$$|f(t)\, e^{-zt}| \leqslant Ce^{(\alpha - \operatorname{Re} z)t} \leqslant Ce^{-\delta t}.$$

Since the integral $\int_0^\infty Ce^{-\delta t} dt$ is convergent, we see, from Weierstrass's test, that the integral in (16.4) converges uniformly in z at $\operatorname{Re} z \geqslant \alpha + \delta$ and the function $F(z)$ is regular in the half-plane. Since $\delta > 0$ is arbitrary, the function $F(z)$ is regular at $\operatorname{Re} z > \alpha$.

The *Fourier transform* of a function $f(t)$ defined on the real axis is given by the following formula:

$$F(z) = \int_{-\infty}^{\infty} e^{izt} f(t)\, dt. \tag{16.6}$$

Theorem 4 *Suppose the function is continuous for $-\infty < t < \infty$ and satisfies the estimates*

$$\begin{aligned} |f(t)| &\leqslant C_1 e^{-\alpha t}, \quad t \geqslant 0, \\ |f(t)| &\leqslant C_2 e^{\beta t}, \quad t \leqslant 0, \end{aligned} \tag{16.7}$$

where $\alpha > 0$ and $\beta > 0$. Then its Fourier transform $F(z)$ is a function regular in the strip $-\alpha < \operatorname{Im} z < \beta$.

Proof. We split the integral in (16.6) into two integrals:

$$F(z) = \int_0^\infty e^{izt} f(t)\, dt + \int_{-\infty}^{0} e^{izt} f(t)\, dt \equiv F_1(z) + F_2(z).$$

In view of condition (16.7) and Theorem 3, the function $F_1(z)$ is regular in the half-plane $\operatorname{Re}(-iz) > -\alpha$ and the function $F_2(z)$

in the half-plane Re $(iz) > -\beta$. The proof of the theorem is complete.

In particular, if $f(t)$ is finite, i.e. $f(t) \equiv 0$ at $|t| > T$, and continuous at $|t| \leqslant T$, its Fourier transform is an entire function. This follows from Theorem 1, since in this case

$$F(z) = \int_{-T}^{T} e^{izt} f(t)\, dt.$$

The *Mellin transform* of a function $f(t)$ defined on the semiaxis $t \geqslant 0$ is defined by the following formula:

$$F(z) = \int_{0}^{\infty} t^{z-1} f(t)\, dt. \tag{16.8}$$

Here $t^z = e^{z \ln t}$, $t > 0$.

Theorem 5 *Let the function $f(t)$ be continuous at $t > 0$ and satisfy the estimates*

$$\begin{aligned} |f(t)| &\leqslant C_1 t^{\alpha}, \quad 0 < t \leqslant 1, \\ |f(t)| &\leqslant C_2 t^{\beta}, \quad 1 \leqslant t < \infty, \end{aligned} \tag{16.9}$$

where $\alpha > \beta$. Then its Mellin transform is a function that is regular in the strip $-\alpha <$ Re $z < -\beta$.

Proof. We split the integral in (16.8) into two integrals:

$$F(z) = \int_{0}^{1} t^{z-1} f(t)\, dt + \int_{1}^{\infty} t^{z-1} f(t)\, dt \equiv F_1(z) + F_2(z).$$

Suppose $0 < t \leqslant 1$ and Re $z \geqslant -\alpha + \delta$, $\delta > 0$. Then

$$|t^{z-1} f(t)| \leqslant C_1 t^{\delta-1}.$$

Since $\int_{0}^{1} t^{\delta-1}\, dt$ converges for $\delta > 0$, by Weierstrass's test the integral $F_1(z)$ is uniformly convergent in z for Re $z \geqslant -\alpha + \delta$. By Corollary 2, the function $F_1(z)$ is regular in the half-plane Re $z > -\alpha$.

Next, for $t \geqslant 1$ and Re $z \leqslant -\beta - \delta$, $\delta > 0$, we have

$$|t^{z-1} f(t)| \leqslant C_2 t^{-\delta-1}.$$

From the convergence of the integral $\int_{1}^{\infty} t^{-\delta-1} dt$ and Corollary 1 it follows that $F(z)$ is regular in the half-plane Re $z < -\beta$. The proof of the theorem is complete.

The Fourier and Mellin transforms are related through the following formula:

$$[M\ (f\ (t))]\ (z) = [F\ (f\ (e^t))]\ (-iz), \tag{16.10}$$

where $[M\ (\varphi\ (t))]\ (z)$ is the Mellin transform and $[F\ (\varphi\ (t))]\ (z)$ is the Fourier transform of the function $\varphi\ (t)$. Indeed, by making the substitution $t = e^\tau$ we obtain

$$[M\ (f\ (t))]\ (z) = \int_0^\infty t^{z-1} f\ (t)\ dt = \int_{-\infty}^\infty e^{z\tau} f\ (e^\tau)\ d\tau$$

(here we assume that all the integrals converge). The last integral coincides with the right-hand side of (16.10).

In particular, employing (16.10), we can derive Theorem 5 from Theorem 4.

16.3 The analytic continuation for the gamma function The *gamma function* $\Gamma\ (x)$ for real $x > 0$ is defined by the formula

$$\Gamma\ (x) = \int_0^\infty t^{x-1} e^{-t}\ dt \tag{16.11}$$

and is the Mellin transform of e^{-t}. (This function was introduced by Euler to interpolate the factorial function $n!$.) Let us consider the integral

$$\Gamma\ (z) = \int_0^\infty t^{z-1} e^{-t}\ dt. \tag{16.12}$$

We have $e^{-t} \leqslant 1$, $0 \leqslant t \leqslant 1$, and $e^{-t} \leqslant c_\beta t^{-\beta}$, $t \geqslant 1$, where β is real and positive, but otherwise arbitrary. By Theorem 5, the function $\Gamma\ (z)$ is regular in the strip $0 < \text{Re}\ z < \beta$ for any $\beta > 0$, so that $\Gamma\ (z)$ is regular in the half-plane $\text{Re}\ z > 0$. Thus, we have found the analytic continuation of $\Gamma\ (x)$ from the semiaxis $(0, +\infty)$ into the right half-plane.

For real $x > 0$ the gamma function satisfies the following functional relationship:

$$\Gamma\ (x + 1) = x\Gamma\ (x).$$

By the uniqueness theorem,

$$\Gamma\ (z + 1) = z\Gamma\ (z)$$

at $\text{Re}\ z > 0$, so that

$$\Gamma\ (z) = \frac{1}{z}\ \Gamma\ (z + 1), \quad \text{Re}\ z > 0. \tag{16.13}$$

The function $\Gamma(z+1)$ is regular in the half-plane $\operatorname{Re} z > -1$. Hence, the function on the right-hand side of (16.13) is regular in the domain $D_1 = \{\operatorname{Re} z > -1,\ z \neq 0\}$ and, hence, $\Gamma(z)$ is continued analytically into D_1. But now the right-hand side of (16.13) is regular in the domain $D_2 = \{\operatorname{Re} z > -2,\ z \neq 0,\ z \neq -1\}$, and we have continued analytically the gamma function into D_2. Continuing this reasoning, we arrive at the following result:

The gamma function allows analytic continuation into the entire complex plane with the points $z = 0, -1, -2, \ldots$ deleted.

This method of analytic continuation is based on the functional relationship (16.13) and therefore has a very restricted area of application. Here is another, more general, method. We split the integral in (16.12) into two:

$$\Gamma(z) = \int_0^1 t^{z-1} e^{-t}\, dt + \int_1^\infty t^{z-1} e^{-t}\, dt \equiv F_1(z) + F_2(z). \qquad (16.14)$$

By Theorem 5, the function $F_2(z)$ is an entire function and $F_1(z)$ is a regular function for $\operatorname{Re} z > 0$. Hence, we need only continue analytically the function $F_1(z)$.

To this end we expand e^{-t} in a Taylor series:

$$e^{-t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n.$$

This series converges uniformly at $0 \leqslant t \leqslant 1$. If we multiply it by t^{z-1}, with $\operatorname{Re} z \geqslant 1$, the new series will also be uniformly convergent at $0 \leqslant t \leqslant 1$. We have

$$F_1(z) = \int_0^1 t^{z-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n\, dt \sum_{n=0}^{\infty} \frac{(-1)^n}{z+n} \frac{1}{n!} \quad (\operatorname{Re} z \geqslant 1). \qquad (16.15)$$

The terms $\frac{(-1)^n}{z+n} \frac{1}{n!}$ of the last series are regular in the entire complex plane except at the points $z = 0, -1, -2, \ldots$. Let D_ρ be the complex z plane with the ρ-neighborhoods of the points $z = 0, -1, -2, \ldots$ deleted. Then $|1/(z+n)| \leqslant 1/\rho$ at $z \in D_\rho$, and the series on the right-hand side of (16.15) is uniformly convergent at $z \in D_\rho$, so that the sum of the series is regular in the complex z plane with the points $z = 0, -1, -2, \ldots$ deleted. Thus, the formula

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z+n} \frac{1}{n!} + \int_1^\infty t^{z-1} e^{-t}\, dt \qquad (16.16)$$

gives the analytic continuation of the gamma function onto the entire complex plane with the points $z = 0, -1, -2, \ldots$ deleted.

16.4 The analytic continuation for the Laplace transform We will now study another important method of analytic continuation of functions defined through integrals, the method of rotation of the integration contour.

Theorem 6 *Suppose that a function $f(\zeta)$ is regular and bounded in the angle $|\arg \zeta| \leqslant \alpha < \pi/2$. Then its Laplace transform $F(z)$ can be continued analytically onto the angle $|\arg z| < \pi/2 + \alpha$.*

Proof. We consider the integral

$$F_\beta(z) = \int_{l_\beta} e^{-z\zeta} f(\zeta)\, d\zeta, \tag{16.17}$$

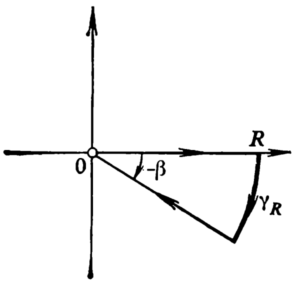


Fig. 51

where l_β is the ray $0 \leqslant |\zeta| < \infty$, $\arg \zeta = -\beta$, with $0 \leqslant \beta < \alpha$. We wish to show that

$$F_\beta(x) = F_0(x) = \int_0^\infty e^{-xt} f(t)\, dt \tag{16.18}$$

at $x > 0$. Consider the closed contour C_R consisting of the segments $[0, R]$ and $[Re^{-i\beta}, 0]$ and the arc $\gamma_R : |\zeta| = R$, $-\beta \leqslant \arg \zeta \leqslant 0$ (Fig. 51).

By Cauchy's integral theorem,

$$\int_{C_R} e^{-x\zeta} f(\zeta)\, d\zeta = 0.$$

We must show that the integral along γ_R tends to zero as $R \to \infty$. By hypothesis, $|f(\zeta)| \leqslant M$ at $0 \leqslant |\zeta| < \infty$ and $|\arg \zeta| \leqslant \alpha$. Moreover, $\zeta = Re^{i\varphi}$, $-\beta \leqslant \varphi \leqslant 0$, at $\zeta \in \gamma_R$. Hence

$$\left| \int_{\gamma_R} e^{-x\zeta} f(\zeta)\, d\zeta \right| \leqslant M \int_{\gamma_R} |e^{-x\zeta}|\, |d\zeta|$$

$$= MR \int_{-\beta}^{0} e^{-xR\cos\varphi}\, d\varphi < MR\beta e^{-xR\cos\beta} \to 0$$

$(R \to \infty,\ x > 0)$. Going over to the limit as $R \to \infty$ in the identity

$$\left(\int_0^R + \int_{\gamma_R} + \int_{Re^{-i\beta}}^{0} \right) e^{-x\zeta} f(\zeta)\, d\zeta = 0,$$

we arrive at (16.18). Moreover, since $\zeta = \rho e^{-i\beta}$, $0 \leqslant \rho < \infty$, on ray l_β, we have

$$F_\beta(z) = \int_0^\infty e^{-\rho(ze^{-i\beta})} f(\rho e^{-i\beta})\, e^{-i\beta}\, d\rho.$$

Employing the fact that $|f(\rho e^{-i\beta})| \leqslant M$ at $0 \leqslant \rho < \infty$, we obtain, via Theorem 3, that the function $F_\beta(z)$ is regular in the half-plane $\operatorname{Re}(ze^{-i\beta}) > 0$, while the function $F_0(z)$ is regular in the half-plane $\operatorname{Re} z > 0$. We put

$$\Phi(z) = \begin{cases} F_0(z), & \operatorname{Re} z > 0, \\ F_\beta(z), & \operatorname{Re}(ze^{-i\beta}) > 0. \end{cases} \tag{16.19}$$

Since $F_0(x) \equiv F_\beta(x)$ at $x > 0$, the function $\Phi(z)$ is regular in the union of the half-planes $\operatorname{Re} z > 0$ and $\operatorname{Re}(ze^{-i\beta}) > 0$, i.e. in the angle $-\pi/2 < \arg z < \pi/2 + \beta$. But β is any number such that $0 \leqslant \beta < \alpha$. Hence, we have continued analytically the function $F(z)$ into the angle $-\pi/2 < \arg z < \pi/2 + \alpha$. Similarly, by selecting β in such a way that $-\alpha \leqslant \beta < 0$, we find that $F(z)$ can be continued analytically into the angle $-\pi/2 - \alpha < \arg z < \pi/2$. The proof of the theorem is complete.

Example 1. Consider the function

$$F(x) = \int_0^\infty \frac{e^{-xt}}{1+t}\, dt \quad (x > 0).$$

The function $f(\zeta) = 1/(1+\zeta)$ is regular and bounded in any angle $|\arg \zeta| \leqslant \alpha < \pi/2$. Hence $F(x)$ allows analytic continuation into the angle $|\arg \zeta| < \pi/2 + \alpha$, i.e. into the complex plane with a cut along the semiaxis $(-\infty, 0]$. □

16.5 The Cauchy integral An integral given by the formula

$$F(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{16.20}$$

is known as the *Cauchy integral*. We will study its analytic properties under the assumption that $f(\zeta)$ is continuous on curve γ.

(1) Suppose γ is a finite curve. Then the complement of γ consists of a finite or infinite number of domains. In each of these domains the Cauchy integral, by Theorem 1, is a regular function. But generally these regular functions are different, i.e. they are not analytic continuations of each other. For instance,

$$\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{1}{\zeta - z}\, d\zeta = \begin{cases} 1, & |z| < 1, \\ 0, & |z| > 1. \end{cases}$$

First let us show that the function represented by a Cauchy integral is regular at the point at infinity. Making the substitution $z = w^{-1}$ and assuming that $F(z) = G(w)$, we obtain

$$G(w) = \frac{w}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta w - 1}\, d\zeta.$$

Since γ is a finite curve, the denominator $\zeta w - 1$ is nonzero for small w's and, by Theorem 1, the function $G(w)$ is regular at point $w = 0$.

(2) Suppose γ is an infinite curve. For the sake of simplicity we will consider the case where γ is the real axis. Then

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt. \tag{16.21}$$

Let us assume that $f(t)$ satisfies the estimate

$$|f(t)| \leqslant C(1 + |t|)^{-\alpha}, \quad -\infty < t < \infty, \quad \alpha > 0. \tag{16.22}$$

Let us show that formula (16.21) defines two functions, $F_+(z)$ and $F_-(z)$, regular in the half-planes $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$, respectively. We will use Corollary 1. Let us consider the case $\operatorname{Im} z > 0$. We assume that $z = x + iy$ lies in the semistrip $\Pi : |x| \leqslant a$, $y \geqslant b$, with $a > 0$ and $b > 0$. For real t and for $z \in \Pi$ we have $|t - z|^2 = (t - x)^2 + y^2 \geqslant t^2 - 2|t|a \geqslant t^2/2$ if $|t| \geqslant 4a$. Hence

$$\left| \frac{f(t)}{t-z} \right| \leqslant \frac{C\sqrt{2}(1+|t|)^{-\alpha}}{|t|} \equiv g(t)$$

$$(z \in \Pi, \quad |t| \geqslant 4a).$$

Since the integral $\int\limits_{|t| \geqslant 4a} g(t)\, dt$ is convergent, by Weierstrass's test the integral in (16.21) is uniformly convergent in $z \in \Pi$. By Corollary 1 the function $F(z)$ is regular at $z \in \Pi$, and since we can select $a > 0$ as large as desired and $b > 0$ as small as desired, the integral in (16.21) represents the function $F_+(z)$, which is regular in the upper half-plane. We can prove similarly that the integral in (16.21) represents $F_-(x)$, which is regular in the lower half-plane.

Example 2. Let a function $f(z)$ be continuous on the semiaxis $t \geqslant 0$ and satisfy the estimate (16.22). Then the Cauchy integral

$$F(z) = \int_0^{\infty} \frac{f(t)}{t-z}\, dt$$

represents a function that is regular in the plane with a cut along the semiaxis $[0, +\infty)$. □

(3) If the function $f(\zeta)$ is regular on the path of integration γ, the Cauchy integral allows analytic continuation across the path of integration. The method used in this process consists of shifting the integration contour.

Example 3. Let

$$F(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=2}\frac{d\zeta}{(\zeta^2+1)(\zeta-z)}\quad(|z|<2).$$

The function $F(z)$ is regular in the circle $|z|<2$. We wish to show that the function can be continued analytically onto the entire complex z plane. We put $R>2$. Then

$$F_R(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=R}\frac{d\zeta}{(\zeta^2+1)(\zeta-z)}\quad(|z|<R).$$

This function is regular in the circle $|z|<R$. Let us demonstrate that

$$F_R(z)\equiv F(z)\quad(|z|<2).\tag{16.23}$$

This should prove our proposition. The integrand $f(\zeta)=(\zeta^2+1)^{-1}\times(\zeta-z)^{-1}$ is regular in the annulus $|z|<|\zeta|<\infty$ if $|z|>1$, since $1/(\zeta^2+1)$ is regular for all $\zeta\neq\pm i$.

Hence, by Cauchy's integral theorem, the integrals along the circles $|\zeta|=2$ and $|\zeta|=R$ of the function $f(\zeta)$ are equal at $|z|<2$, which proves the validity of (16.23). □

This example allows the following generalization. Take the Cauchy integral (16.20) with γ a simple closed curve. Then this integral defines a function regular in the region D_0, the interior of γ.

Suppose $f(\zeta)$ is regular inside a closed domain D bounded by two curves, γ' and γ, with γ' a simple closed curve and γ lying inside γ'. Then the formula

$$F_{\gamma'}(z)=\frac{1}{2\pi i}\int\limits_{\gamma'}\frac{f(\zeta)}{\zeta-z}\,d\zeta$$

gives the analytic continuation of the function $F(z)$ into the domain D', the interior of γ'. Indeed, the function $f(\zeta)/(\zeta-z)$ is regular in D if $z\in D_0$, so that by Cauchy's integral theorem

$$\frac{1}{2\pi i}\int\limits_{\gamma'}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{f(\zeta)}{\zeta-z}\,d\zeta\quad(z\in D_0).$$

The integral on the left-hand side represents a function that is regular in D', while the integral on the right-hand side is $F(z)$. Hence, $F_{\gamma'}(z)=F(z)$ $(z\in D_0)$, and our proposition is valid.

A similar method can be applied to integrals of the (16.21) type.

Theorem 7 *Let the function $f(\zeta)$ be regular in the strip $-a\leqslant \operatorname{Im}\zeta\leqslant 0$ and satisfy the condition*

$$|f(\zeta)|\leqslant C(1+|\zeta|)^{-\alpha},\ \alpha>0\quad(-a\leqslant\operatorname{Im}\zeta\leqslant 0).$$

Then the integral (16.21) *allows analytic continuation into the half-*

plane Im $z > -a$, *and the continuation* $F_a(z)$ *is given by the formula*

$$F_a(z) = \frac{1}{2\pi i} \int_{-ia-\infty}^{-ia+\infty} \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad (\operatorname{Im} z > -a). \tag{16.24}$$

Thus, we have discussed the following methods of analytic continuation for functions represented through integrals:

(i) integration by parts;
(ii) rotation of integration contour;
(iii) shift of integration contour.

Other examples of analytic continuation will be considered in Secs. 21 and 23.

Chapter III

The Laurent Series. Isolated Singular Points of Single-Valued Functions

17 The Laurent Series

17.1 The domain of convergence of the Laurent series A series of the form

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n, \tag{17.1}$$

where a is a fixed point in the complex z plane, and the c_n are given complex numbers, is called a *Laurent series.* The series is said to converge at point z if

$$\sum_{n=0}^{\infty} c_n (z-a)^n, \tag{17.2}$$

$$\sum_{n=-1}^{-\infty} c_n (z-a)^n = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n} \tag{17.3}$$

converge at this point, while the series (17.1) is the sum of the series (17.2) and (17.3).

The series (17.2) is a power series and, hence, its domain of convergence is the circle $|z-a| < R$ (at $R = 0$ the series (17.2) is convergent solely at point a, while at $R = \infty$ its domain of convergence is the entire complex plane). Putting $1/(z-a) = t$ in (17.3), we arrive at the power series $\sum_{n=1}^{\infty} c_{-n}t^n$, whose domain of convergence is the circle $|t| < \alpha$. Hence, the series (17.3) converges in the domain $|z-a| > \rho$, with $\rho = 1/a$. If

$$\rho < R, \tag{17.4}$$

then the series (17.1) converges in the domain

$$\rho < |z-a| < R, \tag{17.5}$$

which is an annulus centered at point z.

At each point lying outside the closed annulus (17.5) the Laurent series (17.1) is divergent, since one of the series, (17.2) or (17.3), is divergent. Thus, if condition (17.4) is met, the domain of convergence of the series (17.1) is the annulus (17.5). At the boundary of this annulus the series may be either divergent or convergent. If $\rho > R$, the series (17.2) and (17.3) have no common domain of convergence and, hence, the series (17.1) is convergent not at a single point.

Remark 1. From Abel's theorem (see Sec. 11) it follows that in each closed annulus $\rho < \rho_1 \leqslant |z-a| \leqslant R_1 < R$ lying in annulus (17.5) the series (17.1) is uniformly convergent and, by Weierstrass's theorem (Theorem 4 of Sec. 12), its sum $f(z)$ is regular in the annulus (17.5). The converse is also true (see Theorem 1 in Sec. 17.2).

17.2 Expanding a regular function in a Laurent series

Theorem 1 *A function $f(z)$ that is regular in an annulus $D: \rho < |z-a| < R$ can be expanded in this annulus in a Laurent series*:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n, \qquad (17.6)$$

where

$$c_n = \frac{1}{2\pi i} \int\limits_{|\zeta-a|=R_0} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta,$$

$$\rho < R_0 < R, \qquad (17.7)$$

$$n = 0, \pm 1, \pm 2, \ldots .$$

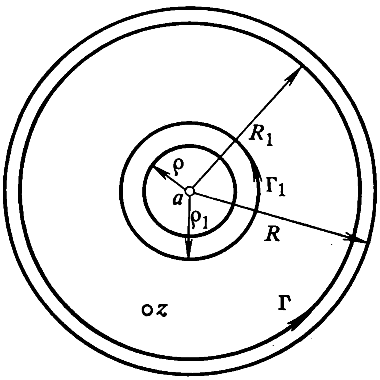


Fig. 52

Proof. Consider the annulus $D_1: \rho_1 < |z-a| < R_1$ (Fig. 52). We denote the outer and inner boundaries of D_1 by Γ and Γ_1, respectively. Let z be a point lying in D_1. By Eq. (10.4),

$$f(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta. \qquad (17.8)$$

Note that (see the proof of Theorem 1 of Sec. 12)

$$\frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{n=0}^{\infty} c_n (z-a)^n, \qquad (17.9)$$

$$c_n = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta. \qquad (17.10)$$

We transform the second term on the right-hand side of (17.8). We have

$$-\frac{1}{\zeta - z}\,\frac{1}{z-a-(\zeta-a)} = \frac{1}{(z-a)\left[1-\dfrac{\zeta-a}{z-a}\right]} = \sum_{k=0}^{\infty} \frac{(\zeta-a)^k}{(z-a)^{k+1}}. \qquad (17.11)$$

If $\zeta \in \Gamma_1$, then $\left|\frac{\zeta-a}{z-a}\right| = \frac{\rho_1}{|z-a|} = q < 1$ and, hence, by Weierstrass's test, the series (17.11) is uniformly convergent in ζ ($\zeta \in \Gamma_1$) for each $z \in D_1$. Since $f(\zeta)$ is continuous on Γ_1, it is bounded on Γ_1. This implies that the series

$$-\frac{f(\zeta)}{\zeta-z} = \sum_{k=0}^{\infty} \frac{f(\zeta)(\zeta-a)^k}{(z-a)^{k+1}} \tag{17.12}$$

is uniformly convergent in ζ on Γ_1. Integrating the series (17.12) term by term and putting $k+1 = -n$, we obtain

$$-\int_{\Gamma_1} \frac{f(\zeta)}{\zeta-z}\, d\zeta = \sum_{n=-1}^{-\infty} c_n (z-a)^n, \tag{17.13}$$

$$c_n = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta. \tag{17.14}$$

Substituting (17.13) and (17.9) into (17.8), we find an expansion (17.6) that is convergent at each point of the annulus D_1. The expansion coefficients are given by (17.10) and (17.14).

By Corollary 2 of Cauchy's integral theorem (see Eq. (9.13)), we can take the circle $|\zeta - a| = R_0$, $\rho_1 < R_0 < R_1$, as the integration contour in Eqs. (17.14) and (17.10), i.e. formula (17.7) holds. Since ρ_1 can be chosen as close to ρ as desired and R_1 as close to R as desired, the series (17.6) converges in the entire annulus D. The proof of the theorem is complete.

17.3 The uniqueness of expansion in a Laurent series

Theorem 2 *The expansion of a function $f(z)$ regular in an annulus $D: \rho < |z-a| < R$ in a Laurent series is unique.*

Proof. Suppose that $f(z)$, which is regular in the annulus D, has two expansions in D:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n = \sum_{n=-\infty}^{\infty} \tilde{c}_n (z-a)^n. \tag{17.15}$$

Multiplying both series by $(z-a)^{-m-1}$, where m is a fixed integer, we obtain

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^{n-m-1} = \sum_{n=-\infty}^{\infty} \tilde{c}_n (z-a)^{n-m-1}. \tag{17.16}$$

Since the two series in (17.16) converge on the circle $|z-a| = R_0$, $\rho < R_0 < R$, integrating them along this circle termwise and allowing for the fact that

$$\int_{|z-a|=R_0} (z-a)^k\, dz = \begin{cases} 0, & k \neq -1 \\ 2\pi i, & k = -1 \end{cases} \quad (k \text{ an integer}),$$

we find that $c_m = \tilde{c}_m$ for each integral m. The proof of the theorem is complete.

From Theorem 2 it follows that the expansion coefficients in the Laurent series of a given function do not depend on the way in which they are obtained.

Example 1. The function

$$f(z)=\frac{1}{(1-z)(z+2)}$$

is regular in the following domains: $D_1: |z|<1$, $D_2: 1<|z|<2$, and $D_3: |z|>2$. Let us find the Laurent expansions for $f(z)$ in these domains. We express $f(z)$ as a sum of two partial fractions:

$$f(z)=\frac{1}{3}\left(\frac{1}{1-z}+\frac{1}{z+2}\right). \tag{17.17}$$

If $|z|<1$, then

$$\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n, \tag{17.18}$$

while if $|z|>1$, then

$$\frac{1}{1-z}=-\frac{1}{z\left(1-\frac{1}{z}\right)}=-\sum_{n=1}^{\infty}\frac{1}{z^n}. \tag{17.19}$$

Similarly, in the circle $|z|<2$ we have

$$\frac{1}{z+2}=\frac{1}{2\left(1+\frac{z}{2}\right)}=\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{2^{n+1}}, \tag{17.20}$$

while if $|z|>2$, then

$$\frac{1}{z+2}=\frac{1}{z\left(1+\frac{2}{z}\right)}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{n-1}}{z^n}. \tag{17.21}$$

(a) By virtue of Eqs. (17.17), (17.18), and (17.20), the function $f(z)$ in the domain $D_1: |z|<1$ can be expanded in the following Laurent series:

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{3}\left[1+\frac{(-1)^n}{2^{n+1}}\right]z^n.$$

This is simply a Taylor series.

(b) In $D_2: 1<|z|<2$ the expansion of $f(z)$ in a Laurent series (see Eqs. (17.17), (17.19), and (17.20)) has the following form:

$$f(z)=\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)\frac{1}{z^n}+\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3\times 2^{n+1}}.$$

This series contains both positive and negative powers of z.

(c) In $D_3 : |z| > 2$ the function $f(z)$ can be expressed as a Laurent series with only negative powers of z (see Eqs. (17.17), (17.19), and (17.21)):

$$f(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{n-1}-1}{3z^n}. \quad \square$$

Remark 2. Note the link between a Fourier series and a Laurent series. Suppose function $f(z)$ is regular in the annulus

$$\delta_1 < |z| < 1+\delta_2, \quad (0 \leqslant \delta_1 < 1, \quad \delta_2 > 0), \qquad (17.22)$$

which contains the unit circle $|z| = 1$. Then we can express this function as a Laurent series:

$$f(z)=\sum_{n=-\infty}^{\infty} c_n z^n,$$

whence, putting $z = e^{i\varphi}$, we arrive at the Fourier expansion of the function

$$F(\varphi)=f(e^{i\varphi})=\sum_{n=-\infty}^{\infty} c_n e^{in\varphi}. \qquad (17.23)$$

Conversely, if we can represent a function $F(\varphi)$ in the form

$$F(\varphi) \equiv f(e^{i\varphi}),$$

where $f(z)$ is regular in the annulus (17.22), the series in (17.23) is the Fourier series for $F(\varphi)$.

17.4 Cauchy's inequalities for the coefficients of the Laurent series

Theorem 3 *Let a function $f(z)$ be regular in an annulus $D : \rho_0 < |z - a| < R_0$. Then the coefficients of the Laurent series*

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

for $f(z)$ in D satisfy the following inequalities:

$$|c_n| \leqslant \frac{M}{R^n}, \quad n=0, \pm 1, \pm 2, \ldots, \qquad (17.24)$$

where $M = \max_{z \in \gamma_R} |f(z)|$, with $\gamma_R : |z - a| = R$ *and* $\rho_0 < R < R_0$. *The inequalities* (17.24) *are called Cauchy's inequalities for the coefficients of the Laurent series.*

Proof. Employing (17.7), we find that

$$|c_n| \leqslant \frac{1}{2\pi} \int\limits_{|\zeta - a| = R} \frac{f(\zeta)}{|\zeta - a|^{n+1}} |d\zeta| \leqslant \frac{M}{2\pi R^{n+1}} \int\limits_{|\zeta - a| = R} |d\zeta| = \frac{M}{R^n}.$$

18 Isolated Singular Points of Single-Valued Functions

18.1 Classification of isolated singular points of single-valued functions

Definition 1. Suppose we have a single-valued function $f(z)$ that is regular in an annulus $0 < |z - a| < \rho$ but is not regular at point a ($a \neq \infty$). Then point a is said to be an *isolated singular point* of $f(z)$, or simply an *isolated singularity.*

The annulus $0 < |z - a| < \rho$, i.e. the circle $|z - a| < \rho$ with its center deleted, will sometimes be called a *punctured neighborhood of point a.*

Similarly, the point at infinity is said to be an *isolated singular point* (*isolated singularity*) of the function $f(z)$ if the latter is regular in the domain $\rho < |z| < \infty$.

Depending on the way in which $f(z)$ behaves near point a three types of isolated singular points are distinguished.

Definition 2. An isolated singular point of a single-valued function $f(z)$ is

(a) a *removable singular point* if $\lim\limits_{z \to a} f(z)$ exists and is finite;

(b) a *pole* if $\lim\limits_{z \to a} f(z) = \infty$;

(c) an *essential singularity* if $\lim\limits_{z \to a} f(z)$ does not exist.

Example 1. For the function $f(z) = \sin z/z$ point $z = 0$ is a removable singular point since $f(z)$ is regular at $z \neq 0$ and

$$\lim_{z \to 0} \frac{\sin z}{z} = \lim_{z \to 0} \frac{z - \frac{z^3}{3!} + \dots}{z} = 1. \ \square$$

Example 2. For the function $f(z) = z/(z+1)$ point $z = -1$ is a pole because $f(z)$ is regular at $z \neq -1$ and $\lim\limits_{z \to -1} f(z) = \infty$. $\square$

Example 3. The point $z = \infty$ is an essential singularity for e^z, $\sin z$, and $\cos z$ since all these functions are regular in the entire complex plane and do not have a limit as $z \to \infty$. Indeed, $\lim\limits_{z \to +\infty} e^x = \infty$ and $\lim\limits_{z \to -\infty} e^x = 0$, while both $\sin x$ and $\cos x$ have no limit as $x \to \infty$. $\square$

Example 4. For the function $f(z) = e^{1/z^2}$ the point $z = 0$ is an essential singularity because $f(z)$ is regular at $z \neq 0$ and has no limit as $z \to 0$. Indeed, if $z = x$, then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} e^{1/x^2} = \infty,$$

while if $z = iy$, then

$$\lim_{z\to 0} f(z) = \lim_{y\to 0} e^{-1/y^2} = 0. \ \square$$

18.2 The Laurent series in a neighborhood of a singular point

Suppose we have a function $f(z)$ that is regular in an annulus $K: 0 < |z - a| < \rho$. Then we can expand this function in the following Laurent series:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n + \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}, \tag{18.1}$$

which converges in K.

Definition 3. The series (18.1) is called the *Laurent series of function $f(z)$ in a neighborhood of point a*, while the two series

$$f_1(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-a)^n}, \tag{18.2}$$

$$f_2(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \tag{18.3}$$

are called, respectively: the *principal part* and the *regular part* of the series (18.1).

Suppose a function $f(z)$ is represented in a neighborhood of the point at infinity, i.e. in the domain $R < |z| < \infty$, by a convergent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n. \tag{18.4}$$

Definition 4. The series (18.4) is called the *Laurent series of function $f(z)$ in a neighborhood of the point at infinity*, while the two series

$$f_1(z) = \sum_{n=1}^{\infty} c_n z^n, \tag{18.5}$$

$$f_2(z) = c_0 + \sum_{n=1}^{\infty} \tag{18.6}$$

are called, respectively, the *principal part* and the *regular part* of the series (18.4).

Remark 1. The principal part of a Laurent series in a neighborhood of a point a (both finite and at infinity) is, by definition, the sum of those and only those terms of the Laurent series that tend to infinity as $z \to a$.

The principal part of a Laurent series is a function that is regular in the entire complex plane except at point a, while the regular part, i.e. the difference between the function $f(z)$ and the principal part $f_1(z)$, is a function regular at point a.

Example 5. The Laurent series of the function $f(z) = z^2 e^{1/z}$ in a neighborhood of point $z = 0$ is

$$f(z) = z^2 \left(1 + \frac{1}{z} + \ldots + \frac{1}{n! z^n} + \ldots\right)$$

$$= z^2 + z + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)!\, z^n}, \qquad (18.7)$$

and, hence, the principal part of the series (18.7) in a neighborhood point $z = 0$ is

$$f_1(z) = \sum_{n=1}^{\infty} \frac{1}{(n+2)!\, z^n},$$

while the regular part is $f_2(z) = z^2 + z + 1/2$.

The series (18.7), which converges in a neighborhood of the point at infinity (this series is convergent in the entire finite plane with point $z = 0$ deleted) is the Laurent series of $f(z)$ in a neighborhood of point $z = \infty$. The principal part of the series (18.7) in a neighborhood of point $z = \infty$ is $z^2 + z$ and the regular part is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)!\, z^n}. \ \square$$

Example 6. Let us find the Laurent series of

$$f(z) = \cos \frac{z}{z+1}$$

in a neighborhood of the point $z = -1$. We have

$$f(z) = \cos\left(1 - \frac{1}{z+1}\right) = \cos 1 \times \cos \frac{1}{z+1} + \sin 1 \times \sin \frac{1}{z+1}.$$

Employing the well-known expansions for $\cos z$ and $\sin z$, we arrive at the Laurent series of $f(z)$ in a neighborhood of point $z = -1$:

$$f(z) = \cos \frac{z}{z+1} = \cos 1 + \frac{\sin 1}{z+1} - \frac{\cos 1}{2!\,(z+1)^2}$$

$$-\frac{\sin 1}{3!\,(z+1)^3}+\ldots+(-1)^n\frac{\cos 1}{(2n)!\,(z+1)^{2n}}$$
$$+(-1)^n\frac{\sin 1}{(2n+1)!\,(z+1)^{2n+1}}+\ldots\,.\ \square \qquad (18.8)$$

Example 7. To find the regular part $g_2(z)$ of the Laurent series of the function $g(z) = z^2 \cos\frac{z}{z+1}$ in a neighborhood of point $z = -1$, we expand z^2 in a Taylor series in powers of $z + 1$. This yields

$$z^2 = [(z + 1) - 1]^2 = (z + 1)^2 - 2(z + 1) + 1. \qquad (18.9)$$

Multiplying expansions (18.8) and (18.9), we find that

$$g_2(z)=\frac{\cos 1}{2}-2\sin 1+(\sin 1-2\cos 1)(z+1)+\cos 1\times(z+1)^2.\ \square$$

Example 8. To find the principal part $f_1(z)$ of the Laurent series of the function $f(z) = 1/(z^2 + 1)$ in a neighborhood of point $z = i$, we represent this function in the form $f(z) = \frac{1}{z-i}\, g(z)$, where

$$g(z)=\frac{1}{z+i}=\frac{1}{2i}+a(z-i)+\ldots\,.$$

Hence,

$$f_1(z)=\frac{1}{2i}\,\frac{1}{z-i}\,.\ \square$$

Example 9. The principal part of the Laurent series of

$$f(z)=\frac{z^6}{(z^2+1)(z^2-4)}$$

in a neighborhood of point $z = \infty$ is z^2 because $f(z) = z^2 + 3 + g(z)$, where $g(z)$ is a proper rational fraction (a function that is regular at the points at infinity). $\square$

18.3 A removable singular point

Theorem 1 *An isolated singular point a of a function $f(z)$ is removable if and only if the principal part of the Laurent series of this function is identically zero in a neighborhood of point a.*

Proof. Necessity. Suppose point a is a removable singular point of $f(z)$. By the definition of a removable singular point,

$$\lim_{z\to a} f(z) = A \neq \infty \qquad (18.10)$$

and, hence, $f(z)$ is regular and bounded in a punctured neighborhood of point a, i.e.

$$|f(z)| \leqslant M, \quad 0 < |z - a| < \rho. \qquad (18.11)$$

If $0 < \rho_1 < \rho$, then (18.11) and Cauchy's inequalities (Eq. (17.24)) imply

$$|c_n| \leqslant \frac{M}{\rho_1^n} \quad (n=0, \pm 1, \pm 2, \ldots). \tag{18.12}$$

Since in this formula we can select ρ_1 as small as desired and since the c_n do not depend on ρ_1, we can write $c_n = 0$ for $n = -1, -2, \ldots$, i.e. the principal part of the Laurent series of $f(z)$ in a neighborhood of point a is identically equal to zero.

Sifficiency. If the principal part of the Laurent series of $f(z)$ in a neighborhood of point a is identically equal to zero, then

$$f(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots, \tag{18.13}$$

with the series (18.13) converging in an annulus $0 < |z-a| < \rho$. But this power series converges in the entire circle $|z-a| < \rho$ and, hence, $\lim\limits_{z\to a} f(z) = c_0$, i.e. point a is a removable singular point of $f(z)$.

The above proof implies that condition (18.10) can be replaced by condition (18.11). We have therefore arrived at

Theorem 2 *An isolated singular point a of a function $f(z)$ is removable if and only if $f(z)$ is regular and bounded in a punctured neighborhood of point a.*

Remark 2. If we continue the function $f(z)$ to point a via the continuity of $f(z)$, i.e. $f(a) = \lim\limits_{z\to a} f(z) = c_0$, we arrive at a function that is regular at a:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \quad |z-a| < \rho.$$

This fact justifies the term "removable singular point". For this reason we will often consider removable singular points as regularity points.

Example 10. The function

$$f(z) = \frac{(e^z-1)^2}{1-\cos z}$$

is regular at point $z=0$ since it is regular in a punctured neighborhood of this point and, as $z \to 0$,

$$e^z - 1 \sim z, \quad (e^z-1)^2 \sim z^2, \quad 1-\cos z \sim \frac{z^2}{2},$$

which implies that $\lim\limits_{z\to 0} f(z) = 2$, or $f(0) = 2$. □

Example 11. The function $f(z) = \cot z - 1/z$ is regular at $z = 0$

since

$$\cot z = \frac{\cos z}{\sin z} = \frac{1 - \frac{z^2}{2!} + \dots}{z - \frac{z^3}{3!} + \dots} = \frac{1}{z} + g(z),$$

where $g(z)$ is regular at point $z = 0$. □

18.4 A pole

Theorem 3 *Point $a \neq \infty$ is a pole of a function $f(z)$ if and only if this function can be represented in the form*

$$f(z) = (z - a)^{-m} \psi(z), \quad \psi(a) \neq 0, \tag{18.14}$$

where $\psi(z)$ is regular at point a, and m is a positive integer (m is called the order of pole a of $f(z)$).

Similarly, the point $z = \infty$ is a pole of a function $f(z)$ if and only if this function can be represented in the form

$$f(z) = z^m h(z), \quad h(\infty) \neq 0, \tag{18.15}$$

where $h(z)$ is regular at point $z = \infty$, and m is a positive integer (called the order of pole $z = \infty$ of $f(z)$).

Proof. Suppose $a \neq \infty$ is a pole of $f(z)$. Then $f(z)$ is regular in a punctured neighborhood of point a and $\lim_{z \to a} f(z) = \infty$, which implies that

$$|f(z)| > 1 \tag{18.16}$$

in an annulus $K: 0 < |z - a| < \rho$ (we select ρ in such a way that $f(z)$ is regular in K).

Consider the function $g(z) = 1/f(z)$. It is regular in K since $f(z)$ is regular in K and does not vanish in this annulus by virtue of (18.16). Hence, a is an isolated singular point of $g(z)$. From (18.16) it follows that $|g(z)| < 1$ in K and, by Theorem 2, we find that a is a removable singular point of $g(z)$. Putting

$$g(a) = \lim_{z \to a} \frac{1}{f(z)} = 0,$$

we find that $g(z)$ is regular in the circle $|z - a| < \rho$ and point a is a zero of $g(z)$.

Let m be the order of this zero. By Theorem 6 of Sec. 12 (Eq. (12.36)), $g(z) = (z - a)^m h(z)$, where $h(z)$ is regular at point a, $h(a) \neq 0$. This yields

$$f(z) = \frac{1}{g(z)} = (z - a)^{-m} \psi(z),$$

where $\psi(z) = 1/h(z)$ is regular at a, $\psi(a) = 1/h(a) \neq 0$.

Conversely, Eq. (18.14) implies that $f(z)$ is regular in a punctured neighborhood of point a and $\lim\limits_{z\to a} f(z) = \infty$, i.e. $z = a$ is a pole of $f(z)$.

Now suppose that $z = \infty$ is a pole of $f(z)$. As in the case of a finite point, $z = \infty$ is a zero of the function $g(z) = 1/f(z)$. To conclude the proof we need only apply Theorem 6 of Sec. 12 (Eq. (12.39)).

Corollary 1 *An isolated singular point $a \neq \infty$ of a function $f(z)$ is an mth order pole of this function if and only if the following asymptotic formula is valid*:

$$f(z) \sim A(z-a)^{-m}, \quad A \neq 0 \quad (z \to a). \tag{18.17}$$

A similar formula is valid in the case where the point at infinity is an mth order pole of $f(z)$:

$$f(z) \sim Bz^m, \quad B \neq 0 \quad (z \to \infty). \tag{18.18}$$

The definitions of the order of a pole and the order of a zero (or Eqs. (18.17), (18.18), (12.40) and (12.41)) imply that an mth order pole can be thought of as a zero of negative order, $-m$.

Theorem 4 *An isolated singular point a of a function $f(z)$ is a pole of this function if and only if the principal part of the Laurent series of $f(z)$ in a neighborhood of point a consists only of a finite number of terms.*

Proof. Suppose that point $a \neq \infty$ is an mth order pole of $f(z)$. Then Eq. (18.14) is valid. Expanding $\psi(x)$ in a Taylor series in a neighborhood of point a, we obtain for $f(z)$ a Laurent series in the same neighborhood:

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \ldots + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \ldots . \tag{18.19}$$

Its principal part $f_1(z) = \sum\limits_{k=1}^{m} \frac{c_{-k}}{(z-a)^k}$ contains only a finite number of terms (the number of terms no higher than m) with $c_{-m} \neq 0$, where m is the order of pole a.

Conversely, from (18.19) it follows that Eq. (18.14) is valid; hence, a is an mth order pole of $f(z)$. We have thus proved the theorem for the case of a finite pole. If point $z = \infty$ is a pole of $f(z)$, we need only employ Eq. (18.15).

Example 12. For the function $f(z) = 1/\sin(1/z)$ the points $z_k = 1/k\pi$ ($k = \pm 1, \pm 2, \ldots$) are first order poles since the function $g(z) = 1/f(z) = \sin(1/z)$ is regular at $z \neq 0$ and points z_k are first order zeros of $g(z)$ ($g'(z_k) \neq 0$). Hence, point $z = 0$ is a nonisolated singular point (a limit or accumulation point for the poles). On the other hand, point $z = \infty$ is a first order pole of $f(z)$ since $f(z) \sim z (z \to \infty)$. □

Example 13. For the function $f(z) = (1 - \cos z)/(e^z - 1)^3$ the point $z = 0$ is a first order pole since both $\varphi(z) = 1 - \cos z$ and $\psi(z) = (e^z - 1)^3$ are regular in a neighborhood of $z = 0$ and, as $z \to 0$, we have $1 - \cos z \sim z^2/2$ and $(e^z - 1)^3 \sim z^3$, i.e. $f(z) \sim 1/2z$. The points $z_k = 2k\pi i$ ($k = \pm 1, \pm 2, \ldots$) are third order poles of $f(z)$ since they are third order zeros of the function $\psi(z)$, while $\varphi(z_k) \neq 0$. The point $z = \infty$ is a nonisolated singular point (a limit or accumulation point for the poles) of $f(z)$. □

18.5 An essential singularity

Theorem 5 *An isolated singular point a of a function $f(z)$ is an essential singularity of this function if and only if the principal part of the Laurent series of $f(z)$ contains an infinite number of terms in a neighborhood of point a.*

The proof follows from Theorems 1 and 4.

Example 14. Point $z = 0$ is an essential singularity of the function

$$f(z) = e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}$$

since the principal part of the Laurent series of $e^{1/z}$ contains an infinite number of terms. □

Example 15. Point $z = \infty$ is an essential singularity of the function

$$f(z) = \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

since the principal part of the Laurent series of $\cos z$ contains an infinite number of terms in a neighborhood of point $z = \infty$. □

The behavior of a function in a neighborhood of an essential singularity is governed by

Theorem 6 (Sochozki's theorem) *Let point a be an essential singularity of a function $f(z)$. Then each complex number A has corresponding to it a sequence of points $\{z_n\}$ that converges to point a and* $\lim_{n \to \infty} f(z_n) = A$.

Proof. (1) $A = \infty$. Note that there is not a single neighborhood of point a in which $f(z)$ is bounded, since otherwise point a, by Theorem 2, would be a removable singular point. This means that for each positive integer n there is a point z_n in the annulus $K_n: 0 < |z - a| < 1/n$ such that $|f(z_n)| > n$, i.e. $z_n \to a$ and $f(z_n) \to \infty$ as $n \to \infty$.

(2) $A \neq \infty$. Note that if for each positive ε and each positive δ there is a point z_δ $(0 < |z_\delta - a| < \delta)$ such that $|f(z_\rho) - A| < \varepsilon$, the theorem is valid (it is sufficient to take $\varepsilon = 1/n$ and $\delta = 1/n$, $n = 1, 2, \ldots$).

Suppose the above statement is not true. Then there is a positive ε_0 and a positive δ_0 such that for all z: $0 < |z - a| < \delta_0$ we have

$$|f(z) - A| \geqslant \varepsilon_0. \tag{18.20}$$

Take the function

$$g(z) = \frac{1}{f(z) - A}. \tag{18.21}$$

From (18.20) and (18.21) we have

$$|g(z)| \leqslant \frac{1}{\varepsilon_0}, \quad 0 < |z - a| < \delta_0. \tag{18.22}$$

Since a is an isolated singular point of $f(z)$, it is an isolated singular point for $g(z)$, too ($g(z) \neq 0$ in the annulus $0 < |z - a| < \delta_0$ by virtue of (18.20)).

By Theorem 2, point a is a removable singular point of $g(z)$ and, hence,

$$\lim_{z \to a} g(z) = B. \tag{18.23}$$

From (18.21) we have

$$f(z) = A + \frac{1}{g(z)}, \quad 0 < |z - a| < \delta_0, \tag{18.24}$$

while from (18.23) and (18.24) it follows that $\lim\limits_{z \to a} f(z)$ does indeed exist (it is finite if $B \neq 0$ and infinite if $B = 0$), i.e. point a is either a removable singular point of $f(z)$ or a pole, which contradicts the hypothesis of the theorem. The proof of Sochozki's theorem is complete.

We will now formulate a deeper theorem that characterizes the behavior of a function in a neighborhood of an essential singularity.

Theorem 7 (Picard's second theorem) *In any neighborhood of an essential singularity the function with such a singularity admits any value (and an infinite number of times) with the exception of, perhaps, one value.*

We will illustrate Picard's second theorem with two examples.

Example 16. Point $z = \infty$ is an essential singularity of the function $f(z) = e^z$ (see Example 3). Consider the equation

$$e^z = A \quad (A \neq 0). \tag{18.25}$$

This equation has the following solutions:

$$z_k = \ln |A| + i(\arg A + 2k\pi), \tag{18.26}$$

where $\arg A$ is a fixed value of the argument of A, and $k = 0, \pm 1, \pm 2, \ldots$. From (18.25) and (18.26) it follows that in any neighborhood of point $z = \infty$ there is an infinite number of points z_k at which the function e^z admits a value equal to A ($A \neq 0$). This function does not admit the value $A = 0$ (such a value is said to be *exceptional* for e^z). □

Example 17. Point $z = \infty$ is an essential singularity of $f(z) = \sin z$, and for each A there is an infinite number of solutions for the equation $\sin z = A$:

$$z_k = \frac{1}{i} \ln (iA + \sqrt{1 - A^2}) + 2k\pi \quad (k \text{ an integer}).$$

Hence, $\sin z$ has no exceptional values. □

In conclusion of this section we will discuss a number of examples connected with the problem of determining the type of an isolated singular point.

Example 18. Suppose two functions, $f(z)$ and $g(z)$, are regular at a point a, with $g(z) \not\equiv 0$. Then for the function $F(z) = f(z)/g(z)$ point $z = a$ is either a pole or a point of regularity. Indeed, if $g(a) \neq 0$, then $F(z)$ is regular at $z = a$. If point $z = a$ is an mth order zero of $g(z)$ and $f(a) \neq 0$, then a is an mth order pole of $F(z)$. Finally, if point a is an nth order zero of $f(z)$ and an mth order zero of $g(z)$, then at $n \geqslant m$ the function $F(z)$ is regular at point a, while at $n < m$ point a is an $(m - n)$th order pole of $F(z)$.

For instance, the function $\tan z$ is regular in the entire complex plane except at points $z_k = \pi/2 + k\pi$ (with k an integer), which are first order poles. Similarly, the function $\cot z$ has first order poles at the points $\tilde{z}_k = k\pi$ (with k an integer) and no other finite singular points. □

Example 19. For a rational function $R(z) = P_n(z)/Q_m(z)$, where $P_n(z)$ and $Q_m(z)$ are polynomials of, respectively, the nth and mth degrees without common zeros, the zeros of the denominator $Q_m(z)$ are the only points that are the poles of $R(z)$, i.e. $R(z)$ has no other singular points in the finite complex plane. The point at infinity, $z = \infty$, is a singular point, namely, an $(n - m)$th order pole if $n > m$ and a point of regularity if $n \leqslant m$. □

Example 20. Suppose point $z = a$ is an essential singularity of a function $f(z)$. Then the same point is either an essential singularity or a nonisolated singular point (a limit or accumulation point of poles) of the function $g(z) = 1/f(z)$. Indeed, if there is an annulus $0 < | z - a | < \delta$ in which $f(z) \neq 0$, then point a is an isolated singular point, precisely, an essential singularity, of $g(z)$. (An example: $f(z) = e^{1/z}$, $g(z) = e^{-1/z}$, $z = 0$.) But if in any neighborhood of point a there are zeros of $f(z)$, then for $g(z)$ these points are poles and, hence, $z = a$ is a limit or accumulation point of poles of the function $g(z)$. (An example: $f(z) = \sin(1/z)$, $g(z) = 1/\sin(1/z)$, $z = 0$.) □

Example 21. For the function $f(z) = e^{1/\sin z}$ the points $z_k = k\pi$

$(k = 0, \pm 1, \pm 2, \ldots)$ are essential singularities. Indeed, $\sin z \sim (-1)^k (z - k\pi)$ as $z \to k\pi$. Suppose k is even. Then $\sin z \to +0$ and $f(z) \to +\infty$ as $z = x \to k\pi + 0$, but $\sin z \to -0$ and $f(z) \to 0$ as $z = x \to k\pi - 0$, i.e. $f(z)$ has no limit at point z_k. The case of odd k's can be considered in a similar manner. The function $f(z)$ has no other singular points in the finite plane. The point at infinity, $z = \infty$, is a limit or accumulation point of essential singularities of $f(z)$. □

Now we will generalize the result of Example 21.

Example 22. Let us show that if point $z = a$ is a pole of a function $f(z)$, then for the function $g(z) = e^{f(z)}$ this point is an essential singularity.

Suppose m is the order of the pole. Then, according to Eq. (18.17), we have $f(z) \sim A(z - a)^{-m}$, $A \neq 0$ $(z \to a)$. Putting $A = |A| e^{i\alpha}$ and $z - a = re^{i\alpha}$, we obtain

$$f(z) \sim |A| r^{-m} e^{i(\alpha - m\varphi)}). \tag{18.27}$$

Let us consider the ray $l_1 : z - a = re^{i\varphi_1}$, where $\varphi_1 = \alpha/m$. Then (18.27) implies that $f(z) \sim |A| r^{-m}$ $(r \to 0,\ z \in l_1)$, whence $\lim\limits_{z\to a,\ z\in l_1} g(z) = \infty$. Similarly, on the ray $z - a = re^{i\varphi_2}$, where $\varphi_2 = (\alpha + \pi)/m$, we have $f(z) \sim -|A| r^{-m}$ and, hence, $\lim\limits_{z\to a,\ z\in l_2} g(z) = 0$. All this means that the function $g(z)$ has no limit as $z \to a$, i.e. point a is an essential singularity of $g(z)$. □

Example 23. For the function $f(z) = \dfrac{z^2}{\sin^2 \dfrac{1}{z+1}}$ the points $z_k = -1 + \dfrac{1}{k\pi}$ $(k = \pm 1, \pm 2, \ldots)$ are second order poles, the point $z = -1$ is a limit or accumulation point of poles, and point $z = \infty$ is a fifth order pole since $\sin \dfrac{1}{z+1} \sim \dfrac{1}{z}$, $f(z) \sim z^5$ $(z \to \infty)$. The function $f(z)$ has no other singular points. □

19 Liouville's Theorem

The reader will recall that a function that is regular in the entire complex is called an *entire* function.

Let us expand an entire function $f(z)$ into a Taylor series:

$$f(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{19.1}$$

The series converges at all values of z and, hence, is the Laurent series of $f(z)$ in a neighborhood of the point at infinity.

The only singular point that an entire function $f(z)$ may have in the extended complex plane is $z = \infty$. If $z = \infty$ is an nth order pole of an entire function $f(z)$, then $f(z)$ is an nth degree polynomial. An entire function for which $z = \infty$ is an essential singularity is known as a *transcendental entire* function. (Examples: e^z, $\sin z$, and $\cos z$.)

If an entire function $f(z)$ is regular at point $z = \infty$, then $f(z) = c_0 = \text{const}$. Thus, the only analytic functions that have no singular points in the extended complex plane are constants.

Theorem 1 (Liouville's theorem) *Let an entire function*

$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$

in the domain $|z| > R_1$ *satisfy the inequality*

$$|f(z)| \leqslant M |z|^n \quad (\textit{with } n \textit{ a nonnegative integer}). \tag{19.2}$$

Then $f(z)$ *is a polynomial of a degree no higher than* n.

Proof. Employing Cauchy's inequalities (see Sec. 17.4), we obtain for $R > R_1$ the following estimate for the expansion coefficients in (19.1):

$$|c_k| \leqslant \frac{MR^n}{R^k} = MR^{n-k}, \quad k = 1, 2, \ldots . \tag{19.3}$$

If $k > n$, then from (19.3) it follows that $c_k = 0$, since we can take R as large as desired and the coefficients c_k do not depend on R. Thus, $c_{n+1} = c_{n+2} = \ldots = 0$, i.e. $f(z)$ is a polynomial of a degree no higher than n. The proof of the theorem is complete.

Corollary 1 *If an entire function* $f(z)$ *is bounded in the entire complex plane, it is a constant*:

$$f(z) \equiv \text{const}.$$

Let us use Liouville's theorem to prove

The fundamental theorem of algebra *Every polynomial* $P_n(z) = c_0 + c_1 z + \ldots + c_n z$ $(c_n \neq 0,\ n \geqslant 1)$ *has at least one zero.*

Proof. Suppose that $P_n(z)$ has not a single zero. Then the function $g(z) = 1/P_n(z)$ is an entire function. Since $g(z) \to 0$ and $z \to \infty$ $(P_n(z) \sim c_n z^n,\ z \to \infty)$, this function is bounded in the entire complex plane, and this means, by Corollary 1, that $g(z) \equiv \text{const}$, which contradicts the hypothesis. Thus, the polynomial $P_n(z)$ has at least one zero.

A more general class of functions compared to entire functions are meromorphic functions.

Definition. A function $f(z)$ is said to be *meromorphic* if in each bounded part of the complex plane it is regular except, perhaps, at a finite number of poles.

In the entire complex plane a meromorphic function may have an infinite number of poles. (Examples: $\cot z$, $1/\sin z$, and $1/(e^z - 1)$.)

A rational function is meromorphic and has only a finite number of poles in the extended complex plane. The converse is also true, i.e. we have

Theorem 2 *A meromorphic function $f(z)$ that in the extended complex plane has only a finite number of poles $a_1, a_2, \ldots, a_s$ (point $z = \infty$ may also be a pole) is rational and can be represented in the form*

$$f(z) = A + f_0(z) + \sum_{k=1}^{s} f_k(z). \tag{19.4}$$

where $f_0(z)$ and $f_k(z)$ are the principal parts of the Laurent series of $f(z)$ in neighborhoods of points $z = \infty$ and a_k, respectively, and

$$A = \lim_{z \to \infty} [f(z) - f_0(z)].$$

Proof. Let

$$f_k(z) = \sum_{j=1}^{m_k} \frac{A_{j,k}}{(z-a_k)^j} \quad \text{and} \quad f_0(z) = A_1 z + \ldots + A_m z^m$$

be the principal parts of the Laurent series of $f(z)$ at a_k and $z = \infty$, respectively. Then

$$g(z) = f(z) - f_0(z) - \sum_{k=1}^{s} f_k(z)$$

is regular in the entire extended plane and, hence, $g(z) \equiv A =$ const. Since $f_k(z) \to 0$ as $z \to \infty$ ($k = 1, 2, \ldots, s$), we conclude that $A = \lim\limits_{z \to \infty} [f(z) - f_0(z)]$.

Remark 1. Formula (19.4) is the partial-fraction expansion of a rational function, which is well-known from courses of mathematical analysis ($A + f_0(z)$ is the entire part of such an expansion). Theorem 2 provides a simple way in which this formula can be derived.

Remark 2. It can be shown (see Bitsadze [1]) that every meromorphic function can be represented in the form of a fraction of two entire functions.

Meromorphic functions obey

Theorem 4 (Picard's first theorem) *A meromorphic function that is not a constant admits any complex value except, perhaps, two values.*

The values that a meromorphic function does not admit are known as *Picard's exceptional values.* For instance, $\tan z$ has two exceptional values, i and $-i$, i.e. $\tan z$ can never be equal to i or $-i$.

Chapter IV

Multiple-Valued Analytic Functions

20 The Concept of an Analytic Function

20.1 Analytic continuation along a sequence of domains The concept of analytic continuation plays an extremely important role in the theory of functions of a complex variable. A generalization

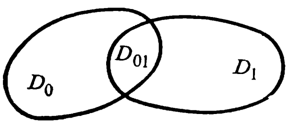


Fig. 53

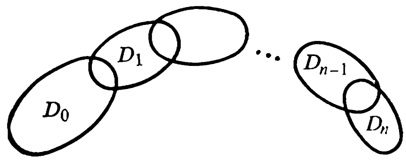


Fig. 54

of this concept leads to the generalization of the concept of a regular function, namely, to the concept of a multiple-valued analytic function.

Suppose we have two domains, D_0 and D_1, whose intersection D_{01} is not empty and is a domain (Fig. 53). Suppose we have functions $f_0(z)$ and $f_1(z)$ that are regular in D_0 and D_1, respectively, and coincide in D_{01}, i.e.

$$f_1(z) \equiv f_0(z), \quad z \in D_{01}.$$

Then the function $f_1(z)$ is said to be a *direct analytic continuation of function $f_0(z)$ from domain D_0 to domain D_1*. According to the uniqueness theorem this continuation is unique.

Now suppose we have a sequence of domains $D_0, D_1, \ldots, D_n$ such that all the intersections $D_j \cap D_{j+1}$, $0 \leqslant j \leqslant n-1$, are not empty and are domains (Fig. 54). Suppose we have functions $f_0(z)$, $f_1(z), \ldots, f_n(z)$ such that each subsequent function $f_{j+1}(q)$ is the direct analytic continuation of the previous function $f_j(z)$ from domain D_j to domain D_{j+1}. This means that the functions $f_j(z)$ are regular in the domains D_j and that $f_j(z) \equiv f_{j+1}(z)$, $z \in D_j \cap D_{j+1}$.

Then the function $f_n(z)$ is said to be an *analytic continuation of the function $f_0(z)$ along the sequence of domains $D_0, D_1, \ldots, D_n$*; and this continuation is unique.

The set of regular functions we have just obtained, $\{f_0(z), f_1(z), \ldots, f_n(z)\}$, defines a function $F(z)$ whose values are given by the formula

$$F(z) = f_j(z), \quad z \in D_j.$$

Note that the "function" $F(z)$ may prove to be multiple-valued. Indeed, the sequence of domains $D_0, D_1, \ldots, D_n$ may form a closed loop, i.e. D_0 may intersect with D_n. But the values of $f_0(z)$ and $f_n(z)$ in $D_0 \cap D_n$ are not obliged to coincide. Also, the multiple-valuedness may appear even at the first step if $D_0 \cap D_1$ consists of more than one domain. Figure 55 depicts the case where $D_0 \cap D_1$ consists of domain D_{01} and the shaded area $\widetilde{D}_{01}$. While for $z \in D_{01}$ the functions $f_0(z)$ and $f_1(z)$ coincide, for $z \in \widetilde{D}_{01}$ these functions do not necessarily coincide, so that for $z \in \widetilde{D}_{01}$ either $F(z) = f_0(z)$ or $F(z) = f_1(z)$, and the function $F(z)$ is, generally speaking, double-valued.

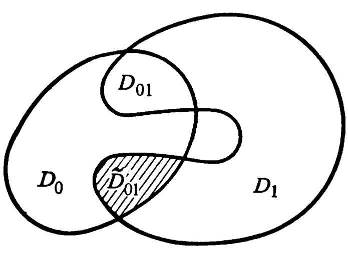


Fig. 55

The (generally) multiple-valued function $F(z)$ is, by definition, "composed" or "sewn together" from single-valued elements, the regular functions $f_0(z), f_1(z), \ldots, f_n(z)$. A collection of elements obtained from the initial element $f_0(z)$ by analytic continuation along the sequences of domains that allow for such continuation is called an *analytic function* $F(z)$. Thus, an analytic function consists of regular elements (or, as they are sometimes called, *regular branches*). It is important that the initial element uniquely defines an analytic function.

A concept that is more convenient than that of an analytic continuation along a sequence of domains is that of an analytic continuation along a curve.

20.2 Analytic continuation along a curve We will say that a function $f(z)$ that is regular in a neighborhood of a point z_0 is an *element* at point z_0. Two elements are said to be *equivalent* if they are fixed at a certain point and coincide in a neighborhood of this point. The equivalence relation between elements is transitive. In what follows we will assume that each element is considered to within an equivalence relation. We are now ready to introduce the concept of an analytic continuation along a curve.

Definition 1. Suppose we have a curve γ and a continuous function $\varphi(z)$ defined on it. In addition, suppose that for each point ζ on curve γ there is an element $f_\zeta(z)$ and that element coincides with $\varphi(z)$ on an arc of curve γ that contains point ζ.

Then the element $f_{z_1}(z)$ at the terminal point z_1 of γ is called an *analytic continuation along curve* γ *of the element* $f_{z_1}(z)$ fixed at the initial point z_0 of γ. It is also said in this case that the element $f_{z_0}(z)$ is *continued analytically along curve* γ, or that this element *admits of an analytic continuation along curve* γ.

Remark 1. A function given on curve γ is a single-valued function of the points of γ (see Sec. 4). Precisely, if curve γ is given by the equation $z = \sigma(t)$, $\alpha \leqslant t \leqslant \beta$, then each point $z_t = \sigma(t)$ of γ has corresponding to it a single number $\varphi(z_t)$, the value of φ at point z_t of γ. But if γ has self-intersections, $\varphi(z)$ as a function of points in the complex z plane may not be single-valued.

Remark 2. If the element $f_{z_0}(z)$ can be continued analytically along γ, it can be continued analytically along a sequence of domains covering γ. Moreover, $f_{z_0}(z)$ can also be continued along any curve γ' that lies sufficiently close to γ and has the same end points as γ. We will prove these facts in Sec. 20.5 (Lemmas 2 and 3). Conversely, if a given element can be continued analytically along a sequence of domains, we can easily prove that it can be continued analytically along any curve lying in this sequence of domains.

An important property of an analytic continuation along a curve is its uniqueness.

Theorem *The analytic continuation of a given element along a given curve is unique.*

Proof. Suppose we have a curve $\gamma : z = \sigma(t)$, $0 \leqslant t \leqslant 1$, and let element $f_0(z)$, which is fixed at the initial point $z_0 = \sigma(0)$ of this curve, be analytically continuable along γ. Then to each point $z_t = \sigma(t)$ there corresponds an element $f_t(z)$, and $\varphi(t) = f_t(z_t)$, $0 \leqslant t \leqslant 1$, is continuous. Suppose that the continuation is not unique. Then there is another set of elements $\tilde{f}_t(z)$ at points z_t of γ, and $\tilde{\varphi}(t) = \tilde{f}_t(z_t)$, $0 \leqslant t \leqslant 1$, is also continuous, but the elements $f_1(z)$ and $\tilde{f}_1(z)$ at the terminal point of γ are not equivalent. If we prove that $\varphi(t) \equiv \tilde{\varphi}(t)$ at $0 \leqslant t \leqslant 1$, we have proved the theorem. Indeed, the elements $f_1(z)$ and $\tilde{f}_1(z)$ coincide on an arc of γ containing point z_1, since by definition these elements coincide with the functions φ and $\tilde{\varphi}$, respectively, on an arc. By the uniqueness theorem these elements are identically equal in a neighborhood of point z_1.

Suppose M is the set of all t's for which $\varphi(t) = \tilde{\varphi}(t)$. This set contains the segment $[0, \delta]$ provided $\delta > 0$ is sufficiently small. Indeed, the elements $f_0(z)$ and $\tilde{f}_0(z)$ at point z_0 are equivalent and hence coincide in a neighborhood of this point and, therefore, on an arc $\tilde{\gamma} : z = \sigma(t)$, $0 \leqslant t \leqslant \delta$, of γ. Suppose $M \neq [0, 1]$. Then there is a positive t^* such that $\varphi(t) \equiv \tilde{\varphi}(t)$, $0 \leqslant t < t^*$, but nevertheless in any neighborhood of point t^* there are points that do not belong to M. The continuity of φ and $\tilde{\varphi}$ implies $\varphi(t^*) = \tilde{\varphi}(t^*)$, so that $t^* \in M$, and if $t^* = 1$, the proof of the theorem is complete. Suppose t^* is less than 1. By hypothesis, $\varphi(t)$ coincides with $\tilde{\varphi}(t)$

at $t \leqslant t^*$; the elements $f_{t^*}(z)$ and $\tilde{f}_{t^*}(z)$ coincide with φ and $\tilde{\varphi}$, respectively, on an arc of γ containing point z_{t^*}, by the definition of analytic continuation. Hence, $f_{t^*}(z) = \tilde{f}_{t^*}(z)$ on an arc $z = \sigma(t)$, $t^* - \alpha \leqslant t \leqslant t^*$, and by the uniqueness theorem these elements are identically equal in a neighborhood of point z_{t^*}. Whence, set M contains a segment $[t^*, t^* + \alpha]$, which contradicts the definition of number t^*. The proof of the theorem is complete.

20.3 The definition of an analytic function Suppose an element $f(z)$ is fixed at z_0. We continue it analytically along all the curves starting at z_0 for which such continuations are possible. The resulting set of elements is called an *analytic function generated by element* $f(z)$. The set of all such curves will be called the *set of admissible curves*.

This definition of an analytic function belongs to the German methematician K.T.W. Weierstrass. By definition, two analytic functions are equal if and only if the initial elements are equivalent. By the theorem of Sec. 20.2, there is only one analytic function generated by a given element. This element is said to be the *germ* of the analytic function. Equivalent elements generate the same analytic function. The range of values that an analytic function $F(z)$ admits of at a point z coincides with the range of values which all of the function's elements admit of.

Other properties of analytic functions will be considered in Sec. 24. This will allow us to study many examples of analytic function meanwhile.

20.4 Analytic continuation of power series Up till now we said nothing of how actually to proceed with an analytic continuation along a curve. Here is the algorithm of analytic continuation based on the principle of re-expansion of power series. Consider the series

$$f_0(z) = \sum_{n=0}^{\infty} c_n (z-a)^n, \tag{20.1}$$

which has a finite convergence radius $R_0 > 0$. The function $f_0(z)$ is regular in the circle K_0: $|z - a| < R_0$, so that $f_0(z)$ is an element at point a. We take a point b belonging to K_0 and expand $f_0(z)$ in a power series in $z - b$. We have

$$(z-a)^n = [(z-b)+(b-a)]^n = \sum_{k=0}^{n} C_n^k (b-a)^{n-k} (z-b)^k.$$

Substituting it into (20.1) and collecting terms with the same powers of $z - b$ yields

$$f_1(z) = \sum_{n=0}^{\infty} d_n (z-b)^n. \tag{20.2}$$

Suppose R_1 is the radius of convergence of (20.2) and K_1 is the circle $|z-b|<R_1$. Then $R_1 \geqslant R_0 - |b-a|$, since R_1 is no less than the distance between point b and the boundary of K_0 (see Sec. 12). If $R_1 = R_0 - |b-a|$, the circle K_1 lies inside K_0 and analytic continuation is impossible. Let $R_1 > R_0 - |b-a|$. Then K_1 cannot lie entirely in K_0 (Fig. 56). By the uniqueness theorem,

$$f_1(z) \equiv f_0(z), \quad z \in K_0 \cap K_1. \tag{20.3}$$

Hence, the series (20.2) is the direct analytic continuation of (20.1) (from circle K_0 into circle K_1).

Suppose there is a sequence of elements (power series) $f_0(z)$, $f_1(z), \ldots, f_n(z)$ such that the element $f_j(z)$ is the direct analytic continuation of $f_{j-1}(z)$, $1 \leqslant j \leqslant n$. Let $K_0, K_1, \ldots, K_n$ be the circles of the series for $f_0(z)$, $f_1(z), \ldots, f_n(z)$ centered at points $z_0, z_1, \ldots, z_n$. Then the element $f_n(z)$ is the analytic continuation of the element $f_0(z)$ along the sequence of circles $K_0, K_1, \ldots, K_n$.

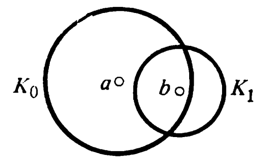


Fig. 56

Analytic continuation by re-expansion of a power series is not very effective. Other methods are available for continuing analytic functions. The main method uses the concept of integral representation of a function.

20.5 Some properties of an analytic continuation along a curve The results of this section will be used in Sec. 24.

Suppose element $f_0(z)$ is analytically continued along a curve γ: $z = \sigma(t)$, $0 \leqslant t \leqslant 1$, and let $f_t(z)$ be the respective element at point $z_t = \sigma(t)$ of curve γ. We assume that $f_t(z)$ can be represented by the power series

$$f_t(z) = \sum_{n=0}^{\infty} c_n(t)(z-z_t)^n.$$

Lemma 1 *The radius of convergence $r(t)$ of the series for $f_t(r)$ is either infinite for all t's or is a continuous function of t.*

Proof. Suppose $r(t_0) < \infty$ for a certain t_0. We select t_1 in such a way that z_t lies within the circle of convergence of the element $f_{t_0}(z)$ and $|z_{t_0} - z_{t_1}| < r(t_0)/2$. Then the radius of convergence of $f_{t_1}(z)$ is no less than the distance from point z_{t_1} to the boundary of the circle of convergence of $f_{t_1}(z)$ (see Sec. 12), so that $r(t_1) \geqslant r(t_0) - |z_{t_0} - z_{t_1}|$. Point z_{t_0} lies within the circle of convergence of $f_{t_1}(z)$, which implies $r(t_0) \geqslant r(t_1) - |z_{t_0} - z_{t_1}|$. Thus,

$$|r(t_0) - r(t_1)| \leqslant |z_{t_0} - z_{t_1}| = |\sigma(t_0) - \sigma(t_1)|,$$

and $r(t)$ is a continuous function of t.

Lemma 2 *Analytic continuation of an element along a curve can* *be replaced by analytic continuation along a finite sequence of circles.*

Proof. Since the radius of convergence $r(t)$ is a continuous, positive function of t at $0 \leqslant t \leqslant 1$, we can write $r(t) \geqslant \delta > 0$ at $t \in [0, 1]$. We select a sequence of values $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $|\sigma(t_{j+1}) - \sigma(t_j)| < \delta$ for $j = 0, 1, \ldots, n-1$. Then the circles K_j: $|z - z_{t_j}| < r(t_j)$, $j = 0, 1, \ldots, n$, comprise a finite sequence, and the elements $f_{t_0}(z), f_{t_1}(z), \ldots, f_{t_n}(z)$ form an analytic continuation along this sequence.

Lemma 3 *If an element can be continued analytically along a curve, it can also be continued along any other curve that is sufficiently close to the initial curve and whose initial and terminal points coincide with those of the initial curve. Moreover, the elements at the terminal point coincide.*

How are we to interpret the fact that two curves are sufficiently close? Take two curves γ_j: $z = \sigma_j(t)$, $0 \leqslant t \leqslant 1$, with $j = 1, 2$. The quantity $\max\limits_{0 \leqslant t \leqslant 1} |\sigma_1(t) - \sigma_2(t)|$ is called the *distance between the curves.* The curves are said to be close if the distance defined above is small.

Proof. We continue the element $f_0(z)$ at the initial point $z_0 = \sigma(0)$ of the curve γ: $z = \sigma(t)$, $0 \leqslant t \leqslant 1$, analytically along γ. We take a sequence of values $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $|\sigma(t_j) - \sigma(t_{j-1})| < \sigma/4$ for $t_{j-1} \leqslant t \leqslant t_j$, $j = 1, 2, \ldots, n$, where the positive quantity δ is the same as in Lemma 2. Then, by Lemma 2, $f_0(z)$ can be continued analytically along the sequence of circles $K_0, K_1, \ldots, K_n$; the center of K_j lies at point z_{t_j} and the radius is δ. Take the curve $\tilde{\gamma}$: $z = \tilde{\sigma}(t)$, $0 \leqslant t \leqslant 1$, for which $\tilde{\sigma}(0) = \sigma(0)$ and $\tilde{\sigma}(1) = \sigma(1)$ and such that the distance between the two curves is less than $\delta/4$.

We wish to show that the element $f_0(z)$ can be continued analytically along $\tilde{\gamma}$. By construction, the arcs γ_j and $\tilde{\gamma}_j$ of γ and $\tilde{\gamma}$ that connect points $\sigma(t_j)$ and $\sigma(t_{j+1})$, and $\tilde{\sigma}(t_j)$ and $\tilde{\sigma}(t_{j+1})$, respectively, lie inside K_j. Suppose point $\tilde{z}_t$ of $\tilde{\gamma}$ lies on arc $\tilde{\gamma}_j$, and let $\tilde{f}_t(z)$ be equal to $f_{t_j}(z)$. Point $\tilde{z}_t$ lies inside the circle of convergence K_j of $f_{t_j}(z)$, so that $\tilde{f}_t(z)$ is the element at point $\tilde{z}_t$. Note that

$$\tilde{f}_1(z) = f_1(z).$$

Thus, at each point $\tilde{z}_t$ of $\tilde{\gamma}$ we have fixed an element $\tilde{f}_t(z)$ and, thereby, a function $\varphi(z)$ is defined on curve $\tilde{\gamma}$ via the formula $\varphi(\tilde{z}_t) = \tilde{f}_t(\tilde{z}_t)$.

Let us show that φ is continuous on $\tilde{\gamma}$. If this is so, then the element $f_0(z)$ is analytically continuable along $\tilde{\gamma}$. By construction, φ

is continuous on the arcs $\tilde{\gamma}_j$ everywhere except at their end points, so that we must only check whether φ is continuous at the points $\tilde{z}_{t_j}$, $1 \leqslant j \leqslant n$. Each point $\tilde{z}_{t_j}$ belongs to $K_j \cap K_{j+1}$, and in this domain $f_{t_j}(z) \equiv f_{t_{j+1}}(z)$, since the element $f_0(z)$ is continued analytically along the sequence of circles $K_0, K_1, \ldots, K_n$. Hence, $\tilde{f}_{t_j}(z_{t_j}) \equiv \tilde{f}_{t_{j+1}}(\tilde{z}_{t_j})$, which proves that φ is continuous at $\tilde{z}_{t_j}$. The proof of the lemma is complete.

21 The Function ln z

21.1 Analytic continuation of the function ln x In courses of mathematical analysis the function $\ln x$ is studied only for real, positive values of x. It is natural to define the function $\ln z$ for complex valued z's as the analytic continuation of $\ln x$. The function $\ln x$ can be expanded in the following Taylor series:

$$\ln x = \ln [1 + (x - 1)] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x - 1)^n,$$

which converges in the interval $0 < x < 2$. Let us take this series for complex valued z's, i.e. we consider the function

$$f_0(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z - 1)^n. \tag{21.1}$$

The series converges in the circle K_0: $|z - 1| < 1$, so that $f_0(z)$ is regular in this circle and $f_0(x) = \ln x$ for $0 < x < 2$. Hence, $f_0(z)$ is the (unique) analytic continuation of $\ln x$ from the interval $0 < x < 2$ into the circle K_0.

By $\ln z$ we will denote the analytic function that is generated by the element $f_0(z)$ at point $z = 1$.

Our task is to establish the curves along which the element $f_0(z)$ is analytically continuable and to obtain practical formulas for $\ln z$. We could use the method of re-expansion of power series (Sec. 20) to carry out the analytic continuation of $f_0(z)$, but this would be a very tedious task. It is more convenient to employ the integral representation

$$\ln x = \int_1^x \frac{dt}{t}, \quad 0 < x < \infty.$$

We would like to know whether the initial element $f_0(z)$ has a similar integral representation.

Lemma 1 *The following formula holds in the circle K_0: $|z - 1| <$*

< 1:

$$f_0(z) = \int_1^z \frac{d\zeta}{\zeta}, \tag{21.1'}$$

where the integral is taken along any curve lying inside K_0.

Proof. The function $f_0(z)$, given by (21.1), is regular in K_0. The integral on the right-hand side of (21.1') is, by Theorem 5 of Sec. 9, also regular in K_0 because the integrand is regular in K_0. For $0 < x < 2$ this integral is equal to $\ln x$, i.e. coincides with the series in (21.1). By the uniqueness theorem, the integral coincides with the series for $z \in K_0$, i.e. formula (21.1') is valid.

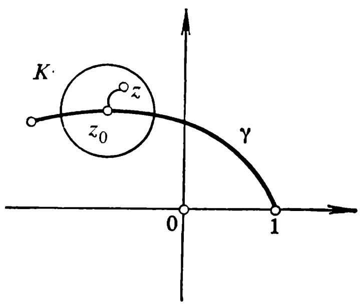


Fig. 57

Lemma 2 *The element* $f_0(z)$ *is analytically continuable along any curve* γ *that starts at point* $z = 1$ *and does not pass through* $z = 0$.

Proof. Assuming that

$$w(z) = \int_1^z \frac{d\zeta}{\zeta},$$

where the integral is taken along an arc of γ, we obtain the function $w(z)$ on γ. Take a circle K centered at point $z_0 \in \gamma$ and not containing point $z = 0$ and assume that at $z \in K$

$$f(z) = \int_1^{z_0} \frac{d\zeta}{\zeta} + \int_{z_0}^z \frac{d\zeta}{\zeta} = w(z_0) + \int_{z_0}^z \frac{d\zeta}{\zeta}, \tag{21.2}$$

where the last integral is taken along any curve lying in K (Fig. 57). By Theorem 5 of Sec. 9, the last integral represents a function of z that is regular in K, since the integrand is regular in K. Hence, the function $f(z)$ is an element at point z_0 on γ. By construction, an element at the initial point $z = 1$ coincides with the initial element $f_0(z)$. To complete the proof, we must see, in accordance with Definition 1 in Sec. 20, whether the values of $w(z)$ and $f(z)$ coincide on an arc γ_0 of γ containing point z_0. We can assume that this arc lies in K. Then at $z \in \gamma_0$ we have

$$w(z) = \int_1^z \frac{d\zeta}{\zeta} = w(z_0) + \int_{z_0}^z \frac{d\zeta}{\zeta}, \tag{21.3}$$

where the path of integration is a part of the arc γ_0. Since we can take the path of integration in (2) the same as in (3), we find that

$f(z) \equiv w(z)$, $z \in \gamma_0$, which is what we set out to prove.

Suppose D is a domain in the extended complex plane, and $f(z)$ is an element at point $z_0 \in D$. Let us assume that $f(z)$ is analytically continuable along all curves that lie in D. As a result of such continuation we arrive at a set of elements, which we call a *function analytic in D*.

This definition and Lemma 2 lead to

Theorem 1 *The function* ln z *is analytic in the domain* $0 < |z| < \infty$.

21.2 The basic properties of ln z From the proof of Lemma 2 it follows that the value of ln z at any point $z \neq 0$ or ∞ is given by the formula

$$\ln z = \int_1^z \frac{d\zeta}{\zeta}, \tag{21.4}$$

where the integral is taken along a curve γ that does not pass through 0 or ∞. Let us evaluate this integral. We have $\zeta = re^{i\varphi}$, with $r = |\zeta|$, so that

$$d\zeta = e^{i\varphi}\, dr + ire^{i\varphi}\, d\varphi, \quad \frac{d\zeta}{\zeta} = \frac{dr}{r} + i\, d\varphi,$$

and the integral in (21.4) is

$$\int_\gamma \frac{dr}{r} + i \int_\gamma d\varphi = \ln |z| + i\Delta_\gamma \arg z,$$

where $\Delta_\gamma \arg z$ is the variation of the argument along γ (see Sec .6). Hence,

$$\ln z = \ln |z| + i\Delta_\gamma \arg z. \tag{21.5}$$

This is the main formula for ln z.

Remark 1. The value of ln z depends not only on point z but also on the curve γ along which the integral in (21.4) is taken. Strictly speaking, this value should have been written as $(\ln z)_\gamma$ or $(\gamma) \ln z$. But such notation is not common and we will use it only infrequently. Instead we will always specify along what path the initial element is continued analytically.

We go back to formula (21.4). It yields

$$\frac{d}{dz} \ln z = \frac{1}{z}. \tag{21.6}$$

Example 1. Let us evaluate ln z at point z_1 by carrying out the analytic continuation of the initial element $f_0(z)$ along a curve γ:

(a) γ is the segment $[1, i]$ and $z_1 = i$;

(b) γ is the semicircle γ_+: $z = e^{it}$, $0 \leqslant t \leqslant \pi$, and $z_1 = -1$;

(c) γ is the semicircle γ_-: $z = e^{-it}$, $0 \leqslant t \leqslant \pi$, and $z_1 = -1$.

In the case (a) we have $\Delta_\gamma \arg z = \pi/2$, so that $\ln i = i\pi/2$. In the case (b) we have $\Delta_\gamma \arg z = +\pi$, so that $\ln(-1) = i\pi$, while in the case (c) $\Delta_\gamma \arg z = -\pi$, so that $\ln(-1) = -i\pi$. □

The following properties of $\ln z$ follow from formula (21.5):

(1) All the values of $\ln z$ at point z are given by the formula

$$\ln z = \ln |z| + i \arg z. \tag{21.7}$$

Here $\arg z$ is a multiple-valued function, namely, $\arg z = (\arg z)_0 + 2k\pi i$, where $(\arg z)_0$ is a fixed value of the argument, and k is an integer. We can also write this formula as

$$\ln(re^{i\varphi}) = \ln r + i\varphi + 2k\pi\, i, \ k = 0, \pm 1, \pm 2, \ldots, \tag{21.8}$$

where $\ln r$ is real.

Hence, $\ln z$ is an infinite-valued function, i.e. at each point $z \neq 0$, ∞ it has an infinitude of values. The real part of this function is single-valued:

$$\operatorname{Re} \ln z = \ln |z|$$

for any $z \neq 0, \infty$ and for any value of $\ln z$.

Formula (21.7) implies that

$$e^{\ln z} = z, \tag{21.9}$$

so that $\ln z$ is the inverse of e^z.

By virtue of (21.7), and two values of $\ln z$ at a point z_0 differ by $2k\pi i$, with k an integer. This leads to the next important property of $\ln z$.

(2) If $f_1(z)$ and $f_2(z)$ are the elements of $\ln z$ at a point z_0, then $f_1(z) - f_2(z) \equiv 2k\pi i$ in a neighborhood of this point, with k an integer.

This implies that any element of the logarithm at any point $z_0 \neq 0$, ∞ is fixed entirely by its value at this point. This statement is generally not true for an arbitrary analytic function.

(3) Suppose $f(z)$ is an element of $\ln z$ such that $f(z_0) = \ln z_0$. Then

$$f(z) = \ln z_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n z_0^n} (z - z_0)^n. \tag{21.10}$$

The series is convergent in the circle $|z - z_0| < |z_0|$.

Note that the Taylor expansion coefficients in (21.10) have the same form as in the case where z and z_0 are real.

We will now prove the validity of (21.10). The Taylor formula yields

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

From (21.6) it follows that $f'(z) = 1/z$ in a neighborhood of point z_0, so that $f^{(n)}(z_0) = (-1)^n (n-1)!/z_0^n$ for $n \geqslant 1$, which means we have proved that (21.10) is valid.

We will now generalize formula (21.5) for the case where the initial value of ln z is fixed at a point that is not point $z = 1$.

(4) Suppose we have fixed the value of ln z at point z_0, i.e. we know ln z_0, and have a curve γ that connects points z_0 and z. Let ln z be the value of the logarithm at point z obtained as a result of analytic continuation along curve γ. Then

$$\ln z = \ln z_0 + \ln \left| \frac{z}{z_0} \right| + i\Delta_\gamma \arg z. \tag{21.11}$$

We can write this formula in another form:

$$\ln z = \ln |z| + i [\operatorname{Im} (\ln z_0) + \Delta_\gamma \arg z]. \tag{21.12}$$

The proof follows from the fact that

$$\ln z = \ln z_0 + \int \frac{d\zeta}{\zeta}.$$

Example 2. Let ln $i = i5\pi/2$ and suppose γ is the segment $[i, 2]$. We continue analytically the element of the logarithm, equal to $i5\pi/2$ at point i, along the curve γ. Then formula (21.12) yields

$$\ln z|_{z=2} = \ln 2 + i \left[\frac{5\pi}{2} + \Delta_\gamma \arg z \right]$$
$$= \ln 2 + \frac{i5\pi}{2} - \frac{i\pi}{2} = \ln 2 + 2\pi i. \quad \square$$

21.3 The branch points of ln z Multple-valued analytic functions may have singular points that differ from those discussed in Chap. III. These singularities are called branch points.

Definition 1. Suppose a function $F(z)$ is analytic in a punctured neighborhood of a point a and is multiple-valued in the neighborhood. Then point a is said to be a *branch point of* $F(z)$.

Example 3. The points 0 and ∞ are the branch points of ln z. $\square$

Here is another definition of a branch point. Suppose $F(z)$ is analytic in the annulus $0 < |z - a| < r$. We take a point z_0 of this annulus and the element $f_0(z)$ at this point and continue the element analytically along the circle $|z - a| = |z_0 - a|$ with the initial and terminal points at z_0. (This process can be formulated briefly in the following way: "Traverse the circle about point a in either the positive or negative direction, depending on the orientation of the circle".) If the element $f_1(z)$, obtained as a result of analytic continuation, does not coincide with the initial element $f_0(z)$ after a full circuit, then point a is a branch point of $F(z)$.

Let us take a point $z_0 \neq 0, \infty$ and the element $f_0(z)$ of ln z and circuit point $z = 0$ in the positive sense. If $f_1(z)$ is the element obtained as a result of analytic continuation, then by (21.11) we have

$$f_1(z) = f_0(z) + 2\pi i.$$

Thus, we have arrived at the next property.

(5) When point $z = 0$ is circuited in the positive direction,

$$\ln z \to \ln z + 2\pi i, \tag{21.13}$$

i.e. the element of $\ln z$ increases by $+2\pi i$. When point $z = 0$ is circuited in the negative direction,

$$\ln z \to \ln z - 2\pi i. \tag{21.13'}$$

Remark 2. Property 5 is characteristic of the logarithm. Indeed, suppose $F(z)$ is analytic in the annulus K: $0 < |z| < r$ and possesses the following property: when point $z = 0$ is circuited in the positive direction,

$$F(z) \to F(z) + c, \quad c \neq 0$$

(i.e. each of its elements receives an increment of $c = \text{const}$). Then

$$F(z) = \frac{c}{2\pi i} \ln z + G(z),$$

with $G(z)$ regular in K.

To prove the above statement, we consider the function $G(z) = F(z) - \frac{c}{2\pi i} \ln z$. This function is analytic and single-valued in K since $G(z) \to G(z)$ when point $z = 0$ is circuited.

The function $\ln z$, just as any multiple-valued analytic function, is "composed" or "sewn together" from single-valued analytic function, its elements. Each element of $\ln z$ is said to be a regular branch of $\ln z$. Similarly, each element of a multiple-valued analytic function is said to be a (regular) branch of the function. The elements can be chosen in different ways, depending on the "composition" of the (multiple-valued) analytic function.

Formula (21.11) and the properties of $\arg z$ (see Sec. 6) bring us to the following property of the logarithm:

(6) Suppose two curves, γ_1 and γ_2, lie in the domain $0 < |z| < \infty$, connect points a and b, and are homotopic in this domain. Suppose that $f(z)$ is an arbitrary element of $\ln z$ at point a. Then as a result of analytic continuation of this element along γ_1 and along γ_2 we arrive at the same element at point b.

Indeed, the variations of the arguments along γ_1 and γ_2 are equal: $\Delta_{\gamma_1} \arg z = \Delta_{\gamma_2} \arg z$, so that, according to (21.11), the analytic continuation along γ_1, and γ_2 leads to the same value of the logarithm at point b.

Let D be an arbitrary simple connected domain not containing points 0 and ∞. We fix a point $z_0 \in D$ and the value $\ln z_0$ at this point. Next we continue analytically the element $f(z)$ of ln z ($f(z_0) =$ ln z_0) along all the paths that start at point z_0 and lie in D. We have thus constructed a function $f(z)$ that is analytic and single-valued in D. This follows from Property 6 and from the fact that any two curves that lie in a simple connected domain and have a

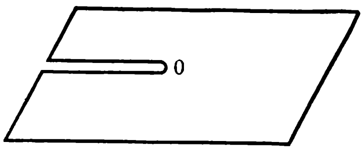


Fig. 58

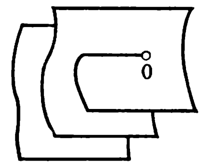


Fig. 59

common end and a common beginning are homotopic. The resulting single-valued analytic function is called a *regular branch of the logarithm in domain D*. If we select another value of the logarithm at point z_0, we have another regular branch of ln z in this domain.

Let us take the complex z plane with a cut along the ray $(-\infty, 0]$ as D (Fig. 58). The function ln z in this domain separates into an infinite number of branches. They are expressed by the formula

$$f_k(z) = \ln |z| + i(\arg z)_0 + 2k\pi i, \quad k = 0, \pm 1, \pm 2, \ldots \quad (21.14)$$

Here $(\arg z)_0$ is the branch on which

$$-\pi < (\arg z)_0 < \pi.$$

Instead of studying an infinite number of regular functions (branches) in one domain D we can take an infinite number of identical copies of this domain. We denote these domains by D_k, $k = 0, \pm 1, \ldots$, and assume that the regular function (branch) $f_k(z)$ is given in D_k.

The next step is to "paste" the D_k (the *sheets*, as they are commonly known) into one surface. Let l_k be the cut along $(-\infty, 0]$ on sheet D_k and l_k^+ and l_k^- the upper and lower banks of this cut. If $z = x < 0$, then

$$f_k(x) = \ln |x| + (2k + 1)\pi i, \quad x \in l_k^+,$$

$$f_k(x) = \ln |x| + (2k - 1)\pi i, \quad x \in l_k^-,$$

since $(\arg x)_0) = \pm\pi$, $x \in l_k^{\pm}$. Hence,

$$f_k(x)\,|_{l_k^+} = f_{k+1}(x)\,|_{l_{k+1}^-}.$$

We will paste the lower bank l_{k+1}^- with the upper bank l_k^+, $k = 0, \pm 1, \pm 2, \ldots$. Then the function ln z will be single-valued on a surface of infinitely many sheets.

The surface we have just constructed is depicted in Fig. 59 and is called the *Riemann surface for* $\ln z$. It resembles an infinitely high winding staircase. Note that the Riemann surface for $\ln z$ is simply connected.

Remark 3. The logarithm can be "cut" into regular branches in various ways. Precisely, for D we may take the complex z plane cut along any simple curve γ connecting points 0 and ∞. The choice of the cut is determined by the problem that must be solved. For instance, when evaluating integrals of the type $\int_0^\infty R(x)\ln x\,dx$ it is convenient to cut the plane along the ray $[0, +\infty)$ (Sec. 29).

Since the reader already knows the conformal mappings of several domains performed by e^z, we can easily obtain some conformal mappings performed by $\ln z$. The function e^z maps conformally and in a one-to-one manner the strip Π: $0 < \operatorname{Im} z < a$ of width $a \leqslant 2\pi$ onto the sector S: $0 < \arg w < a$. Hence, the inverse function $z = \ln w$ maps conformally and in a one-to-one manner the sector S: $0 < \arg w < a$ onto the strip Π: $0 < \operatorname{Im} z < a$ (Fig. 60). But here we must be careful since the inverse function is not single-valued. The sector S is a simply connected domain not containing points 0 and ∞, which means that in this domain the function $z = \ln w$ splits into (single-valued) branches. The mapping $S \to \Pi$ is performed by one branch, $z_0(w)$. This branch can be specified in two ways: either $0 < \operatorname{Im} z_0(w) < 2\pi$ in S or $z_0(1) = 0$ (in the latter case the value of the branch on the boundary is specified).

The function $w = \ln z$ (precisely, its branch $f_0(z)$ given by (21.14)) maps the complex z plane with a cut along the semiaxis $(-\infty, 0)$ conformally and in a one-to-one manner onto the strip $-\pi < \operatorname{Im} w < \pi$.

Other branches of the logarithm map the sector S onto other strips. Precisely, if $f_k(z)$ is the branch of the logarithm in S fixed by (21.14), the function $w = f_k(z)$ maps sector S into strip Π_k conformally and in a one-to-one manner (see Fig. 49):

$$2k\pi < \operatorname{Im} w < 2k\pi + a.$$

In courses of mathematical analysis there exists the following functional relationship for $\ln x$: $\ln(x_1x_2) = \ln x_1 + \ln x_2$. Since $\ln z$ is not a single-valued function, we must interpret the similar relationship

$$\ln(z_1z_2) = \ln z_1 + \ln z_2 \quad (z_1, z_2 \neq 0) \tag{21.15}$$

somewhat differently. Precisely, if $w_1 = \ln z_1$ is any value of $\ln z$ at point z_1 and $w_2 = \ln z_2$ is also any value of $\ln z$ at point z_2, their sum $w_1 + w_2$ is one of the values of $\ln(z_1z_2)$. This follows from the identity

$$e^{w_1+w_2} = e^{\ln z_1} e^{\ln z_2} = z_1 z_2.$$

Moreover, if $w_0 = \ln(z_1 z_2)$ is one of the values of $\ln z$ at point

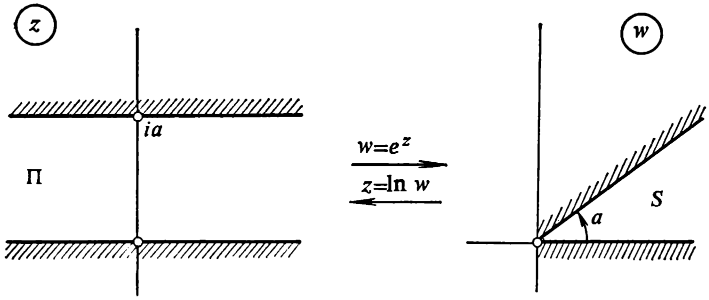


Fig. 60

$z_1 z_2$, then there are values $w_j = \ln z_j$, $j = 1, 2$, such that $w_0 = w_1 + w_2$, i.e. formula (21.15) is valid. To prove this, we fix the values $w_0 = \ln(z_1 z_2)$ and $w_1 = \ln z_1$. Then

$$e^{w_0 - w_1} = e^{w_0} e^{-w_1} = z_1 z_2 z_1^{-1} = z_2,$$

i.e. $w_2 - w_1$ coincides with one of the values of $\ln z_2$.

Formula (21.15) is obviously invalid if we substitute arbitrary values of $\ln z$ at points z_1, z_2, and $z_1 z_2$. For instance, $z_1 = z_2 = 1$, $\ln(z_1 z_2) = 0$, $\ln z_1 = 0$, and $\ln z_2 = 2\pi i$.

22 The Power Function. Branch Points of Analytic Functions

22.1 Operations on analytic functions In Sec. 21 we introduced the elementary multiple-valued analytic function $\ln z$. All the other elementary analytic functions can be expressed in terms of the logarithm through the arithmetic functions, substitutions of a function into another function, and the operation of inversion. Let us define these operations for analytic functions.

The various operations on analytic functions are introduced via the operations on their initial elements. Suppose we have two elements, $f(z)$ and $g(z)$, fixed at the same point z_0, and let $F(z)$ and $G(z)$ be the two analytic functions generated by these elements. Then the functions

$$f(z) \pm g(z), \quad f(z)\, g(z), \quad \frac{f(z)}{g(z)}$$

are also elements at point z_0 (for the quotient, $g(z_0)$ must not be zero). These elements generate the analytic functions denoted by

$$F(z) \pm G(z), \quad F(z)\,G(z), \quad \frac{F(z)}{G(z)},$$

respectively. But if the elements $f(z)$ and $g(z)$ are fixed at different points, their sum, difference, product, and quotient are not defined, so that these operations on the analytic function $F(z)$ and $G(z)$ are not defined either.

By $F'(z)$ we denote the analytic function generated by the element $f'(z)$ at point z_0.

By definition, the result of these operations on analytic functions is an analytic function, too. Let us now consider the important case where the function $F(z)$ and $G(z)$ are analytic in the same domain.

Theorem 1 *Suppose $F(z)$ and $G(z)$ are analytic in a domain D. Then the functions*

$$F'(z), \ G'(z), \quad F(z) \pm G(z), \quad F(z)\,G(z), \quad \frac{F(z)}{G(z)}$$

are analytic in D, too (in the case of the quotient, $G(z) \neq 0$ at $z \in D$).

Proof. Let us prove the analyticity of $F(z) + G(z)$. (The analyticity of the other functions can be proved in a similar manner.) Let $f(z)$ and $g(z)$ be the initial elements of these functions fixed at point z_0, while the curve γ lies in D and has its initial point at z_0. If we continue analytically the elements $f(z)$ and $g(z)$ along γ, we obtain the elements $f_\zeta(z)$ and $g_\zeta(z)$ at each point ζ of γ. Their sum $h_\zeta(z) = f_\zeta(z) + g_\zeta(z)$ is regular at point ζ; hence, the element $h(z) = f(z) + g(z)$ at point z_0 is continued analytically along curve γ.

This theorem makes it possible to broaden somewhat the supply of elementary analytic functions. For instance, the following functions are analytic (the domains of analyticity are specified in the parentheses):

$$\ln^2 z \ (0 < |z| < \infty), \ z \ln \ (0 < |z| < \infty),$$

$$z + \ln z \ \ (0 < |z| < \infty), \quad \frac{\ln z + 1}{\ln z - 1} \quad (z \neq 0, \ e, \ \infty).$$

Example 1. Consider the function $F(z) = z \ln z$ (the initial element of the logarithm is fixed at point $z = 1$, $\ln 1 = 0$). We will try to establish the singular points of this function. Specifically, we will prove that $z = 0$ is a branch point of $F(z)$. Suppose γ is the circle $|z| = 1$ starting at point $z = 1$ and oriented in the positive sense. As we circuit point z along γ (i.e. as we continue analytically the initial element $f(z)$ along γ), $\ln z \to \ln z + 2\pi i$, so that $f(z) \to f(z) + 2\pi i z$. Hence, point $z = 0$ is a branch point. As point $z = 0$ is circuited n times, $f(z) \to f(z) + 2n\pi i z$. Point $z = \infty$ is also a branch point of $F(z)$, since if we circuit point $z = 0$ in the positive direction, we at the same time circuit point $z = \infty$ in the negative direction. □

Let us now define the composite function. Suppose the analytic functions $F(z)$ and $G(z)$ are generated by the elements $f(z)$ and $g(z)$ fixed at points z_0 and $w_0 = f(z_0)$, respectively. The composite analytic function $G(F(z))$ is then generated by the element $g(f(z))$.

Theorem 2 *Suppose $F(z)$ is analytic in a domain D, its values lie in a domain $\tilde{D}$, and $G(z)$ is analytic in $\tilde{D}$. Then $G(F(z))$ is analytic in D.*

Proof. We select a curve γ with the initial point at a point $z_0 \in D$. We continue analytically the element $f(z)$ along γ. Then at each point ζ of γ we have the element $f_\zeta(z)$ and the function $w(z)$ on γ, i.e. $w(z) = f_z(z)$. This function maps γ onto a curve $\tilde{\gamma}$ lying in $\tilde{D}$, with the initial point at point $w_0 = f(z_0)$. By hypothesis, the initial element $g(w)$ of $G(w)$ can be continued analytically along $\tilde{\gamma}$. This yields an element $g_w(w)$ at each point w of $\tilde{\gamma}$. If w belongs to $\tilde{\gamma}$ and $w = f_\zeta(\zeta)$, the function $g_w(f_\zeta(z)) = h_\zeta(z)$ is regular at point $\zeta \in \gamma$ and, hence, is an element at this point. This proves that we have continued analytically the element $g(f(z))$ along γ, so that $G(F(z))$ is a function that is analytic in D.

Example 2. The function $\ln(z - a)$ is analytic in the domain $0 < |z - a| < \infty$. □

Example 3. The function $\ln \frac{z-1}{z+1}$ is analytic in the extended complex z plane with points $+1$ and -1 deleted.

Indeed, the function $w = \frac{z-1}{z+1}$ in the specified domain D is regular and does not assume the values 0 and ∞. The function $\ln w$ is analytic in the domain $\tilde{D}$: $0 < |w| \infty$. □

Here we will mention the following identities: $e^{\ln(z-a)} = z - a$ and $\operatorname{Re} \ln(z - a) = \ln |z - a|$, which hold for $z \neq 0, \infty$.

Remark 1. Strictly speaking, the formula $F(z) = \ln(z - a)$ does not define an analytic function completely, since, by the very definition of an analytic function, we must fix its initial element. This is due to the fact that the formula may specify not one but several analytic functions if the initial element is not specified.

Example 4. The formula $F(z) = \ln e^z$ defines an infinitude of analytic functions:

$$F_k(z) = z + 2k\pi i, \quad k = 0, \pm 1, \pm 2, \ldots . \quad \square$$

Other examples of this type are given in Sec. 22.5.

22.2 The power function For real and positive x's and for a fixed real number α the following formula is valid: $x^\alpha = e^{\alpha \ln x}$. Let us generalize this formula so that it incorporates complex valued z's and α (α is fixed) by putting, by definition,

$$z^{\alpha} = e^{\alpha \ln z}. \tag{22.1}$$

We take the element $g_0(z) = e^{\alpha f_0(z)}$ at point $z = 1$, where $f_0(z)$ is the initial element of $\ln z$ at point $z = 1$ (see formula (21.1)), as the initial element of the function z^{α}. Then

$$g_0(z) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (z-1)^k, \quad \binom{\alpha}{k} \equiv \frac{\alpha(\alpha-1)\ \ldots\ (\alpha-k+1)}{k!}, \tag{22.2}$$

Indeed,

$$\frac{d^k}{dz^k} g_0(z)|_{z=1} = k! \binom{\alpha}{k}.$$

From this relationship and the Taylor formula

$$g_0(z) = \sum_{k=0}^{\infty} \frac{g_0^{(k)}(z_0)}{k!} (z-z_0)^k$$

follows formula (22.2).

The properties of the logarithm yield the following property of the power function.

Theorem 3 *The function z^{α} is analytic in $0 < |z| < \infty$.*

Proof. The function $\ln z$ is analytic in D: $0 < |z| < \infty$, which means that the function $\alpha \ln z$ possesses the same property. Since e^z is an entire function, we conclude, from Theorem 2, that the function $z^{\alpha} = e^{\alpha \ln z}$ is analytic in D since it is a composite of two analytic functions.

The derivative of the power function is calculated by the same formulas as in the case of real numbers:

$$\frac{d}{dz} z^{\alpha} = \alpha z^{\alpha-1}. \tag{22.3}$$

Remark 2. The above formula must be understood in the sense that

$$\frac{d}{dz} z^{\alpha} = \frac{\alpha z^{\alpha}}{z}, \tag{22.3'}$$

where the values of z^{α} on both sides are the same.

All formulas for z^{α} follow from the formulas for the logarithm and (22.1). The basic formula for the power function follows from (22.1) and (21.12).

(1) Suppose curve γ connects points z_0 and z_1 and does not pass through the points 0 and ∞. Then at z_0 we have fixed the element $f(z)$ of z^{α} in such a way that $f(z_0) = z_0^{\alpha}$. If we continue analytically

this element along γ, its value at point z_1 will be

$$z_1^\alpha = z_0^\alpha \exp\left[\alpha \ln \left|\frac{z_1}{z_0}\right| + i\alpha\Delta_\gamma \arg z\right]. \tag{22.4}$$

This formula is rather tedious and will almost never be used in our narrative. It simplifies if we are dealing with real α's. At the same time this case is the most important for applications.

(2) Any element of z^α at each point $z_0 \neq 0, \infty$ is completely fixed by its value at the point. Any two elements, $f_1(z)$ and $f_2(z)$, at each point $z_0 \neq 0, \infty$ differ only by a numerical factor:

$$f_2(z) \equiv e^{i2\pi k\alpha} f_1(z), \tag{22.5}$$

where k is an integer.

This property follows from (22.1) and Property 2 of the logarithm (see Sec. 21).

(3) All the values of z^α (α is real) at point $z = re^{i\varphi}$ are given by the formula

$$z^\alpha = (re^{i\varphi})^\alpha = r^\alpha e^{i(\varphi+2k\pi)\alpha}, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{22.6}$$

In particular, for real α's the function $|z^\alpha|$ is single-valued:

$$|z^\alpha| = |z|^\alpha. \tag{22.7}$$

Formulas (22.4) and (21.11) yield the following basic formula for z^α (α is real):

(4) Suppose at point $z_0 = r_0 e^{i\varphi_0}$ the value of z^α is $z_0^\alpha = r_0^\alpha e^{i\alpha\varphi_0}$. Let z_1^α be the value of z^α at point z_1 obtained as the result of analytic continuation along a curve γ that connects points z_0 and z_1. Then

$$z_1^\alpha = |z_1|^\alpha e^{i\alpha(\varphi_0+\Delta_\gamma \arg z)}. \tag{22.8}$$

In particular, under such continuation,

$$\Delta_\gamma \arg z^\alpha = \alpha\Delta_\gamma \arg z. \tag{22.9}$$

Here are some examples.

Example 5. All values of the function $\sqrt[n]{z} = z^{1/n}$, with n a positive integer not less than 2, are given at point $z = re^{i\varphi}$, $r \neq 0$, by the formula

$$\sqrt[n]{z} = \sqrt[n]{re^{i\varphi}} = \sqrt[n]{r}\, e^{\frac{i}{n}(\varphi+2k\pi)}, \quad k = 0, 1, \ldots, n-1. \tag{22.10}$$

Indeed, the values of $\sqrt[n]{z}$ at $k = 0, 1, \ldots, n-1$ are all different, since the numbers $e^{i\varphi_k}$, $\varphi_k = (\varphi + 2k\pi)/n$ for these values of k are different. Moreover, any integer k can be represented as $k = nm + l$, where m and l are integers and $0 \leqslant l \leqslant n-1$. Since $e^{i\varphi_k} = e^{i2\pi m} e^{i\varphi_l} = e^{i\varphi_l}$, we see that (22.10) gives all the values of $\sqrt[n]{z}$.

Thus, the function $\sqrt[n]{z}$ in $0 < |z| < \infty$ is n-valued, i.e. at each point of this domain it has exactly n different values.

Formula (22.10) yields the following identity:

$$(\sqrt[n]{z})^n \equiv z. \qquad (22.11)$$

Hence, the function $\sqrt{z}$ is the (right) inverse of z^n. □

Example 6. All values of the function $\sqrt{z}$ at a point $z = re^{i\varphi}$ are given by the formula $\sqrt{z} = \sqrt{re^{i\varphi}} = \pm\sqrt{r}e^{i\varphi/2}$. Hence, the function $\sqrt{z}$ is double-valued in $0 < |z| < \infty$. □

Example 7. If α is real and irrational, the function z is infinite-valued in $0 < |z| < \infty$.

Indeed, all values of z^α at point $z = re^{i\varphi}$ are given by (22.6). Let us see whether different values of z^α correspond to different values of k. Suppose that this is not so. Then there are numbers k_1 and k_2 that are not equal and yet $e^{ik_2 2\pi\alpha} = e^{ik_1 2\pi\alpha}$. This means that $(k_1 - k_2)\alpha = m$, $m \neq 0$, is an integer, i.e. α is rational, which contradicts the hypothesis. □

Remark 3. If α is not real, the function z^α is infinite-valued in $0 < |z| < \infty$.

Example 8. Suppose that at point $z = 1$ we have fixed the element $f(x)$ of $\sqrt{z}$ in such a way that $f(1) = 1$, and γ is the segment $[1, i]$. We wish to calculate $\sqrt{i}$ obtained as a result of analytic continuation along γ.

We have $\sqrt{i} = 1$ and $\Delta_\gamma \arg z = \pi/2$. The formula (22.8) yields $\sqrt{i} = e^{i\pi/4}$.

Now suppose γ is the arc $z = e^{-it}$, $0 \leqslant t \leqslant 3\pi/2$. Then $\Delta_\gamma \arg z = -3\pi/2$, so that $\sqrt{i} = e^{-i3\pi/4} = -e^{i\pi/4}$. □

Here is another formula for z^α:

(5) Suppose $f(z)$ is the element of z^α at a point $z_0 \neq 0$ such that $f(z_0) = z_0^\alpha$. This element can be expanded in a Taylor series:

$$f(z) = z_0^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(z-z_0)^k}{z_0^k}, \qquad (22.12)$$

which converges in the circle $|z - z_0| < |z_0|$ (i.e. the circle's center is at z_0 and the circle's radius is equal to the distance from point z_0 to point $z = 0$).

Indeed, according to the Taylor formula,

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k.$$

Moreover, from (22.3′) it follows that $f'(z) = \alpha z^\alpha/z$, so that $f^{(k)}(z) = k! \binom{\alpha}{k} z^\alpha z^{-k}$. Substituting the expressions for the various derivatives into the Taylor formula, we arrive at (22.12).

Note that (22.12) has the same form as the Taylor formula from mathematical analysis for a power function (for z_0, z, and α real).

When α is real and x_1 and x_2 are real and positive,

$$(x_1x_2)^\alpha = x_1^\alpha x_2^\alpha.$$

However, a similar relationship

$$(z_1z_2)^\alpha = z_1^\alpha z_2^\alpha \tag{22.13}$$

for z_1 and z_2 nonzero and complex-valued must be interpreted differently because of the multiple-valuedness of z^α. We must interpret (22.13) in the same sense as we did (21.15) for the logarithm. Precisely, if w_1 and w_2 are values of z^α at points z_1 and z_2, then w_1w_2 is one of the values of z^α at point z_1z_2. Moreover, if w_0 is a value of z^α at point z_1z_2, then there are values $w_1 = z_1^\alpha$ and $w_2 = z_2^\alpha$ such that $w_0 = w_1w_2$. The proof follows directly from (22.6).

22.3 The branch points of the power function The definition of the power function and Property 5 of the logarithm (Sec. 21) yields the following property of the power function:

(6) Suppose $f(z)$ is the element of z^α at a point $z_0 \neq 0, \infty$. Then, when the point $z = 0$ is circuited in the positive sense, the element is multiplied by $e^{i2\pi\alpha}$, i.e.

$$f(z) \to e^{i2\pi\alpha} f(z), \tag{22.14}$$

while when the point $z = 0$ is circuited in the negative direction, the element is multiplied by $e^{-i2\pi\alpha}$, i.e.

$$f(z) \to e^{-i2\pi\alpha} f(z). \tag{22.14$'$}$$

In Sec. 21 we introduce the concept of a branch point. Property 6 implies that the points 0 and ∞ are the branch points of z^α if α is not an integer. Below we introduce the following classification of isolated branch points.

Definition 1. Suppose a function $F(z)$ is analytic in the annulus K: $0 < |z - a| < \rho$, and suppose that at each point of this annulus $F(z)$ has exactly $n \geqslant 2$ different elements of $F(z)$. Then we say that $F(z)$ at point a has a *branch point of degree of ramification* $n - 1$ or *multiplicity* n. The branch point at $z = \infty$ can be treated similarly.

If n is finite, the branch point at $z = a$ is said to be algebraic, while if $n = \infty$, the branch point at $z = a$ is said to be of *infinite multiplicity* or is called a *logarithmic branch point.*

Remark 4. We can show (see Sec. 26) that if a function $F(z)$ that is analytic in an annulus K has exactly n different elements at a point of the annulus, then this is true for all points of the annulus (this is also true for the case where $n = \infty$).

Example 9. Points 0 and ∞ are branch points of multiplicity n of the function $\sqrt[n]{z}$. For instance, for $\sqrt{z}$ these points are branch points of multiplicity 2.

Indeed, suppose $f_0(z)$ is an element of $\sqrt[n]{z}$ at a point $z_0 \neq 0, \infty$. Then all the elements at this point have the form

$$f_k(z) = e^{i2\pi k/n} f_0(z), \quad k = 0, 1, \ldots, n-1,$$

i.e. there are exactly n of such elements. □

Example 10. The function $F(z) = 1/\sqrt{z}$ is analytic in the annulus $0 < |z| < \infty$; points 0 and ∞ are branch points of multiplicity 2 for this function. □

Remark 5. Here is a typical mistake involving the singularities of the function $F(z) = 1/\sqrt{z}$. It is often stated that point $z = 0$ is a pole of $1/\sqrt{z}$ since $\lim\limits_{z\to 0} (1/\sqrt{z}) = \infty$. This, however, is not true since poles are singularities of single-valued functions.

Example 11. The function $F(z) = \sqrt{\dfrac{z-1}{z+1}}$ has two branch points of multiplicity 2: $z = \pm 1$. Point $z = \infty$ is not a singular point. Indeed,

$$F(z) = \sqrt{G(z)}, \quad G(z) = \frac{1-(1/z)}{1+(1/z)}.$$

Function $G(z)$ is regular at point $z = \infty$ since $G(\infty) = 1 \neq 0$. By Theorem 2, $F(z)$ is analytic in the extended complex z plane with points $+1$ and -1 deleted. □

In Sec. 21 we found that in every simply connected domain that does not contain points 0 and ∞ the function $\ln z$ splits into regular branches. Since $z^\alpha = e^{\alpha \ln z}$, we can say that in every such domain the function z^α splits into regular branches. Any two such branches differ by a factor of $e^{i2\pi\alpha k}$, where k is an integer (see (22.6)).

Example 12. Suppose S is the sector $0 < \arg z < \beta \leqslant 2\pi$. In this sector the function z^α splits into regular branches. One such branch is determined (for α real) by the formula

$$f_0(z) = |z|^\alpha e^{i\alpha \arg z}, \quad 0 < \arg z < 2\pi. \tag{22.15}$$

The other branches of this function have the form

$$f_k(z) = e^{i2\pi\alpha k} f_0(z), \tag{22.16}$$

where k is an integer.

For instance, $\sqrt{z}$ splits into two branches, $f_0(z) = \sqrt{|z|}\, e^{(i/2)\arg z}$ and $f_1(z) \equiv -f_0(z)$, where $0 < \arg z < 2\pi$ (cf. Sec. 13).

Now suppose that $\alpha > 0$ and $0 < \alpha\beta \leqslant 2\pi$. Then the branch $w = f_0(z)$ of z^α maps the sector S: $0 < \arg z < \beta$ in a one-to-one manner onto the sector $\tilde{S}$: $0 < \arg w < \alpha\beta$ in the complex w plane (Fig. 61), i.e. unfolds the initial sector (S) α times. Indeed, from

(22.6) it follows that if $w = \rho e^{i\psi}$ and $z = re^{i\varphi}$ $(0 < \varphi < \beta)$, then

$$\rho = r^{\alpha}, \quad \psi = \alpha\varphi,$$

so that points w fill up sector $\tilde{S}$ (Fig. 61). □

Example 13. Suppose D is the complex z plane with a cut along the semiaxis $[0, +\infty)$ (Fig. 47). The function $F(z) = \sqrt{z}$ splits in this domain into two regular branches

$$f_1(re^{i\varphi}) = \sqrt{r}\, e^{i\varphi/2}, \quad f_2(z) \equiv -f_1(z).$$

Here $z = re^{i\alpha}$, $0 < \varphi < 2\pi$. The function $w = f_1(z)$ maps D in a one-to-one manner and conformally onto the upper half-plane

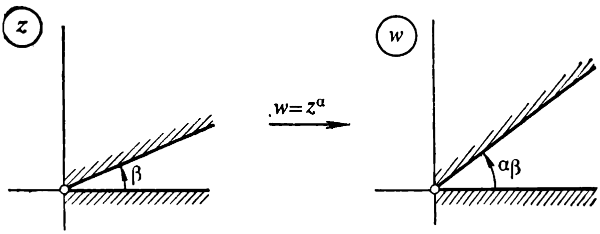


Fig. 61

$\operatorname{Im} w > 0$, while the function $w = f_2(z)$ maps D onto the lower half-plane $\operatorname{Im} w < 0$ (see Fig. 47).

We take $z = x + i0$, with $x > 0$ (i.e. point z lies on the upper bank of the cut). Then $f_1(x + i0) = \sqrt{x} > 0$. But if $z = x - i0$ (i.e. point z lies on the lower bank), then $f_1(x - i0) - \sqrt{x}$. □

22.4 The Riemann surface for z^{α} If α is such that z^{α} is an infinite-valued function, the Riemann surface for z^{α} will be the same as for the logarithm. A new type of Riemann surface appears when z^{α} is finite-valued.

Let us build the Riemann surface of $\sqrt{z}$. Suppose D is the complex z plane with a cut along the ray $(-\infty, 0]$. Then the function $\sqrt{z}$ splits in D into two branches, $f_1(z)$ and $f_2(z)$, such that $f_1(1) = 1$ and $f_2(z) \equiv -f_1(z)$. We take two copies D_1 and D_2 of D and assume that the function $f_k(z)$ is defined in D_k $(k = 1, 2)$. Then for a point z belonging to D_k we have

$$f_{1,2}(re^{i\varphi}) = \pm\sqrt{r}\, e^{i\varphi/2}, \quad -\pi < \varphi < \pi.$$

Let l_k be the cut in the sheet D_k, with l_k^+ and l_k^- the upper and lower banks of the cut. Since $\varphi = \pm\pi$ on $l_k^{\pm}$, we have

$$f_1(z)|_{z\in l_1^+} = f_2(z)|_{z\in l_2^-},$$

$$f_1(z)|_{z\in l_1^-} = f(z)|_{z\in l_2^+}.$$

For this reason, to obtain a surface on which $\sqrt{z}$ is single-valued we must paste the upper bank l_1^+ to the lower bank l_2^- and the lower bank l_1^- to the upper bank l_2^+ (criss-cross). This means we have the Riemann surface for $\sqrt{z}$ (Fig. 62) with a self-intersection.

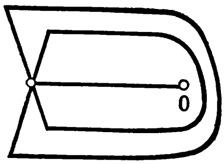


Fig. 62

The Riemann surface for $\sqrt[n]{z}$ can be constructed along the same lines. Take n copies $D_0, D_1, \ldots, D_{n-1}$ of D, which is the complex z plane with a cut along the ray $(-\infty, 0]$. In each domain D_k we consider the regular function

$$f_k(z) = \sqrt[n]{r}e^{(i/n)(\varphi+2k\pi)}$$

$$(z = re^{i\varphi}, \quad -\pi < \varphi < \pi).$$

Then $f_k(z)\,|_{z\in l_k^+} = f_{k+1}(z)\,|_{z\in l_k^-}$. We paste the bank l_0^+ to the bank l_1^-, then l_1^+ to l_2^-, and so on, and finally l_{n-1}^- to l_0^+. We have arrived at the Riemann surface of $\sqrt[n]{z}$ with self-intersections.

Note that the Riemann surface for $\sqrt[n]{z}$ is simply connected for any integral value of n.

22.5 **Examples** The definition of a branch point of multiplicity n requires that at each point of the annulus $0 < |z - a| < r$ there be exactly n different elements (not values!) of the analytic function $F(z)$ considered. Here are some examples that show that this requirement cannot be replaced by the condition that at each point there be exactly n values of $F(z)$.

Example 14. The function $F(z) = \sqrt{z}\sin z$ has exactly two singular points, 0 and ∞, which are branch points of multiplicity 2. But at $z_k = k\pi$, $k = \pm 1, \pm 2, \ldots$ this function assumes only one value: $F(z_k) = 0$. □

Example 15. Consider the function

$$F(z) = z^z = e^{z\ln z}$$

All values of this function at $z = 1/n$ (n a positive integer) are given by the formula

$$F\left(\frac{1}{n}\right) = \frac{1}{\sqrt[n]{n}} e^{i2k\pi/n}, \quad k = 0, 1, \ldots, n-1.$$

Hence, at $z = 1/n$ the function z^z has exactly n different values. Now suppose α is an irrational number. Then $F(\alpha)$ has an infinitude of values:

$$F(\alpha) = \alpha^{\alpha}e^{i\alpha 2k\pi}, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{22.17}$$

Thus, at different points of the domain $0 < |z| < \infty$ this function has different numbers of values.

Let us now show that point $z = 0$ (and, hence, point $z = \infty$) is a logarithmic branch point for $F(z)$. We take the element $f_0(z)$ of this function at a point $z_0 \neq 0, \infty$ and circuit point $z = 0$ in the positive direction. Since the element of $\ln z$ will increase by $+2\pi i$, after one circuit the element becomes $e^{2\pi iz} f_0(z)$. Hence, all the elements of z^z at point $z_0 \neq 0, \infty$ are given by the formula

$$f_k(z) = f_0(z)\, e^{i2k\pi z}, \quad k = 0, \pm 1, \pm 2, \ldots, . \ \square$$

Example 16. Let us calculate i^i, i.e. the values of z^z at point i. Formula (22.17) yields

$$i^i = e^{-\frac{\pi}{2}+2k\pi}, \quad k=0, \pm 1, \pm 2, \ldots .$$

Note that all the values of i^i are real. $\square$

Here are additional examples of the type of Example 4.

Example 17. The expression of the type $F(z) = \sqrt{z^2}$ determines two analytic functions, $F_1(z) = z$ and $F_2(z) = -z$. $\square$

Examples 4 and 17 show that we must be very careful when dealing with formal expressions for multiple-valued functions. The operations on analytic functions were properly introduced at the beginning of this section. On the other hand, the examples given below show that not all expressions involving the root sign or the logarithm are multiple-valued functions.

Example 18. The function $F(z) = \cos\sqrt{z}$ is analytic in the domain $0 < |z| < \infty$, according to Theorem 2. We will show that this function is single-valued. We select a point z_0 and any element $f(z)$ of $\sqrt{z}$ at this point and circuit point $z = 0$. As a result, $f(z) \to -f(z)$, $\cos f(z) \to \cos f(z)$, in view of the fact that the cosine is an even function. This implies that $\cos\sqrt{z}$ is a single-valued function. Point $z = 0$ is a removable singularity in this case; hence, $\cos\sqrt{z}$ is an entire function. The only singularity at $z = \infty$ is an essential singularity. $\square$

Example 19. The function $F(z) = (\sin\sqrt{z})/\sqrt{z}$ is also an entire function. (Here $\sqrt{z}$ in the numerator and denominator means the same analytic function.) $\square$

The function z^α is indeed a complicated function.

Example 20. Consider the equation $z^\alpha = 1$, $\alpha \neq 0$. Solving it, we obtain $\alpha \ln z = 2k\pi i$, whence

$$z_k = e^{(i/\alpha)2k\pi}, \quad k=0, \pm 1, \pm 2, \ldots .$$

Let α be a real number. Then the roots z_k lie on the unit circle. If α is irrational, these roots everywhere densely fill the unit circle $|z| = 1$.

Such a situation is impossible for a regular function because if a function $f(z)$ is regular in a neighborhood of the unit circle $|z| =$

$= 1$, then on the circle there can be no more than a finite number of solutions of the equation $f(z) = 1$. The given example can be interpreted as follows. We cut the complex z plane along, say, the semiaxis $(-\infty, 0]$. The function splits into an infinitely large number of regular branches in such a plane. For each branch there is only a finite number of points on the unit circle $|z| = 1$ at which the function assumes the value of 1. In other words, the roots of the equation $z^\alpha = 1$ lie on different sheets of the Riemann surface. □

23 The Primitive of an Analytic Function. Inverse Trigonometric Functions

23.1 The primitive of an analytic function Suppose we have an analytic function $F(z)$ generated by the element $f_0(z)$ at a point $z_0 \neq \infty$. Let us take a small circle K centered at z_0 and consider the function

$$g_0(z) = \int_{z_0}^{z} f_0(\zeta)\, d\zeta, \quad z \in K, \tag{23.1}$$

where the integral is taken along a path lying in K. Then $g_0(z)$ is regular in K.

An analytic function $G(z)$ generated by the element $g_0(z)$ at a point z_0 is called a *primitive of* $F(z)$. We will use the notation

$$G(z) = \int_{z_0}^{z} F(\zeta)\, d\zeta. \tag{23.2}$$

Theorem 1 *If a function $F(z)$ is analytic in a domain D, its primitive $G(z)$ is also analytic in D.*

The proof of this theorem is exactly the same as that of Lemma 2 of Sec. 21. Supoose curve γ lies in D and starts at point z_0. We take a point ζ of γ and denote the arc along γ connecting points z_0 and ζ by γ_ζ. Now $f(z)$ is the element of $F(z)$ at point ζ obtained by continuing analytically the initial element $f_0(z)$ along γ_ζ. We take a small circle K centered at point ζ and select a point $z \in K$. Then

$$g(z) = \int_{\gamma_\zeta} F(\zeta')\, d\zeta' + \int_{\zeta}^{z} f(\zeta')\, d\zeta',$$

where the second integral on the right-hand side is taken along a curve lying in K. By Theorem 5 of Sec. 9, $g(z)$ is regular in K, i.e. is the element at point ζ, and if $\zeta = z_0$, we have $g(z) = g_0(z)$. Thus, at each point ζ of γ we have constructed an element, and the consistency of these elements is verified in the same way as in Lemma 2 of Sec. 21. This proves that the element $g_0(z)$ is analytically continuable along γ, so that the function $G(z)$ generated by this element is analytic in D.

It is obvious that

$$G'(z) = F(z).$$

Moreover, if $G_1(z)$ and $G_2(z)$ are two primitives of a single analytic function $F(z)$, then $G_1(z) - G_2(z) \equiv \text{const}$.

23.2 The functions arctan z arccot z, artanh z, and arcoth z The function arctan x (x real) allows for the following integral representation:

$$\arctan x = \int_0^x \frac{dt^2}{1+t^2}.$$

The function $(1+z^2)^{-1}$ is regular in the entire complex plane except at the poles $z = \pm i$. We assume that

$$\arctan z = \int_0^z \frac{d\zeta}{1+\zeta^2}. \tag{23.3}$$

By Theorem 1, arctan z is analytic in the complex z plane with points $+i$ and $-i$ deleted.

Let us express arctan z in terms of the logarithm. We have

$$\int_0^z \frac{d\zeta}{1+\zeta^2} = \frac{1}{2}\int_0^z \left(\frac{1}{1-i\zeta} + \frac{1}{1+i\zeta}\right) d\zeta = \frac{1}{2i}\ln\frac{1+iz}{1-iz}.$$

Hence,

$$\arctan z = \frac{1}{2i}\ln\frac{1+iz}{1-iz}, \tag{23.4}$$

which shows that arctan z is analytic in the extended complex z plane with points $+i$ and $-i$ deleted. The analyticity of arctan z at point $z = \infty$ follows from the representation

$$\arctan z = \frac{1}{2i}\ln\frac{i+z^{-1}}{z^{-1}-i}.$$

The two points $z = +i$ and $z = -i$ are logarithmic branch points.

The function arctan z is the inverse of tan z, i.e.

$$\tan(\arctan z) = z$$

for all $z \neq \pm i, \infty$.

Remark 1. If we wish to be precise, we must say that arctan z is the right inverse of tan z. Indeed, the multiple-valued expression $F(z) = \arctan(\tan z)$ defines not one but an infinitude of analytic functions $F_k(z) = z + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

Let $f_0(z)$ be the element of **arctan** z **at point** $z = 0$, so that $f_0(0) = 0$. Then

$$f_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}. \tag{23.5}$$

The series converges in the circle $|z| < 1$.

The formula for the derivative of arctan z, known from mathematical analysis for real z's, remains unchanged:

$$\frac{d}{dz} \arctan z = \frac{1}{1+z^2}.$$

The analytic functions arccot z, artanh z, and arcoth z are introduced in a similar manner. Since all these functions can be expressed in terms of the logarithm, calculating their values results in calculating values of the logarithm. For this reason they play no independent role in the theory of functions of a complex variable.

23.3 The functions arcsin z, **arccos** z, **arsinh** z, **and arcosh** z For real $x \in [-1, 1]$ the function arcsin x allows for the following integral representation:

$$\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

Let us continue this function analytically to complex values of the independent variable. To this end we use Theorem 1. The function $F(z) = 1/\sqrt{1-z^2}$ is analytic in the complex z plane with points $+1$ and -1 deleted (these are the branch points of $F(z)$). By Theorem 1, the function

$$\arcsin z = \int_0^z \frac{d\zeta}{1-\zeta^2} \tag{23.6}$$

is analytic in the complex plane with points ± 1 deleted. The integral is taken along any path not passing through points ± 1.

We will fix the initial element $f_0(z)$ of the function arcsin z at point $z = 0$. We can do this either via the series

$$f_0(z) = \arcsin z = z + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} z^{2n+1},$$

or via the integral representation

$$f_0(z) = \arcsin z = \int_0^z \frac{d\zeta}{\sqrt{1-\zeta^2}}, \quad z \in D.$$

Here D is the complex plane with cuts along the rays $(-\infty, -1]$ and $[1, \infty)$, the integral is taken along a path lying in D, and the branch of the square root is the one on which

$$\sqrt{1-\zeta^2}\,|_{\zeta=0}=1.$$

The function $\arcsin z$ can also be expressed in terms of the logarithm. For all $z \neq \pm 1, \infty$ we have

$$\sin w = z, \quad w = \arcsin z.$$

Solving the equation

$$e^{iw} - e^{-iw} = 2iz$$

for z, we find that

$$\arcsin z = -i \ln\left(iz+\sqrt{1-z^2}\right). \tag{23.7}$$

Let us study the special features of $\arcsin z$ concerning its multiple-valuedness. Let γ_+ and γ_- be simple closed curves with their initial points at $z=0$; points $z = 1$ and $z = -1$ lie inside γ_+ and γ_- respectively (Fig. 63). The curves γ_- and γ_+ are oriented in the positive and negative senses, respectively. For instance, we can take the circles $|z \mp 1| = 1$ for $\gamma_\pm$ (Fig. 63). Finally, suppose $f_0(z)$ is the initial element of $\arcsin z$ at point $z=0$.

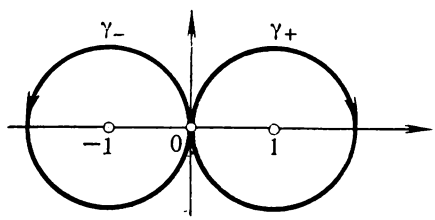


Fig. 63

(1) We continue $f_0(z)$ analytically along curve γ_+. We select a point z lying in a small neighborhood of point $z = 0$. Then the element $f(z)$ obtained as a result of analytic continuation is equal to the integral along a path γ that connects points 0 and z and consists of γ_+ and the segment $\gamma_z = [0, z]$, i.e. $\gamma = \gamma_+\gamma_z$. As the branch point $z = 1$ is circuited, $\sqrt{1-z^2} \to -\sqrt{1-z^2}$. Hence,

$$f(z) = -f_0(z) + \int_{\gamma_+} \frac{d\zeta}{\sqrt{1-\zeta^2}}.$$

The branch of the square root is the one on which $\sqrt{1-\zeta^2}\,|_{\zeta=0} = 1$ (at the initial point of γ_+). By Cauchy's integral theorem, the integral along γ is equal to the integral along the cut $[0, 1]$. On the upper bank of the cut $\sqrt{1-x^2} > 0$, while on the lower bank $\sqrt{1-x^2} < 0$. Hence,

$$\int_{\gamma_+} \frac{d\zeta}{\sqrt{1-\zeta^2}} = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi.$$

The final result is that when curve γ_+ is traversed once,

$$f_0(z) \to [-f_0(z) + \pi]. \tag{23.8}$$

Reasoning along the same lines, we find that when curve γ_- is traversed once,

$$f_0(z) \to [-f_0(z) - \pi]. \tag{23.9}$$

For instance, after curve γ_+ is traversed twice, $f_0(z) \to f_0(z)$.
(2) Now we continue $f_0(z)$ analytically along $\gamma_+\gamma_-$. We have

$$f_0(z) \to [f_0(z) + 2\pi]. \tag{23.10}$$

But if we continue $f_0(z)$ analytically along $\gamma_-\gamma_+$,

$$f_0(z) \to [f_0(z) - 2\pi].$$

This implies, for one, that $\gamma_-\gamma_+$ and $\gamma_+\gamma_-$ are not homotopic in the plane with deleted points $z = \pm 1$ (otherwise the analytic continuation of $f_0(z)$ along these two curves would lead to the same element; see the monodromy theorem of Sec. 24). Moreover, point $z = \infty$ is also a branch point of arcsin z, since we circuit this point when we traverse $\gamma_+\gamma_-^{-1}$ during analytic continuation. This branch point is of infinite multiplicity, since

$$f_0(z) \to f_0(z) + 2k\pi$$

under analytic continuation along the curve $(\gamma_+\gamma_-^{-1})^k$ $(k = \pm 1, \pm 2, \ldots)$. The analytic functions arccos z, arsinh z, and arcosh z are introduced in a similar manner. All these functions can be expressed in terms of the logarithm.

24 Regular Branches of Analytic Functions

24.1 The monodromy theorem We will now show that the concepts of a single-valued analytic function and of a regular function are identical. Suppose we have a function $F(z)$ that is regular in a domain D. At each point $z_0 \in D$ the element $f_{z_0}(z)$ is naturally fixed, i.e. it is the function $F(z)$ proper. We fix the point $z_0 \in D$ and the element $f_{z_0}(z)$ at this point. If a curve γ connects points z_0 and z and lies in D, then $f_{z_0}(z)$ in a quite obvious manner allows for an analytic continuation along γ, namely, we can take the function $F(z)$ as the element at a point $z^* \in \gamma$.

If $F(z)$ is analytic in D and is single-valued in D, then it is regular in D. Indeed, in a neighborhood of any point in D, the values of $F(z)$ coincide with those of a (unique) element, so that $F(z)$ is regular at every point of D.

The following theorem plays an exceptional role in the theory of multiple-valued analytic functions.

The monodromy theorem *Suppose D is a simply connected domain in the extended complex plane and suppose $f(z)$, the element of this function at a point z_0, allows for analytic continuation along all curves starting at z_0 and lying in D. Then the analytic function $F(z)$, which is the result of analytic continuation of $f(z)$ along all such curves, is regular in D.*

By the hypothesis of this theorem, $f(z)$ generates a function that is analytic in D. Hence an alternative formulation of the monodromy theorem:

A function that is analytic in a simply connected domain is regular in this domain.

Proof. Let γ_0 and γ_1 be curves given by the equations $z = \sigma_0(t)$ and $z = \sigma_1(t)$, $0 \leqslant t \leqslant 1$, lying in D, and connecting points z_0 and z_1. We wish to show that the analytic continuation of the element $f(z)$ along γ_0 and γ_1 results in the same element at z_1. We will then have proved that the analytic function $F(z)$ generated by the element $f(z)$ is single-valued in D, and by the above-made remark the function $F(z)$ is regular in D.

For the sake of simplicity we assume that D is a bounded domain. Since D is simply connected, γ_0 and γ_1 are homotopic in D, i.e. there is a function $\Phi(s, t)$ with the following properties (Sec. 3):

(1) The function $\Phi(s, t)$ is defined and continuous in the square K: $0 \leqslant s, t \leqslant 1$ and its values lie in D.

(2) $\Phi(s, 0) = z_0$ and $\Phi(s, 1) = z_1$ for $0 \leqslant s \leqslant 1$; $\Phi(0, t) = \sigma_0(t)$ and $\Phi(1, t) = \sigma_1(t)$.

For each fixed $s \in [0, 1]$ the equation $z = \Phi(s, t)$, $0 \leqslant t \leqslant 1$, determines a curve γ_s lying in D and connecting points z_0 and z_1. If the numbers $s, s' \in [0, 1]$ are near each other, then the distance between the curves γ_s and $\gamma_{s'}$ is small, a fact that follows from the definition of distance between curves, $\rho(\gamma_s, \gamma_{s'}) = \max\limits_{0 \leqslant t \leqslant 1} |\Phi(s, t) - \Phi(s', t)|$, and the uniform continuity of Φ in K. Hence, by Lemma 3 of Sec. 20, for every $s \in I = [0, 1]$ there is a positive $\delta(s)$ such that if s' lies in the interval $I_s = (s - \delta(s), s + \delta(s))$, then the analytic continuation of $f_0(z)$ along all such curves $\gamma_{s'}$ results in the same element at point z_1. By the Heine-Borel lemma, we can always select a finite number of intervals I_{s_j}, $0 = s_0 < s_1 < \ldots$ $\ldots < s_n = 1$ that cover the segment I and are such that the intervals I_{s_j} and $I_{s_{j+1}}$, $0 \leqslant j \leqslant n - 1$, have a nonempty intersection. If $s \in I_{s_0} \cap I_{s_1}$, the analytic continuation of $f(z)$ results in the same element at point z_1; the same is true for $s \in Z_{s_1} \cap I_{s_2}$; and so on. Continuing this line of reasoning, we find that the analytic continuation of the element $f(z)$ along any curve γ_s, $0 \leqslant s \leqslant 1$, results in the same element at point z_1.

The above proof results in the following corollary, which is known as the monodromy theorem, too.

The monodromy theorem (an alternative formulation) *Let the element $f(z)$ be fixed at a point z_0 and allow for analytic continuation along any curves that start at z_0 and lie in a domain D. If two curves, γ_0 and γ_1, start at point z_0 and are homotopic in D, the analytic continuation of $f(z)$ along γ_0 and γ_1 results in the same element.*

Domain D can be multiply connected.

24.2 Isolating regular branches Any element of an analytic function is said to be a *regular branch* of this function. The monodromy theorem gives a simple and convenient algorithm for isolating the branches of a multiple-valued analytic function. Precisely, suppose we have a function $F(z)$ that is analytic in a finitely connected domain D. We cut this domain in such a way that it becomes a simply connected domain $\tilde{D}$; this can be done in the same way as in Sec. 9 (Fig. 43). We fix the element $f_0(z)$ at a point $z_0 \in \tilde{D}$. By the monodromy theorem this element generates in $\tilde{D}$ a regular function $F_0(z)$ that is a regular branch of $F(z)$. Different elements at point z_0 generate different regular branches of $F(z)$. In this way $F(z)$ splits in $\tilde{D}$ into regular branches. Note that the cuts that make D a simply connected domain can be made in different ways, which means that a multiple-valued analytic function can be "cut" into regular branches in different ways.

Here are some examples illustrating the process of splitting a multiple-valued function into regular branches.

Example 1. Let us cut the complex z plane along a simple curve γ connecting points 0 and ∞. The resulting domain D is simply connected. Let us show that the function $\ln z$ splits in D into regular branches. We fix point $z_0 \in D$ and the element $f(z)$ of the logarithm at this point. Since this element allows for analytic continuation along any curve not passing through 0 and ∞, the monodromy theorem implies that this element generates in D a regular branch of n z. In D the function $\ln z$ splits into an infinite number of regular branches. If $f_1(z)$ and $f_2(z)$ are two regular branches of the logarithm, then $f_1(z) - f_2(z) \equiv 2k\pi i$ at $z \in D$, with k an integer (see Sec. 21.2).

Similarly, the function z^α splits in D into regular branches. If α is a real, irrational number or if $\operatorname{Im} \alpha \neq 0$, there is an infinitude of regular branches. And if $\alpha = p/q$, where p and q are coprime numbers and $q \geqslant 1$, then there are exactly q different regular branches (Sec. 22). For example, the function $\sqrt[n]{z}$ splits in D into n regular branches.

If $f_1(z)$ and $f_2(z)$ are two different regular branches of z^α in D, then $f_2(z) \equiv e^{2k\pi i\alpha} f_1(z)$, with k nonzero integer (see Property 2 in Sec. 22). $\square$

Example 2. Let $f(z)$ be the regular branch of the function $\ln z$ in D

(Fig. 31) on which $f(1) = 0$. We wish to calculate the values of $f(-2)$, $f(3)$, and $f(-4)$. By Eq. (21.5), at $z \in D$,

$$f(z) = \ln |z| + i\Delta_\gamma \arg z,$$

where the curve γ connects the points 1 and z and lies in D. Hence,

$$f(-2) = \ln 2 + \pi i, \; f(3) = \ln 3 + 2\pi i,$$

$$f(-4) = \ln 4 + 3\pi i. \; \square$$

Example 3. Let us expand the regular function $f(z)$ of Example 2 in a Taylor series in powers of $z - 3$. Equation 21.10 then yields

$$f(z) = \ln 3 + 2\pi i + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(z-3)^k}{k3^k}. \; \square$$

Example 4. No regular branches of the functions $\ln z$ and $\sqrt[n]{z}$ $(n > 1)$ can be isolated in D: $0 < |z| < \infty$. This fact follows from Property 5 in Sec. 21 and Property 6 in Sec. 22. $\square$

Example 5. Let D_α be the complex z plane with a cut along the ray $z = re^{i\alpha}$, $0 \leqslant r < \infty$, $0 < \alpha < 2\pi$. Then, by the monodromy theorem, the function $\sqrt{z}$ splits in D_α into two regular branches. We normalize the branch $f_\alpha(z)$ by the condition that $f_\alpha(1) = 1$ and calculate $f_\alpha(i)$ $(\alpha \neq \pi/2)$. We have

$$f_\alpha(i) = e^{i\varphi/2}, \; \varphi = \Delta_\gamma \arg z,$$

where γ lies in D and connects points 1 and i.

(1) $\pi/2 < \alpha < 2\pi$. In this case the segment $[1, i]$ can be taken as γ, so that $\varphi = \pi/2$ and $f_\alpha(i) = e^{i\pi/4}$.

(2) $0 < \alpha < \pi/2$. In this case the arc $z = e^{-it}$, $0 \leqslant t \leqslant 3\pi/2$, can be taken as γ, so that

$$\varphi = -3\pi/2, \; f_\alpha(i) = -e^{i\pi/4}. \; \square$$

Before we proceed with the other examples, let us explain the meaning of the expressions $\ln f(z)$, $\sqrt{f(z)}$, and $(f(z))^\alpha$, where $f(z)$ is a regular function. The formula $F(z) = \ln f(z)$ cannot completely define an analytic function since we must fix the initial element or, in view of the properties of the logarithm, the value of the function at a point (see also Example 4 in Sec. 22). On the other hand,

$$F(z) = \ln f(z), \quad F(z_0) = w_0,$$

where w_0 is one of the values of $\ln f(z_0)$, completely defines the analytic function $F(z)$.

Example 6. Suppose we have a function $f(z)$ that is regular and nonzero in a simply connected domain D. Then the function

$$F(z) = \ln f(z), \quad F(z_0) = w_0,$$

is regular in D. Here $e^{w_0} = f(z_0)$.

Indeed, the function $F(z)$ is analytic in D (Theorem 2 of Sec. 22), and by the monodromy theorem this function is regular in D. The function $F(z)$ is uniquely determined by the relationships

$$e^{F(z)} = f(z), \quad F(z_0) = w_0$$

and the condition that it is regular in D. Its values are given by the formula

$$F(z) = \ln |f(z)| + i[\operatorname{Im} w_0 + \Delta_\gamma \arg f(z)], \tag{24.1}$$

where curve γ lies in D and connects points z_0 and z. □

Example 7. Suppose we have a function $f(z)$ that is regular and nonzero in a simply connected domain D. Then the function

$$F(z) = \sqrt{f(z)}, \quad F(z_0) = w_0,$$

where $w_0^2 = f(z_0)$, is regular in D. □

Let us consider various functional relationships for the logarithm and the power function. Suppose $f_1(z)$ and $f_2(z)$ are two functions that are regular and nonzero in a simply connected domain D. Then the functions

$$\begin{aligned} F_1(z) &= \ln f_1(z), & F_1(z_0) &= w_1; \\ F_2(z) &= \ln f_2(z), & F_2(z_0) &= w_2; \\ F_0(z) &= \ln (f_1(z) f_2(z)), & F_0(z_0) &= w_0, \end{aligned}$$

are regular in D (see Example 6).

Lemma 1 *If* $w_0 = w_1 + w_2$, *then*

$$F_0(z) \equiv F_1(z) + F_2(z) \tag{24.2}$$

in D.

Proof. The function $F(z) = F_1(z) + F_2(z)$ is regular in D and $e^{F(z)} = f_1(z) f_2(z)$, $F(z_0) = w_0$. These conditions define a unique function that is regular in D (Example 6). Since $F_0(z)$ satisfies these conditions, too, i.e.

$$e^{F_0(z)} = f_1(z) f_2(z), \quad F_0(z_0) = w_0,$$

we conclude that $F(z) \equiv F_0(z)$ in D. The proof is complete.

We can write (24.2) formally as

$$\ln (f_1(z) f_2(z)) = \ln f_1(z) + \ln f_2(z), \quad z \in D. \tag{24.3}$$

The exact meaning of (24.3) was discussed in Lemma 1.

The following relationships can be proved valid in a similar manner:

$$\ln \frac{f_1(z)}{f_2(z)} = \ln f_1(z) - \ln f_2(z),$$
$$\sqrt{f_1(z) f_2(z)} = \sqrt{f_1(z)}\,\sqrt{f_2(z)}, \qquad (24.4)$$
$$\sqrt{\frac{f_1(z)}{f_2(z)}} = \frac{\sqrt{f_1(z)}}{\sqrt{f_2(z)}},$$
$$(f_1(z) f_2(z))^\alpha = (f_1(z))^\alpha (f_2(z))^\alpha.$$

Here $f_1(z)$ and $f_2(z)$ are two functions that are regular and nonzero in a simply connected domain D and all z's belong to D.

The exact meaning of Eqs. (24.4) is as follows. On the right- and left-hand sides of all the relationships in (24.4) the functions are regular in D, which means that in each relationship the values of the right- and left-hand sides coincide at a point $z_0 \in D$. Take, for example, the second relationship. It should be interpreted in the following way:

$$F_0(z) = F_1(z) F_2(z),$$

where

$$F_0(z) = \sqrt{f_1(z) f_2(z)}, \quad F_0(z_0) = w_0;$$
$$F_j(z) = \sqrt{f_j(z)}, \quad F_j(z_0) = w_j, \quad j = 1, 2.$$

and $w_0 = w_1 w_2$.

Formulas (24.3) and (24.4) lead to important relationships for the argument of the product and quotient of two functions.

Corollary *Suppose $f_1(z)$ and $f_2(z)$ are regular and nonzero in a domain D and curve γ lies in D. Then*

$$\Delta_\gamma \arg (f_1(z) f_2(z)) = \Delta_\gamma \arg f_1(z) + \Delta_\gamma \arg f_2(z), \qquad (24.5)$$

$$\Delta_\gamma \arg \left(\frac{f_1(z)}{f_2(z)}\right) = \Delta_\gamma \arg f_1(z) - \Delta_\gamma \arg f_2(z). \qquad (24.6)$$

Proof. Let us prove the validity of (24.5). Let D be simply connected, γ connect points z_0 and z, and $F_j(z)$, $j = 0, 1, 2$, be the same functions as in Lemma 1. Then at $z \in D$ we have

$$F_j(z) = \ln |f_j(z)| + i\,[\mathrm{Im}\, w_j + \Delta_\gamma \arg f_j(z)], \quad j = 1, 2,$$

$$F_0(z) = \ln |f_1(z) f_2(z)| + i\,[\mathrm{Im}\, w_0 + \Delta_\gamma \arg (f_1(z) f_2(z))].$$

Substituting these expressions into (24.2) and bearing in mind that $w_0 = w_1 + w_2$, we arrive at (24.5).

Now suppose D is a multiply connected domain. We cover γ with a finite number of circles $K_0, K_1, \ldots, K_n$ (all lying in D) whose centers $z_0, z_1, \ldots, z_n$ lie on γ in an ordered fashion, with the z_0 the initial point and z_n the terminal point. We divide γ into arcs γ_0, γ_1, . . ., γ_n, where arc γ_j lies inside K_j, so that $\gamma = \gamma_0\gamma_1 \ldots$ $\ldots \gamma_{n-1}\gamma_n$. Then (24.5) holds for each γ_j, as proved earlier. Since the variation of the argument along γ is equal to the sum of the variations of the argument along γ_0, . . ., γ_n, we find that (24.5) is indeed valid.

Formula (24.6) is proved similarly, with the help of the first formula in (24.4).

Since the fact that two analytic functions are equal means that their initial elements are equal, formulas of the type (24.2) and (24.3) are valid for analytic functions, too. Let us assume, for the sake of simplicity, that the functions $f_1(z)$ and $f_2(z)$ are regular and nonzero in D. We select a point $z_0 \in D$. Then (see Example 7) the functions

$$g_j(z) = \sqrt{f_j(z)}, \qquad g_j(z_0) = w_j, \quad j = 1, 2,$$

$$g_0(z) = \sqrt{f_1(z) f_2(z)}, \quad g_0(z_0) = w_0,$$

are regular in a neighborhood U of this point. Suppose the w_j are such that $w_0 = w_1w_2$. Then, as we have just proved,

$$g_0(z) \equiv g_1(z)\, g_2(z), \quad z \in U. \tag{24.7}$$

If $F_j(z)$ is the analytic function generated by the element $g_j(z)$ fixed at point z_0, $j = 0, 1, 2$, then, by virtue of (24.7),

$$F_0(z) = F_1(z)\, F_2(z), \tag{24.8}$$

where the equality is understood in the sense of that of two analytic functions. We can also write (24.8) as

$$\sqrt{f_1(z) f_2(z)} = \sqrt{f_1(z)}\sqrt{f_2(z)};$$

the exact meaning of this relationship has been explained above. Formula (24.3) and the remaining formulas in (24.4) can be interpreted in a similar manner.

24.3 Regular branches of analytic functions in multiply connected domains The monodromy theorem does not say how to separate regular branches of a function $F(z)$ in a multiply connected domain D. Here is a way of solving this problem. Suppose $f(z)$ is an element of $F(z)$ at a point $z_0 \in D$ and γ is a closed curve lying in D and starting at z_0. If we continue $f(z)$ along γ analytically, we arrive at element $g(z)$ at point z_0. Briefly this fact can be put as follows:

$$f(z) \rightarrow g(z)$$

as curve γ is traversed.

Lemma 2 *If*

$$f(z) \rightarrow f(z)$$

as any arbitrary curve γ *in* D *is traversed, then element* $f(z)$ *generates a branch of* $F(z)$ *that is regular in* D. *In other words, there exists a function* $F_0(z)$ *that is regular in* D *and*

$$F_0(z) \equiv f(z)$$

in a neighborhood of z_0.

This lemma is quite obvious intuitively. A rigorous proof can be found in Shabat [1].

But if the traversal of a closed curve γ in D leads to

$$f(z) \to g(z),$$

with $g(z) \neq f(z)$, the function $F(z)$ does not allow for isolation of a regular branch in D.

This is the procedure of isolation of regular branches in the general case. It can be considerably simplified when we are dealing with such analytic functions as $\sqrt{f(z)}$, $\ln f(z)$, and $(f(z))^\alpha$, where $f(z)$ is a function that is regular in a domain D. This follows from the fact that these functions have the same property as $\ln z$ and z^α (see Secs. 21 and 22); precisely, any element of $\ln f(z)$ (and $(f(z))^\alpha$) at any point is completely defined by its value at the point. For this reason we have

Lemma 3 *Suppose a function* $f(z)$ *is regular and nonzero in a domain* D, *and the analytic function* $F(z) = \ln f(z)$ *is generated by the element* $F_0(z)$ *at a point* $z_0 \in D$. *If all values of* $F(z_0)$ *obtained as a result of traversal of all closed curves* γ *lying in* D *and starting at* z_0 *coincide with* $F_0(z_0)$, *then the analytic function* $F(z)$ *is regular in* D.

The same proposition is valid for $(f(z))^\alpha$.

Proof. We start with the first proposition. Suppose γ is a closed curve that lies in D and starts at z_0. Then $F_0(z) \to F_1(z)$ when γ is traversed (here $F_1(z)$ is the element at point z_0). By hypothesis, $F_1(z_0) = F_0(z_0)$. Since any element of $F(z) = \ln f(z)$ at point z_0 is uniquely defined by its value at this point, we find that $F_1(z) \equiv F_0(z)$. By Lemma 2 the element $F_0(z)$ generates a function that is regular in D. The second proposition can be proved in a similar manner.

Example 8. Let us show that the analytic function $F(z) = \sqrt{z^2 - 1}$ splits into two regular branches in a domain D_0 that is the complex z plane cut along the segments $(-\infty, -1]$ and $[1, +\infty)$.

Since $z^2 - 1 \neq 0$ in D_0, by Theorem 2 of Sec. 22 we can say that $F(z)$ is analytic in D_0. Since this domain is simply connected, the monodromy theorem states that $F(z)$ splits in D_0 into regular branches. A regular branch is completely specified if we fix its value at a point $z_0 \in D_0$:

$$F_0(z) = \sqrt{z^2 - 1}, \quad F_0(z_0) = w_0,$$

where $w_0^2 = z_0^2 - 1$ (i.e. w_0 is one of the values of $\sqrt{z_0^2 - 1}$). There are exactly two regular branches of $F(z)$ in D_0, and they are related thus: $F_1(z) \equiv -F_0(z)$, $z \in D_0$.

Similarly, the function $G(z) = \ln \frac{z-1}{z+1}$ split in D_0 into regular branches; there is an infinite number of such branches, each of which is uniquely specified by its value at a point $z_0 \in D$. For instance, one way to describe all the branches $G_k(z)$ is to write

$$G_0(z) = \ln \frac{z-1}{z+1}, \qquad G_0(0) = \pi i,$$

$$G_k(z) \equiv G_0(z) + 2k\pi i$$

$$(k = 0, \pm 1, \pm 2, \ldots). \ \square$$

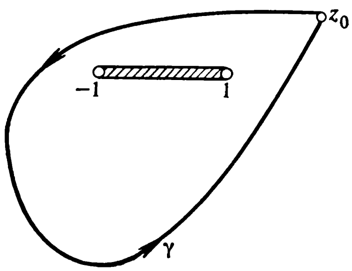


Fig. 64

Example 9. Suppose D is the complex z plane (not in the extended sense) with a cut along $[-1, 1]$ (Fig. 64). Let us show that the analytic function $F(z) = \sqrt{z^2 - 1}$ splits in D into two regular branches. Note that D is multiply connected.

Proof 1. Suppose the initial element $F_0(z)$ of $F(z)$ is fixed at a point z_0 and let γ be a simple closed curve starting at z_0 and lying in D. The value of $F(z_0)$ obtained as a result of traversal of γ is

$$F(z_0) = F_0(z_0)\, e^{i\varphi/2},$$

where $\varphi = \Delta_\gamma \arg(z^2 - 1)$. Since $z^2 - 1 = (z-1)(z+1)$, we can write

$$\varphi = \varphi_1 + \varphi_2, \quad \varphi_1 = \Delta_\gamma \arg(z-1), \quad \varphi_2 = \Delta_\gamma \arg(z+1).$$

(1) If the segment $[-1, 1]$ does not lie inside γ, then $\varphi_1 = \varphi_2 = 0$, so that $F(z_0) = F_0(z_0)$.

(2) If the segment $[-1, 1]$ does lie inside γ (Fig. 64) and γ is oriented in the positive sense, then $\varphi_1 = \varphi_2 = 2\pi$, so that $\varphi = 4\pi$ and again $F(z_0) = F_0(z_0)$. If γ is oriented in the negative sense, then $\varphi = -4\pi$.

We have just proved that the function

$$f_0(z) = \sqrt{z^2 - 1}, \quad z \in D; \quad f_0(z_0) = w_0,$$

is regular in D (here w_0 is one of the values of $\sqrt{z_0^2 - 1}$). Similarly the function

$$f_1(z) = \sqrt{z^2 - 1}, \ z \in D; \quad f_1(z_0) = -w_0,$$

is regular in D and

$$f_1(z) \equiv -f_0(z), \quad z \in D,$$

so that the function $\sqrt{z^2-1}$ splits in D into two regular branches.

Proof 2. We write

$$F(z) = zG(z), \quad G(z) = \sqrt{1-\frac{1}{z^2}},$$

where $G(z)$ is fixed by the condition that

$$G_0(z_0) = \frac{F_0(z_0)}{z_0},$$

with $F_0(z)$ and $G_0(z)$ the initial elements of $F(z)$ and $G(z)$ at point z_0. The function z is regular in D. The function $G(z)$ is analytic in the simply connected domain $\tilde{D} = D \cup \{z = \infty\}$ on Riemann's sphere. According to the monodromy theorem, the function $G(z)$ is regular in $\tilde{D}$ and, hence, in D. Consequently, the function

$$F(z) = zG(z), \quad z \in D; \quad F(z_0) = F_0(z_0),$$

is regular in D. □

Example 10. Suppose D is the complex z plane with a cut along the segment $[-1, 1]$ and $f(z)$ is the branch of $\sqrt{z^2-1}$ that is regular in D and such that $f(2) = \sqrt{3}$. We wish to calculate the values of this function for real $z = x$. If $z \in D$, then

$$f(z) = \left|\sqrt{z^2-1}\right| e^{i(\varphi_1+\varphi_2)/2},$$
$$\varphi_1 = \Delta_\gamma \arg(z-1),$$
$$\varphi_2 = \Delta_\gamma \arg(z+1),$$

where curve γ connects points 2 and z and lies in D.

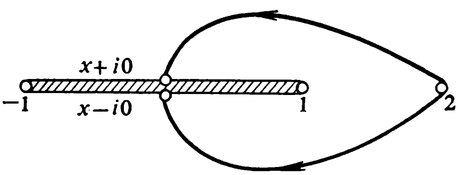


Fig. 65

(1) Let x be greater than unity. Then we can take the segment $[2, x]$ as γ, so that $\varphi_1 = \varphi_2 = 0$ and $f(x) = \sqrt{x^2-1}$. (Here and below all the roots are assumed to be arithmetic roots.)

(2) Now suppose that point $x \in (-1, 1)$ and lies on the upper bank of the cut. Then we can take a curve that lies in the upper half-plane and connects points 2 and x as γ (Fig. 65). Then $\varphi_1 = \pi$ and $\varphi_2 = 0$, so that $f(x+i0) = i\sqrt{1-x^2}$.

(3) Now let x be less than -1. Then any curve that connects points 2 and x and lies in the upper half-plane can be taken as γ. We have $\varphi_1 = \varphi_2 = +\pi$, so that $f(x) = -\sqrt{x^2-1}$.

(4) Finally, suppose x lies on the lower bank of the cut, then a curve lying in the lower half-plane and connecting points 2 and x can be taken as γ (Fig. 65). We have $\varphi_1 = -\pi$ and $\varphi_2 = 0$, so that $f(x-i0) = -i\sqrt{1-x^2}$.

Suppose that $z = iy$ with $y > 0$. Let us take the segment $[2, iy]$

as γ. Then $\varphi_1 + \varphi_2 = \pi$, so that $f(iy) = i\sqrt{y^2+1}$ $(y > 0)$. Similarly, $f(-iy) = -i\sqrt{y^2+1}$ $(y > 0)$.

Let us calculate $f'(z)$. Formula (22.3) yields

$$f'(z) = \frac{z}{\sqrt{z^2-1}} = \frac{z}{f(z)}. \quad \square$$

Example 11. Suppose D and $f(z)$ are the same as in the previous example. Let us expand $f(z)$ in a Laurent series in a neighborhood of point $z = \infty$.

The identity $\sqrt{z^2-1} = z\sqrt{1-1/z^2}$ implies that $f(z) = zg(z)$, with $g(z)$ a branch of $\sqrt{1-1/z^2}$ that is regular in D. Since $f(z) > 0$ at $x > 1$ (see Example 10), we have $g(x) > 0$ at $x > 1$, so that $\lim_{x\to+\infty} g(x) = 1$. Hence, $g(\infty) = 1$ and the sought-for expansion has the form

$$f(z) = z\sum_{n=0}^{\infty} \binom{1/2}{n} (-1)^n z^{-2n}. \quad \square$$

Example 12. Let us show that the analytic function

$$F(z) = \ln\frac{1-z}{1+z}$$

splits into regular branches in the domain D (Fig. 64), where D is the complex z plane cut along the segment $[-1, 1]$.

Proof 1. Suppose $F_0(z)$ is the initial element of $F(z)$ at a point z_0, and γ is a simple closed curve lying in D and starting at z_0. The value $F(z_0)$, obtained as a result of traversal of γ, is

$$F(z_0) = F_0(z_0) + i\Delta_\gamma \arg\frac{1-z}{1+z}.$$

Moreover,

$$\Delta_\gamma \arg\frac{1-z}{1+z} = \varphi_1 - \varphi_2, \quad \varphi_1 = \Delta_\gamma \arg(1-z),$$
$$\varphi_2 = \Delta_\gamma \arg(1+z).$$

If the segment $[-1, 1]$ does not lie inside γ, then $\varphi_1 = \varphi_2 = 0$. But if the segment $[-1, 1]$ does lie inside γ and the curve is oriented in the positive sense, $\varphi_1 = 2\pi$ and $\varphi_2 = 2\pi$, so that again $\varphi_1 - \varphi_2 = 0$. Hence $F(z_0) = F_0(z_0)$ and the function

$$f(z) = \ln\frac{1-z}{1+z}, \quad z \in D; \; f(z_0) = F_0(z_0),$$

is regular in D.

The number of branches into which $F(z)$ splits in D is countable.

A single formula can be given to describe these branches:

$$f_k(z) = \ln\left|\frac{1-z}{1+z}\right| + i\Delta_\gamma \arg\frac{1-z}{1+z} + i\,\mathrm{Im}\,w_0 + 2k\pi i,$$

$$k = 0,\ \pm 1,\ \pm 2, \ldots.$$

Here $w_0 = \ln\frac{z_0+1}{z_0-1}$ is a fixed value of the logarithm, and the curve γ connects the points z_0 and z and lies in D.

Proof 2. The function

$$F(z) = \ln\frac{-1+\frac{1}{z}}{1+\frac{1}{z}}, \quad F(z_0) = F_0(z_0),$$

is analytic in the domain $\widetilde{D} = D \cup \{z = \infty\}$. This domain is simply connected; hence, by the monodromy theorem, $F(z)$ is regular in $\widetilde{D}$ and, therefore, in D. $\square$

Example 13. Let D be the complex z plane with a cut along the segment $[-1, 1]$ and $f(z)$ is the regular (in D) branch of $\ln\frac{1-z}{1+z}$ on which $f(0 + i0) = 0$ (i.e. the value of $f(z)$ at the point $z = 0$ on the upper bank of the cut is zero). We wish to calculate the values of $f(z)$ on the real and imaginary axes.

In D,

$$f(z) = \ln\left|\frac{1-z}{1+z}\right| + i(\varphi_1 - \varphi_2),$$

$$\varphi_1 + \Delta_\gamma \arg(1-z), \quad \varphi_2 = \Delta_\gamma \arg(1+z),$$

with curve γ lying in D and connecting point 0 (on the upper bank) with point z.

(1) Suppose point $z = x \in (-1, 1)$ and lies on the upper bank. Then $\varphi_1 = \varphi_2 = 0$, so that

$$f(x + i0) = \ln\frac{1-x}{1+x}$$

(this is a real number).

(2) Now let $z = x$ be greater than unity. Then $\varphi_1 = -\pi$ and $\varphi_2 = 0$, so that

$$f(x) = \ln\frac{x-1}{x+1} - i\pi.$$

If $z = x < -1$, then $\varphi_1 = 0$ and $\varphi_2 = \pi$, so that

$$f(x) = \ln\frac{x-1}{x+1} - i\pi.$$

(3) Suppose point $z = x \in (-1, 1)$ and lies on the lower bank of

the cut. Then $\varphi_1 = -2\pi$ and $\varphi_2 = 0$, so that

$$f(x-i0) = \ln \frac{1-x}{1+x} - 2\pi i.$$

(4) Suppose $z = iy$ with $y > 0$. Then $\varphi_1 = -\varphi_2$, with $\varphi_2 = \arctan y$, so that at $y > 0$ we have

$$f(iy) = -2i \arctan y,$$

since $|1 - iy| = |1 + iy|$. Similarly, $f(iy) = -2i(\pi + \arctan y)$ at $y < 0$. □

Example 14. We take D and $f(z)$ the same as in the previous example. Let us expand $f(z)$ in a Laurent series in a neighborhood of point $z = \infty$.

At $y > 0$ we have (see Example 13, case (4)) $f(iy) = -2i \arctan y$, so that

$$f(\infty) = \lim_{y\to+\infty} f(iy) = -\pi i.$$

Hence, in a neighborhood of point $z = \infty$ we have

$$f(z) = \ln \frac{-1+(1/z)}{1+(1/z)} = -\pi i + \ln\left(1 - \frac{1}{z}\right) - \ln\left(1 + \frac{1}{z}\right),$$

where the logarithms on the right-hand side are regular functions at point $z = \infty$ and vanish at this point. Expanding these functions in Laurent series, we obtain

$$f(z) = -\pi i - \sum_{n=1}^{\infty} \frac{1}{nz^n} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{nz^n} = -\pi i - \sum_{n=0}^{\infty} \frac{2}{(2n+1) z^{2n+1}}.$$

This last series converges in the annulus $1 < |z| < \infty$. □

Let a and b be two complex numbers, $a \neq b$, and D the complex z plane with a cut along the segment $[a, b]$. We can show, by reasoning along the same lines as in Examples 9 and 12, that the analytic functions

$$F(z) = \sqrt{(z-a)(z-b)}, \quad G(z) = \ln \frac{z-a}{z-b}$$

split in D into regular branches.

24.4 Regular branches of analytic functions in multiply connected domains (continued)

Example 15. Let D be the complex z plane with a cut along the segment $[-1, 1]$ (Fig. 64). We wish to show that for real α's the function

$$F(z) = \left(\frac{1-z}{1+z}\right)^{\alpha}$$

splits in D into regular branches.

Let the initial element $F_0(z)$ of $F(z)$ be fixed at a point z_0 and suppose that γ is a simple closed curve starting at z_0 and lying in D. The value $F(z_0)$ obtained as a result of traversal of γ is

$$F(z_0) = F_0(z_0)\, e^{i\alpha\varphi}, \quad \varphi = \Delta_\gamma \arg \frac{1-z}{1+z}.$$

Just as in Example 12, we have $\varphi = 0$, so that $F(z_0) = F_0(z_0)$, and, according to Lemma 3, the function

$$f(z) = \left(\frac{1-z}{1+z}\right)^\alpha, \quad z \in D; \quad f_0(z_0) = F_0(z_0),$$

is regular in D. Two different branches of $F(z)$ in D differ by the factor $e^{i2\pi\alpha k}$, with k an integer. □

Example 16. Suppose D is the complex z plane with a cut along the segment $[-1, 1]$, and $f(z)$ is the regular (in D) branch of $\left(\frac{1-z}{1+z}\right)^\alpha$, with α a real number, on which $f(0+i0) = 1$ (i.e. $f(z_0) = 1$ at point $z_0 = 0$ on the upper bank of the cut). Let us calculate the values of $f(z)$ on the real axis. At $z \in D$ we have

$$f(z) = \left|\frac{1-z}{1+z}\right|^\alpha e^{i\alpha\varphi}, \quad \varphi = \varphi_1 - \varphi_2,$$

where $\varphi_1 = \Delta_\gamma \arg(1-z)$ and $\varphi_2 = \Delta_\gamma \arg(1+z)$, and γ is a curve that connects points $0+i0$ and z and lies in D. The numbers φ_1 and φ_2 are calculated in the same way as in Example 13.

(1) If point $z = x \in (-1, 1)$ and lies on the upper bank, then $\varphi_1 = \varphi_2 = 0$, so that

$$f(x+i0) = \left(\frac{1-x}{1+x}\right)^\alpha > 0.$$

(2) If $z = x > 1$, then $\varphi_1 = -\pi$ and $\varphi_2 = 0$, so that

$$f(x) = \left(\frac{x-1}{x+1}\right)^\alpha e^{-i\alpha\pi},$$

But if $z = x < -1$, then $\varphi_1 = 0$ and $\varphi_2 = \pi$, so that

$$f(x) = \left(\frac{x-1}{x+1}\right)^\alpha e^{-i\alpha\pi}.$$

In these formulas $\left(\frac{x-1}{x+1}\right)^\alpha$ is positive.

(3) If point $z = x \in (-1, 1)$ and lies on the lower bank, then $\varphi_1 = -2\pi$ and $\varphi_2 = 0$, so that

$$f(x-i0) = \left(\frac{1-x}{1+x}\right)^\alpha e^{-i2\alpha\pi}. \ \square$$

Example 17. Both D and $f(z)$ are the same as in Example 16. We wish to calculate the first two terms in the Laurent expansion

Suppose the initial element $F_0(z)$ of $F(z)$ is fixed at a point $z_0 \in D$ and γ is a simple closed curve starting at z_0 and lying in D. The traversal of γ leads to the following:

$$F(z_0) = F_0(z_0)\, e^{i\varphi/2}, \quad \varphi = \Delta_\gamma \arg [(z^2-1)(z^2-4)].$$

Moreover,

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4, \quad \varphi_j = \Delta_\gamma \arg (z - z_j),$$

where $z_1 = -2$, $z_2 = -1$, $z_3 = 1$, and $z_4 = 2$. If the cuts do not lie inside γ, then all the φ_j vanish, so that $F(z_0) = F_0(z_0)$. If only

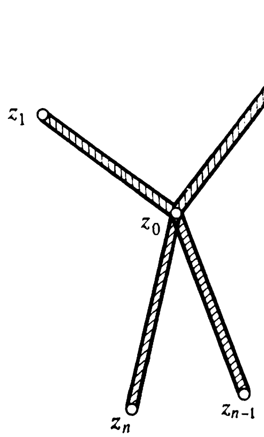


Fig. 66

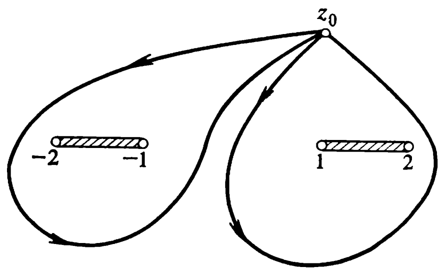


Fig. 67

the segment $[-2, -1]$ lies inside γ, then $\varphi_3 = \varphi_4 = 0$ and $\varphi_1 = \varphi_2 = 2\pi$ (if γ is oriented in the positive sense), so that $\varphi = 4\pi$ and again $F(z_0) = F_0(z_0)$. In the same manner we can prove that $F(z_0) = F_0(z_0)$ if the segment $[1, 2]$ lies inside γ. By Lemma 3, the function

$$f(z) = \sqrt{(z^2-1)(z^2-4)}, \quad z \in D; \quad f(z_0) = F_0(z_0),$$

is regular in D. □

Example 22. Suppose $z_1, z_2, \ldots, z_{2n}$ are different complex numbers and

$$F(z) = \sqrt{(z-z_1)(z-z_2)\ldots(z-z_{2n})}.$$

We cut the complex z plane along the simple nonconcurrent curves l_j connecting the points z_{2j-1} and z_{2j}, $1 \leqslant j \leqslant n$. Then in the resulting domain the function $F(z)$ splits into two regular branches. (The proof is the same as in Example 21.) □

Example 23. Suppose $P_n(z)$ is a polynomial of degree n, i.e.

$$P_n(z) = a(z-z_1)(z-z_2)\ldots(z-z_n), \quad a \neq 0,$$

whose zeros lie in the circle $|z| < R$. At what integral values of m can a regular branch of

$$F(z) = \sqrt[m]{P_n(z)}$$

be isolated in the domain D: $R < |z| < \infty$?

Suppose the initial element $F_0(z)$ of $F(z)$ is specified at a point z_0 and γ is a simple closed curve starting at z_0 and lying in D. Traversal of γ yields

$$F(z_0) = F_0(z_0) e^{i\varphi/m}, \quad \varphi = \varphi_1 + \varphi_2 + \ldots + \varphi_n,$$
$$\varphi_j = \Delta_\gamma \arg(z - z_j).$$

If the interior of γ lies in D, then all the φ_j vanish. If γ is oriented in the positive sense and contains the circle $|z| < R$, then all the $\varphi_j = 2\pi$, so that $\varphi = 2n\pi$.

Hence, the function allows for isolation of a regular branch in D if and only if $e^{i2n\pi/m} = 1$, i.e. m must be a divisor of n. □

Example 24. The function

$$F(z) = \ln \frac{(z-1)(z+4)}{(z-3)(z+2)}, \quad F(0) = w_0,$$

where w_0 is one of the values of $\ln z|_{z=2/3}$, is analytic in D_0, which is the extended complex z plane with points -4, -2, 1, and 3 deleted. Indeed, the function

$$f(z) = \frac{(z-1)(z+4)}{(z-3)(z+2)}$$

is regular and nonzero in D_0, so that the function

$$F(z) = \ln f(z), \quad F(0) = w_0,$$

is analytic in D_0, according to Theorem 2 of Sec. 22. Here are examples of domains in which a regular branch of $F(z)$ can be isolated.

(1) We cut the complex z plane along nonconcurrent rays starting at points -4, -2, 1, and 3. The resulting domain D_1 is simply connected, and by the monodromy theorem the function $F(z)$ splits into regular branches in D_1 (see also Example 6).

(2) We cut the complex z plane along the segments $(-\infty, -4)$, $[-2, 1]$, and $[3, \infty)$. Let us show that in the resulting domain D_2 the function $F(z)$ splits into regular branches. Indeed, if γ is a simple closed curve that lies in D_2 and contains in its interior the segment $[-2, 1]$, then

$$\Delta_\gamma \arg f(z) = \varphi_1 + \varphi_2,$$

$$\varphi_1 = \Delta_\gamma \arg \frac{z+4}{z-3}, \quad \varphi_2 = \Delta_\gamma \arg \frac{z-1}{z+2}.$$

It is obvious that $\varphi_1 = 0$, since $z + 4 \neq 0$ and $z - 3 \neq 0$ in the interior of γ. Also, $\varphi_2 = 0$, a fact that can be proved in the same way of $f(z)$ in a neighborhood of point $z = \infty$. We have

$$f(\infty) = \operatorname*{mil}_{x \to +\infty} f(x) = e^{-i\alpha\pi}$$

(see Example 16, case (2)). Moreover,

$$f(z) = \left(\frac{-1 + \frac{1}{z}}{1 + \frac{1}{z}} \right)^{\alpha} = e^{-i\alpha\pi} g(z), \quad g(z) = \left(\frac{1 - \frac{1}{z}}{1 + \frac{1}{z}} \right)^{\alpha},$$

where $g(z)$ is regular at point $z = \infty$ and is equal to unity at this point. Expanding $g(z)$ in a Laurent series in powers of $1/z$, we obtain

$$f(z) = e^{-i\alpha\pi} \left(1 - \frac{2\alpha}{z} + \ldots \right). \; \square$$

Example 18. Let D be the complex z plane with a cut along the segment $[0, 1]$. The analytic function

$$F(z) = \sqrt[3]{\frac{z}{1-z}}$$

splits in D into three regular branches. This can be proved in the same manner as we did in Example 15.

Let $f(z)$ be a regular branch of $F(z)$ in D that is positive on the upper bank of the cut. We expand $f(z)$ in a Laurent series in a neighborhood of point $z = \infty$. At $z \in D$ we have

$$f(z) = \left| \sqrt[3]{\frac{z}{1-z}} \right| e^{i\varphi/3}, \quad \varphi = \varphi_1 - \varphi_2,$$

where $\varphi_1 = \Delta_\gamma \arg z$ and $\varphi_2 = \Delta_\gamma \arg (1 - z)$, and curve γ lies in D and connects point $z_0 = 1/2 + i0$ (on the upper bank) with point z. Suppose $z = x$ is greater than unity. Then $\varphi_1 = 0$ and $\varphi_2 = -\pi$, so that

$$f(x) = \sqrt[3]{\frac{x}{x-1}}\, e^{i\pi/3}.$$

Hence,

$$f(\infty) = \lim_{x \to +\infty} f(x) = e^{i\pi/3}.$$

In a neighborhood of point $z = \infty$ we have

$$f(z) = e^{i\pi/3} \sqrt[3]{\frac{1}{1 - 1/z}} = e^{i\pi/3} \left(1 - \frac{1}{z} \right)^{-1/3},$$

where the value of the root at $z = \infty$ is unity. Hence, the sought-for expansion is

$$f(z) = e^{i\pi/3} \sum_{n=0}^{\infty} \binom{-1/3}{n} (-1)^n z^{-n}. \; \square$$

Example 19. Suppose $P_n(z)$ is a polynomial of degree n:

$$P_n(z) = a(z - z_1)(z - z_2)\ldots(z - z_n), \quad a \neq 0,$$

where $z_1, \ldots, z_n$ are different complex numbers, and $R > \max\limits_{1\leqslant k\leqslant n} |z_k|$, i.e. all the zeros of $P_n(z)$ lie within the circle $|z| < R$. We wish to show that the analytic function

$$F(z) = \sqrt[n]{P_n(z)}$$

splits in the annulus D: $R < |z| < \infty$ into n regular branches.

Suppose that the initial element $F_0(z)$ of $F(z)$ is fixed at a point z_0 and that γ is a simple closed curve starting at z_0 and lying in D. The value of the element obtained as a result of traversal of γ is

$$F(z_0) = F_0(z_0)\, e^{i\varphi/n}, \quad \varphi = \Delta_\gamma \arg P_n(z).$$

Moreover,

$$\varphi = \varphi_1 + \varphi_2 + \ldots + \varphi_n, \quad \varphi_j = \Delta_\gamma \arg(z - z_j).$$

If the interior of γ lies in D, then all the φ_j vanish, so that $\varphi = 0$ and $F(z_0) = F_0(z_0)$. If the circle $|z| < R$ lies inside γ and this curve is oriented counterclockwise, then all the $\varphi_j = 2\pi$, so that $\varphi/n = 2\pi$ and again $F(z_0) = F_0(z_0)$. By Lemma 3, $F(z)$ splits in D into regular branches. If $f_0(z)$ is one such branch, the other regular branches are

$$f_k(z) = e^{i2\pi k/n} f_0(z), \; k = 1, 2, \ldots, n-1$$

Note that

$$f(z) = \sqrt{P_n(z)} \sim \sqrt[n]{a}\, z \quad (z \to \infty),$$

where $\sqrt[n]{a}$ is a value of the root (different on each branch).

Here is another way of proving this fact. We have

$$F(z) = zG(z), \quad G(z) = \sqrt[n]{a\left(1 - \frac{z_1}{z}\right)\ldots\left(1 - \frac{z_n}{z}\right)},$$

where $z_0 G(z_0) = F_0(z_0)$. The function $G(z)$ is analytic in the simply connected domain $\widetilde{D} = D \cup \{z = \infty\}$ and, by the monodromy theorem, is regular in this domain. Since $\widetilde{D} \supset D$, we conclude that $G(z)$ is regular in D. □

Example 20. Let $F(z)$ be the function of Example 19. We fix point z_0 and connect it with segments to all the z_j, $1 \leqslant j \leqslant n$; let D be the exterior of the resulting "star" (Fig. 66). Reasoning along the same lines as in Example 19, we can show that $F(z)$ splits in D into n regular branches. □

Example 21. The function $F(z) = \sqrt{(z^2 - 1)(z^2 - 4)}$ splits into two regular branches in the domain D that is the complex z plane with cuts along the segments $[-2, -1]$ and $[1, 2]$ (Fig. 67).

as we did in Example 12. Hence, $\Delta_\gamma \arg f(z) = 0$. From this, just as in Example 12, it follows that $F(z)$ splits in D_2 into regular branches. In particular, this proposition is true for the annulus D_3: $3/2 < |z + 1/2| < 7/2$.

(3) We take the domain D_4, which is the complex z plane with cuts along the segments $[-4, -2]$ and $[1, 3]$. In this domain $F(z)$ also splits into regular branches. Indeed, if γ is a simple closed curve lying in D_4, then

$$\Delta_\gamma \arg f(z) = \Delta_\gamma \arg \frac{z+4}{z+2} + \Delta_\gamma \arg \frac{z-1}{z-3} \equiv \varphi_1 + \varphi_2.$$

If the cut along $[-4, -2]$ lies in the interior of γ and the cut along $[1, 3]$ in the exterior of γ, then, obviously, $\varphi_2 = 0$, while $\varphi_1 = 0$ in view of Example 12. The case where the cut along $[1, 3]$ lies inside γ can be considered in a similar manner.

For one, the function $F(z)$ splits into regular branches in the circle $|z + 1/2| < 3/2$ and in the domain $|z + 1/2| > 7/2$. □

Example 25. Suppose D is the complex z plane with a cut along the segment $[0, 1]$, and

$$f(z) = \sqrt[4]{z(1-z)^3}, \quad f(x+i0) > 0, \quad x \in (0, 1).$$

The function $f(z)$ is regular in D (see Example 19). Let us calculate $f(-1)$, $f'(-1)$, and $f''(-1)$. We have

$$f(z) = \sqrt{g(z)}, \quad g(z) = z(1-z)^3.$$

The reader will recall that $\frac{d}{dw}(w^\alpha) = \frac{\alpha w^\alpha}{w}$ (see Eq. 22.3')), where the values of w^α on both sides are the same. Hence,

$$f'(z) = \frac{1}{4} \frac{g'(z) f(z)}{g(z)},$$

$$f''(z) = \frac{1}{4} \frac{g''(z) f(z)}{g(z)} - \frac{3}{16} \frac{(g'(z))^2 f(z)}{g^2(z)}.$$

We select a curve γ that connects a point $x \in (0, 1)$ on the upper bank of the cut with point -1. Then

$$\Delta_\gamma \arg g(z) = \Delta_\gamma \arg z + 3\,\Delta_\gamma \arg(1-z) = \pi.$$

Hence,

$$f(-1) = e^{i\pi/4} \left| \sqrt[4]{g(-1)} \right| = e^{i\pi/4} \sqrt{8}.$$

Moreover, $g(-1) = -8$, $g'(-1) = 20$, and $g''(-1) = -36$, so that

$$f'(-1) = -\frac{5}{8} f(-1) = -\frac{5}{8^{3/4}} e^{i\pi/4},$$

$$f''(-1) = -\frac{3}{64} f(-1) = -\frac{3}{8^{7/4}} e^{i\pi/4}. \ \square$$

Example 26. Let us show that in D: $1 < |z| < \infty$ we cannot isolate a single regular branch of the function

$$F(z) = \ln (z + \sqrt{z^2 - 1}).$$

The function $G(z) = \sqrt{z^2 - 1}$ splits in D into two regular branches, $g_0(z)$ and $g_1(z)$ (see Example 9); for the sake of definiteness we will assume that $g_0(2) > 0$ and $g_1(2) < 0$. Thus, the function $F(z)$ splits in D into two analytic functions:

$$F_j(z) = \ln h_j(z), \quad j = 0, 1,$$
$$h_j(z) = z + g_j(z).$$

Note that $g_1(z) \equiv -g_0(z)$. In view of Example 11,

$$g_0(z) = z - \frac{1}{2z} + O\left(\frac{1}{z^2}\right) \quad (z \to \infty),$$

so that as $z \to \infty$,

$$h_0(z) = z\left(2 + O\left(\frac{1}{z}\right)\right), \quad h_1(z) = \frac{1}{z}\left(\frac{1}{2} + O\left(\frac{1}{z}\right)\right).$$

Let γ be a circle $|z| = \rho > 1$ oriented in the positive sense and with the initial point at point $z = \rho$. If w_j is the initial value of $F_j(z)$ at point ρ, then after traversal of γ we get

$$w_j = i\varphi_j, \quad \varphi = \Delta_\gamma \arg h_j(z).$$

We have

$$\varphi_0 = 2\pi + \psi_0, \quad \psi_0 = \Delta_\gamma \arg\left(2 + O\left(\frac{1}{z}\right)\right).$$

If ρ is large, then $\psi_0 = 0$, so that $F_0(z)$ is multiple-valued. Similarly,

$$\varphi_1 = -2\pi + \psi_1, \quad \psi_1 = \Delta_\gamma \arg\left(\frac{1}{2} + O\left(\frac{1}{z}\right)\right),$$

so that $\psi_1 = 0$ if ρ is large. □

25 Singular Boundary Points

Let the function $f(z)$ be regular in a domain D whose boundary Γ is a simple piecewise smooth curve and let $\zeta \in \Gamma$. Point ζ is said to be a *singular point* of $f(z)$ if this function cannot be analytically continued along a curve γ that lies in D and whose terminal point is ζ.

From this definition and the properties of analytic continuation (Sec. 20) it follows that the possibility of continuing the function $f(z)$ analytically to the boundary point ζ of D does not depend on the choice of curve γ lying in D.

We wish to prove that if

$$\lim_{\substack{z\to\zeta \\ z\in\gamma}} f^{(k)}(z)=\infty \tag{25.1}$$

for a $k \geqslant 0$, then point $\zeta \in \Gamma$ is a singular point for $f(z)$ (at $k=0$ we put $f^{(0)}(z) \equiv f(z)$). Indeed, if $f(z)$ can be continued analytically to the boundary point ζ of D along γ, then, by the very definition of analytic continuation (Sec. 20), there is a function $f_\zeta(z)$ that is regular in a circle K: $|z-\zeta|<\rho$ and is such that $f_\zeta(z) \equiv f(z)$ if $z \in K \cap \gamma$. This means that

$$\lim_{\substack{z\to\zeta \\ z\in\gamma}} f^{(k)}(z)=f_\zeta^{(k)}(\zeta)\neq\infty$$

for any $k \geqslant 0$, which contradicts condition (25.1).

Let us study the case of singular boundary points when D is a circle. We start with the following

Theorem *The boundary of the circle of convergence of the power series*

$$f(z)=\sum_{n=0}^{\infty} c_n (z-a)^n \tag{25.2}$$

contains at least one singular point of the sum $f(z)$ of this series.

Proof. Let K: $|z-a|<R$ be the circle of convergence of (25.2), $0<R<\infty$. We assume there are no singular points of $f(z)$ on the circle γ_R : $|z-a|=R$. Then this function can be analytically continued to every point lying on γ_R. We denote the result of analytic continuation by $F(z)$, so that $F(z) \equiv f(z)$ if $z \in K$. By the definition of analytic continuation, to every point $\zeta \in \gamma_R$ there corresponds a circle K_ζ centered at point ζ; inside every such circle $F(z)$ is regular. Thus, the circle γ_R is covered by an infinite number of circles with centers lying on γ_R.

By Heine-Borel's lemma (e.g. see Kudryavtsev [1]), out of this infinitude of curve we can always select a finite covering, i.e. there is a set of circles K_{ζ_j} ($j=1, 2, \ldots, n$), $\zeta_j \in \gamma_R$, such that each point $\zeta \in \gamma_R$ belongs to at least one of such circles. Let z_j be the intersection point of two adjacent circles K_{ζ_j} and $K_{\zeta_{j+1}}$ ($j=1, 2, \ldots, n$; $K_{\zeta_{n+1}} \equiv K_{\zeta_1}$) that lies outside K and let $R_0=\min\limits_{1\leqslant j\leqslant n}|z_j-a|$. Then the function $F(z)$, which coincides with $f(z)$ in K, is regular in the larger circle K_0: $|z-a|<R_0$, with $R_0>R$. From this it follows (see Corollary 3 of Sec. 12) that the function $F(z)$ can be represented in K_0 by a convergent series (25.2), i.e. the radius of convergence of the series (25.2) is greater than R, which contradicts the hypothesis. The proof of the theorem is complete.

Example 1. The radius of convergence of the series $\sum\limits_{n=0}^{\infty}(-1)^n z^{2n}$

is equal to unity. The boundary of the circle of convergence of this series contains two singular points of the sum of the series $1/(1 + z^2)$, namely, points i and $-i$. □

Corollary *The radius of convergence of the series* (25.2) *is equal to the distance between point a and the nearest singular point of* $f(z)$.

In many cases this proposition enables us to effectively determine the radius of convergence of a power series without using the Cauchy-Hadamard formula (Sec. 11).

Example 2. The radius of convergence of

$$\frac{1}{(z+2)(z-3)} = \sum_{n=0}^{\infty} c_n z^n$$

is 2 since the singular point of the sum of the series nearest to $z = 0$ is -2. □

Example 3. The raduis of convergence of

$$\tan z = \frac{\sin z}{\cos z} = \sum_{n=0}^{\infty} c_n z^n$$

is $\pi/2$ since the singular points of $\tan z$ nearest to $z = 0$ are the poles at $z_1 = \pi/2$ and $z_2 = -\pi/2$. □

Remark 1. The function e^x of the real variable x can be differentiated an infinite number of times on the entire real axis, and the series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{25.3}$$

converges for all values of x. On the other hand, the function $1/(1 + x^2)$ can also be differentiated an infinite number of times on the entire real axis, but the radius of convergence of the series

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \tag{25.4}$$

is 1. The reason for this becomes clear if we move from the real axis into the complex plane. Indeed, the function e^z is an entire function and the radius of convergence of the series in (25.3) is $R = \infty$. On the other hand, the function $1/(1 + z^2)$ has two singular points, $z_1 = i$ and $z_0 = -i$. Hence, the radius of convergence of the series in (25.4) is 1.

Remark 2. The convergence of the power series (25.2) at the boundary points of the circle of convergence is not related to the regularity of the sum of this series at these points (see Examples 4-6).

Example 4. The series $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ is divergent at all the points on the boundary of a unit circle, while point 1 is the singular point of the sum of the series and all the other points on the boundary are points of regularity of the sum. □

Example 5. The series $f(z) = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$ is convergent at point 1 and its sum is regular at this point since $f(z)$ is a regular branch of the function $\ln(1+z)$. □

Example 6. The series $f(z) = \sum\limits_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ is convergent at all points on the boundary of a unit circle, including the point 1, but this point is singular for $f(z)$. Indeed, $f(z)$ is a regular branch of the function $1 + \frac{1-z}{z} \ln(1-z) = F(z)$, and point $z = 1$ is a logarithmic branch point for $F(z)$. □

Here is an example in which every point on the boundary of the circle of convergence is singular.

Example 7. Consider the series

$$f(z) = \sum_{k=0}^{\infty} z^{2^k}.$$

The radius of convergence of this series is 1. First let us show that point $z = 1$ is singular for $f(z)$. We take the segment $[0, 1]$ as γ. If $z \in \gamma$, then $z^{2^k} = x^{2^k} \to 1$ as $z = x \to 1 - 0$ $(k = 0, 1, 2, \ldots)$ and, hence, $f(x) = \sum\limits_{k=1}^{\infty} x^{2^k} \to \infty$ as $x \to 1 - 0$. Thus, condition (25.1) is met $(k = 0)$ and point $z = 1$ is singular for $f(z)$.

Moreover, the fact that

$$f(z) = z^2 + z^4 + \ldots + z^{2n} + [1 + (z^{2^n})^2 + (z^{2^n})^4 + \ldots]$$

leads to the following functional relationship for $f(z)$:

$$f(z) = z^2 + z^4 + \ldots + z^{2n} + f(z^{2n}), \quad n = 1, 2, \ldots \tag{25.5}$$

Since point $z = 1$ is singular for $f(z)$, the functional relationship implies that for any natural number n all the points that satisfy the condition

$$z^{2^n} = 1 \tag{25.6}$$

are also singular for $f(z)$. The roots of Eq. (25.6) form an everywhere dense set of points on the boundary of a unit circle. From this it follows that all the points on the boundary of a unit circle are singular for $f(z)$. Indeed, if a point ζ on this boundary is not singular, then there is an arc of the boundary that contains this point and other nonsingular points. But this is impossible. □

Example 8. Suppose there is only one singular point z_0 (a first-order pole) lying on the boundary of the circle of convergence of the power series $f(z) = \sum_{n=0}^{n} c_n z^n$. Let us find an asymptotic estimate for the c_n.

The hypothesis implies that

$$f(z) = g(z) + \frac{A}{z - z_0}, \quad A \neq 0, \tag{25.7}$$

where $g(z)$ is regular in the circle $|z| < R$, $R > |z_0|$. In view of the fact that the series

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \tag{25.8}$$

is convergent at z_0 we have

$$b_n z_0^n \to 0 \quad (n \to \infty). \tag{25.9}$$

Since

$$\frac{1}{z - z_0} = -\sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}}, \tag{25.10}$$

from (25.7) and (25.8) it follows that

$$c_n = -\frac{A}{z_0^{n+1}} + b_n = -\frac{A}{z_0^{n+1}} \left(1 - \frac{z_0}{A} b_n z_0^n\right),$$

whence in view of (25.9) we arrive at the following asymptotic estimate:

$$c_n \sim -\frac{A}{z_0^{n+1}} \quad (n \to \infty). \tag{25.11}$$

One result that follows from (25.11) is

$$\lim_{n \to \infty} \frac{c_n}{c_{n+1}} = z_0. \ \square$$

Finally, we will formulate a result known as

Pringsheim's theorem *If the radius of convergence of* $\sum_{n=0}^{\infty} c_n z^n$ *is 1 and if* Re $c_n \geqslant 0$ *for all n's, the point* $z = 1$ *is singular for the sum of the series.*

26 Singular Points of Analytic Functions. The Concept of a Riemann Surface

26.1 Branches of analytic functions. Singular points An analytic function, by definition, is the set of all elements obtained from a certain element by the process of analytic continuation. This set is connected in the sense that any two elements of the analytic function can be obtained through analytically continuing one of the elements along a certain curve.

We introduce the concept of a singular point of an analytic function in the following manner. In Sec. 25 we introduced the concept of a singular boundary point for a regular function. Since an analytic function consists of elements (regular functions), we say that z_0 is a *singular point of an analytic function* if it is a singular boundary point of an element of this function.

This idea is rather complex and will not be employed in what follows in such a general form. An important type of singular points is the isolated singular point of an analytic function.

We start by introducing the concept of a branch of an analytic function. Suppose we have fixed an element $f(z)$ at a point z_0. If we continue $f(z)$ analytically along all the curves that start at z_0 and for which such continuation is possible, the resulting set of elements forms an analytic function $F(z)$. But if we continue $f(z)$ analytically only along some of the curves for which such continuation is possible, we arrive at a branch $F_0(z)$ of the analytic function $F(z)$.

In other words, a connected subset of elements of an analytic function is said to be a *branch* of this function.

A branch of an analytic function may be a multiple-valued function of z. A single-valued branch of an analytic function is a regular function. Such branches were considered in Sec. 24; here we consider multiple-valued branches of analytic functions.

An example of a branch of an analytic function is a function that is analytic in a domain D. Let us recall this notion (Sec. 20). Suppose D is a domain in the extended complex plane and $f(z)$ is an element at point $z_0 \in D$ that can be continued analytically along any curve lying in D. The set of elements obtained as a result of all such continuations is called a *function $F(z)$ analytic in D*.

Definition 1. Suppose a function $F(z)$ is analytic in a punctured neighborhood of point a but is not regular at point a. Then point a is said to be an *isolated singular point of $F(z)$*.

This function, $F(z)$, is generally a branch of an analytic function $G(z)$. If point a is a singularity for a branch of an analytic function, it is a singularity for the analytic function, too.

If a is an isolated singularity for a branch $F(z)$, it is a pole or an essential singularity or a branch point.

Example 1. Let K be the annulus $0 < |z| < r$ and point $z_0 \in K$.

We fix the element $f(z)$ of $\ln z$ at point z_0 and continue it analytically along all the curves lying in K. Then we arrive at a function $F_0(z)$ that is analytic in K and is a branch of the analytic function $\ln z$.

Similarly, knowing the element $g(z)$ of $\sqrt[n]{z}$ fixed at a point $z_0 \in K$, we can build a branch $G_0(z)$ of this function that is analytic in K. □

Two analytic functions, according to the definition given in Sec. 20, are equal if and only if their initial elements are equivalent. For branches another definition of equality proves to be more convenient. Precisely, two branches of an analytic function are, by definition, equal if they consist of the same elements.

For this viewpoint, in Example 1 there is exactly one branch of $\ln z$ that is analytic in the annulus K (the situation is the same for $\sqrt[n]{z}$ and z^{α}).

Example 2. Point $z = 0$ is a logarithmic branch point for the branch $F_0(z)$ of the logarithm (see Example 1). The same point is a branch point of multiplicity n for the branch $G_0(z)$ of the function $\sqrt[n]{z}$ (see Example 1). □

A point in the complex plane may be a singular point for some branches of an analytic function and a regular point for the other branches of the function.

Example 3. Let us take the function $F(z) = \frac{1+\sqrt{z}}{z-1}$. By the monodromy theorem, in a neighborhood of point $z = 1$ the function $\sqrt{z}$ splits into two regular branches, $f_1(z)$ and $f_2(z)$. We put $f_1(1) = +1$ and $f_2(1) = -1$. In the same neighborhood the function $F(z)$ splits into two single-valued branches

$$F_j(z) = \frac{1+f_j(z)}{z-1}, \quad j = 1, 2.$$

The branch $F_1(z)$ has a simple pole at point $z = 1$, while the branch $F_2(z)$ is regular at point $z = 1$ because $1 + f_2(1) = 0$. □

Example 4. In the complex z plane with a cut along the segment $[-1, 1]$ the analytic function $F(z) = z + \sqrt{z^2 - 1}$ splits into two branches, $f_0(z)$ and $f_1(z)$ (see Example 9 in Sec. 24). We put $f_0(2) = 2 + \sqrt{3}$ and $f_1(2) = 2 - \sqrt{3}$. Then

$$f_0(z) \sim 2z, \quad f_1(z) \sim \frac{1}{2z} \quad (z \to \infty)$$

(see Example 26 in Sec. 24). Hence, the point $z = \infty$ is a first order pole for the branch $f_0(z)$ and a first order zero for the branch $f_1(z)$. □

Below we will show (see Example 12) that the points $z = \pm 1$ are branch points of multiplicity 2 for the function $F(z)$ of Example 4.

In many cases we can establish the nature of branch points by employing

Theorem 1 *Suppose a function $F(z)$ is analytic in a punctured neighborhood U of a point $z = a$, another function $f(z) \not\equiv 0$ in this neighborhood, and point a is a branch point of multiplicity n for $F(z)$ (here $n \leqslant \infty$). Then point a is a branch point of multiplicity n of each of the following functions (analytic in U):*

$$f(z)+F(z), \quad f(z)F(z), \quad \frac{f(z)}{F(z)}$$

(in the last case we must be sure that $F(z)$ is nonzero for $z \in U$).

Proof. Let us consider the function $G(z) = f(z) + F(z)$ (which is analytic in U). We fix a point $z_0 \in U$. Then, by hypothesis, there are exactly n different elements $f_1(z), \ldots, f_n(z)$ of $F(z)$ at this point. Hence, there are exactly n different elements $f(z) + f_1(z), \ldots, f(z) + f_n(z)$ of $G(z)$ at this point, too, so that point a is a branch point of multiplicity n. The situation with $f(z) F(z)$ and $f(z)/F(z)$ can be shown to be the same quite similarly. The proof of the theorem is complete.

Example 5. Point $z = 0$ is a branch point of multiplicity 2 for each of the following functions: $1/\sqrt{z}$, $z\sqrt{z}$, $z + \sqrt{z}$, $1/(1 + \sqrt{z})$, and $\sqrt{z} \sin z$ (these functions are analytic in a punctiured neighborhood of point $z = 0$). The functions $\sqrt{z} + \sin z$, $e^z \sqrt{z}$, $1/\sqrt{z}$, and $1/(1 + \sqrt{z})$ are analytic in a punctured neighborhood of point $z = \infty$, which is a branch point of multiplicity 2 for these functions. □

Example 6. The points 0 and ∞ are branch points of multiplicity n for the analytic functions

$$\frac{1}{\sqrt[n]{z}}, \quad \sqrt[n]{z}+e^z, \quad \frac{1}{1+\sqrt[n]{z}}, \quad \sqrt[n]{z}\sin z. \ \square$$

Example 7. The points 0 and ∞ are logarithmic branch points for the analytic functions

$$z+\ln z, \quad \frac{\ln z}{z-1}, \quad \frac{1}{\ln z}, \quad \frac{1}{\ln z+1}, \quad e^z \ln z. \ \square$$

Example 8. Suppose that $f(z) \not\equiv 0$ and is regular in a punctured neighborhood of a point $z = a$. Then this point is a branch point of multiplicity n for each of the following functions:

$$\sqrt[n]{z-a}+f(z), \quad \sqrt[n]{z-a}\, f(z), \quad \frac{f(z)}{\sqrt[n]{z-a}},$$

and a branch point of infinite multiplicity for each of the following functions:

$$\ln(z-a)+f(z), \quad f(z)\ln(z-a), \quad \frac{f(z)}{\ln(z-a)}.$$

where all functions are analytic in a punctured neighborhood of point a. □

Example 9. Let us study the singular points of the function $F(z) = 1/(1+\sqrt{z})$. This function is analytic in the extended complex plane with points 0, 1, and ∞ deleted.

Points 0 and ∞ are branch points of multiplicity 2 (see Example 5). In a small neighborhood U of point $z = 1$ the function $\sqrt{z}$ splits into two regular branches, $f_1(z)$ and $f_2(z)$ (according to the monodromy theorem); we put $f_1(1) = 1$ and $f_2(1) = -1$. Corresondingly, the function $F(z)$ splits into two branches $F_j(z) = 1/(1+f_j(z))$, $j = 1, 2$, at $z \in U$. The branch $F_1(z)$ is regular at point $z = 1$, while the branch $F_2(z)$ has a pole at this point since (see formula (22.12))

$$1+f_2(z) = 1+\sqrt{1+(z-1)} = 1+(-1)\left(1+\frac{z-1}{2}+\dots\right)$$
$$= -\frac{z-1}{2}+\dots$$

in a neighborhood of point $z = 1$. □

Example 10. Let us study the singular points of the function $F(z) = \frac{\ln z}{z-1}$, which is analytic in the extended complex plane with points 0, 1, and ∞ deleted. The points 0 and ∞ are logarithmic branch points (see Example 7). In a small neighborhood of point $z = 1$ the function $\ln z$, according to the monodromy theorem, splits into regular functions $f_k(z)$, $k = 0, \pm 1, \pm 2, \dots$; we put $f_k(1) = 2\pi i k$. Point $z = 1$ is a simple pole for the branches $F_k(z) = f_k(z)/(z-1)$, $k \neq 0$, and a point of regularity for the branch $F_0(z)$. □

26.2 Combinations of the root, the logarithm, and regular functions (the singular points) Suppose a function $f(z)$ is regular and nonzero in a domain D. Let us study the function

$$F(z) = \ln f(z), \quad F(z_0) = w_0, \tag{26.1}$$

analytic in D (see Sec. 24), where $z_0 \in D$, and $e^{w_0} = f(z_0)$ (i.e. w_0 is one of the values of the logarithm). The values of $F(z)$ at $z \in D$ can be calculated via formula (24.1):

$$F(z) = \ln |f(z)| + i\,[\mathrm{Im}\, w_0 + \Delta_\gamma \arg f(z)]. \tag{26.2}$$

Here curve γ lies in D and connects points z_0 and z; the values of $F(z)$ depend not only on z but on curve γ as well.

Similarly, we can consider the function

$$G(z) = \sqrt[n]{f(z)}, \quad G(z_0) = \zeta_0 = \rho_0 e^{i\varphi_0}, \tag{26.3}$$

analytic in D, where $z_0 \in D$, and $\zeta_0^n = f(z_0)$. The values of this function at $z \in D$ can be calculated via the formula

$$G(z) = |\sqrt[n]{f(z)}|\, e^{i\left(\varphi_0 + \frac{1}{n}\varphi\right)}, \quad \varphi = \Delta_\gamma \arg f(z), \tag{26.4}$$

For one, if $f(z) \not\equiv 0$ is regular at point a or has a pole at the point, the functions $\ln f(z)$ and $\sqrt[n]{f(z)}$ are analytic in a punctured neighborhood of point a.

We recall an important property of $\ln f(z)$ and $\sqrt[n]{f(z)}$: to fix an element at a point z_0 we must only fix its value at this point.

Theorem 2 *Suppose a function $f(z) \not\equiv 0$ is either regular at a point a and $f(a) = 0$ or has a pole at this point. Then*

(1) *point a is a logarithmic branch point for* $\ln f(z)$;

(2) *if point a is a simple zero or a simple pole for $f(z)$, it is a branch point of multiplicity n for $\sqrt[n]{f(z)}$.*

Proof. If D is a small punctured neighborhood of point a, then

$$f(z) = (z - a)^m h(z), \quad z \in D,$$

where $m \neq 0$ is an integer, and $h(z)$ is regular and nonzero in $D_1 = D \cup \{a\}$ (see Secs. 12 and 18). Suppose γ is a simple closed curve that starts at z_0, lies in D, has point a in its interior, and is oriented in the positive sense. Then

$$\Delta_\gamma \arg f(z) = m\Delta_\gamma \arg (z - a) + \Delta_\gamma \arg h(z) = 2\pi m,$$

since $\Delta_\gamma \arg h(z) = 0$. Indeed,

$$\Delta_\gamma \arg h(z) = \int_\gamma d \arg h(z) = \operatorname{Im} \int_\gamma \frac{h'(z)}{h(z)}\, dz = 0,$$

since the function $h'(z)/h(z)$ is regular in the simply connected domain D_1. Hence,

$$\Delta_{\gamma^k} \arg f(z) = 2\pi k m,$$

where $\gamma^k = \gamma\gamma\ldots\gamma$ (k times). Since any closed curve that starts at z_0 and lies in D is homotopic to γ^k (k is an integer), we conclude that all values of $F(z) = \ln f(z)$ and $G(z) = \sqrt[n]{f(z)}$ at point z_0 are given by the following formulas:

$$\begin{aligned} F_k(z_0) &= \ln |f(z_0)| + i\,[\operatorname{Im} w_0 + 2k\pi m], \\ G_k(z_0) &= |\sqrt[n]{f(z_0)}|\, e^{i\left(\varphi_0 + \frac{1}{n}\cdot 2\pi k m\right)}, \end{aligned} \tag{26.5}$$

$k = 0, \pm 1, \pm 2, \ldots$. Hence, $F(z)$ is an infinite-valued function in D, so that a is a logarithmic branch point for $F(z)$.

Let $m = \pm 1$. Then at point z_0 there are exactly n values $G_k(z_0)$, $0 \leqslant k \leqslant n - 1$, of the function $G(z)$, i.e. there are exactly n different elements at this point.

Here is another proof of the same theorem. For $z \in D$ the following formulas are valid (see Sec. 24.2):

$$\ln f(z) = m \ln (z-a) + \ln h(z), \quad \sqrt[n]{f(z)} = \sqrt[n]{(z-a)^m}\, \sqrt[n]{h(z)},$$

where $\ln h(z)$ and $\sqrt[n]{h(z)}$ are branches that are regular in $D \cup \{a\}$. Then we need only use Theorem 1.

Example 11. Suppose $R(z)$ is a rational function. Then its zeros and poles (and only these points) are the singular points of $F(z) = \ln R(z)$, $F(z_0) = w_0$ ($e^{w_0} = R(z_0)$, and z_0 is neither a zero nor a pole of $R(z)$), which is analytic in the extended complex plane punctured at the above-mentioned points. All are logarithmic branch points. □

Example 12. Suppose $R(z)$ is a rational function that has only simple zeros and poles. All these points are the branch points of multiplicity n of the function $\sqrt[n]{R(z)}$. □

Example 13. Suppose $f(z) = z^m h(z)$, where $h(z)$ is regular and nonzero in the circle D: $|z| < r$, with $m \neq 0$ an integer. Then the function

$$F(z) = \sqrt[n]{f(z)}, \quad F(z_0) = w_0, \quad z_0 \neq 0,$$

is analytic in the annulus K: $0 < |z| < r$. Let us investigate the nature of the singular point $z = 0$.

Let γ be a positively oriented circle $|z| = |z_0|$ starting at point z_0. After k traversals of γ,

$$F(z_0) \to F_k(z_0), \quad F_k(z_0) = w_0 e^{i2\pi km/n}, \tag{26.6}$$

by virtue of (26.4), since $\Delta_\gamma \arg f(z) = m\Delta_\gamma \arg z + \Delta_\gamma \arg h(z) = 2\pi m$.

(a) Suppose m and n are coprime numbers. Then $F(z)$ has at point z_0 exactly n different values, and point 0 is a branch point of multiplicity n for $F(z)$.

(b) Suppose d is the greatest common divisor of m and n, i.e. $m = pd$ and $n = qd$, where p and q are coprime numbers, $q \geqslant 1$. Then among the numbers $F_k(z_0)$, $k = 0, \pm 1, \ldots$, there are exactly q different numbers. If $q \geqslant 2$, point $z = 0$ is a branch point of multiplicity q. But if $q = 1$, i.e. m can be divided by n, point $z = 0$ is not a branch point.

For instance, points 0 and 1 are branch points of multiplicity 3 for the function $\sqrt[3]{z^2/(z-1)}$. □

Here is a more complex example. But first we note the following. Let U be a punctured neighborhood of point a and let all the elements of an analytic function $F(z)$ given at various points of U allow for analytic continuation along all curves lying in U. Then the function splits in U into analytic branches, i.e. functions analytic in U. Indeed, we take a point $z_0 \in U$ and an element $f_0(z)$ at this point, and continue the latter analytically along all the curves in U. We then arrive at a function $F_0(z)$ that is analytic in U and is a branch of $F(z)$. If at point z_0 there exists an element $f_1(z)$ (of function $F(z)$) that is not an element of branch $F_0(z)$, then $f_1(z)$ generates in U another branch $F_1(z)$, and so on.

In the examples we considered earlier the analytic function $F(z)$ either split in U into regular branch or consisted of only one branch. Here is an example of a function of another type.

Example 14. We study the singular points of the function

$$F(z) = \ln \frac{1-\sqrt{z}}{1+\sqrt{z}}.$$

This is a composite function $F(z) = H(G(z))$ consisting of the following functions:

$$G(z) = \frac{1-\sqrt{z}}{1+\sqrt{z}}, \quad H(w) = \ln w$$

We fix the initial element $g(z)$ of $G(z)$ at, say, point $z = 4$: $g(4) = -1/3$, i.e. $\sqrt{z}\,|_{z=4} = 2$. Let D be the extended complex plane with points 0, 1, and ∞ deleted. Then $G(z)$ is analytic in D and does not assume the values 0 and ∞. At $w = -1/3$ we fix the element $h(w)$ of the function $H(w) = \ln w$; let $h(-1/3) = -\ln 3 + \pi i$. Then, by Theorem 2 of Sec. 22, the function $F(z) = H(G(z))$, generated by the element $h(g(z))$, is analytic in D.

(a) $z = 1$. Let us show that in a small punctured neighborhood K: $0 \leqslant |z-1| < r$ of point $z = 1$ the function $F(z)$ splits into two analytic branches $F_{1,2}(z)$, with $F_2(z) \equiv -F_1(z)$ and for each of these branches point $z = 1$ is a logarithimc branch point. Indeed, the function $\varphi(z) = \sqrt{z}$ in the circle $\tilde{K}$: $|z-1| < r$ splits, according to the monodromy theorem, into two regular branches $\varphi_j(z)$, $j = 1, 2$, with $\varphi_2(1) = 1$ and $\varphi_2(z) \equiv -\varphi_1(z)$. Correspondingly, the function $G(z)$ splits in K into two regular branches

$$G_j(z) = \frac{1-\varphi_j(z)}{1+\varphi_j(z)}, \quad j = 1, 2; \quad G_2(z) = \frac{1}{G_1(z)}$$

By Taylor's formula, $\varphi_1(z) = 1 + \frac{z-1}{2} + \ldots$, so that

$$G_1(z) \sim -\frac{1}{4}(z-1), \quad G_2(z) \sim \frac{-4}{z-1} \quad (z \to 1).$$

Hence, the branch $G_1(z)$ has at $z = 1$ a simple zero, while the branch $G_2(z)$ has a simple pole at this point. The function $F(z)$ splits in K into two analytic branches $F_j(z) = \ln G_j(z)$, $j = 1, 2$, with $F_2(z) \equiv -F_1(z)$, and point $z = 1$ is a logarithmic branch point for $F_{1,2}(z)$ (Theorem 2).

(b) $z = 0$. Let us show that in a small annulus $K: 0 < |z| < r$ the function $F(z)$ splits into a countable set of branches, for each of which the point $z = 0$ is a branch point of multiplicity 2.

The function $\widetilde{G}(\zeta) = \ln \frac{1-\zeta}{1+\zeta}$ in a small neighborhood U of point $\zeta = 0$ splits into regular branches $\widetilde{G}_k(\zeta)$, $k = 0, \pm 1, \ldots$ ..., with $\widetilde{G}_k(0) = 2k\pi i$. Hence, each $F_k(z) = \widetilde{G}_k(\sqrt{z})$ is analytic in a punctured neighborhood of point $z = 0$. For $|\zeta|$ small we have

$$\widetilde{G}_k(\zeta) = 2k\pi i - 2\zeta + O(\zeta^2),$$

so that for $|z|$ small we have

$$F_k(z) = 2k\pi i - 2\sqrt{z} + O(z).$$

Hence, $F_k(z)$ is a double-valued function and $z = 0$ is a branch point of multiplicity 2 for this function.

(c) $z = \infty$. The function $F(z)$ in a neighborhood of this point has the same structure as in a neighborhood of point $z = 0$. Indeed, substitution of $1/\zeta$ for z yields

$$\widetilde{F}(\zeta) \equiv F(z) = \ln \frac{\sqrt{\zeta}-1}{\sqrt{\zeta}+1},$$

where ζ lies in a neighborhood of point $\zeta = 0$. □

Example 15. Let us study the singular points of the function

$$F(z) = \sqrt[3]{z} + \sqrt{z+1}.$$

This function is the sum of two analytic functions, $G(z) = \sqrt[3]{z}$ and $H(z) = \sqrt{z+1}$. We will fix their initial elements $g(z)$ and $h(z)$ at point $z = 1$, i.e. $\sqrt[3]{z}\,|_{z=1} = 1$ and $\sqrt{z+1}\,|_{z=1} = \sqrt{2} > 0$. The function $F(z)$ is analytic in the extended complex plane punctured at points -1, 0, and ∞.

(a) $z = 0$. Let us show that in a small punctured neighborhood K_1: $0 < |z| < r$ of point $z = 0$ the function $F(z)$ splits into two analytic branches, $F_1(z)$ and $F_2(z)$, for each of which point $z = 0$ is a branch point of multiplicity 3. Indeed, the function $H(z) = \sqrt{z+1}$ splits, according to the monodromy theorem, into two regular branches, $H_1(z)$ and $H_2(z) \equiv -H_1(z)$. For this reason $F(z)$ splits in K_1 into two analytic branches:

$$F_1(z) = G(z) + H_1(z), \quad F_2(z) = G(z) + H_2(z),$$

for each of which $z = 0$ is a branch point of multiplicity 3 (see Example 8).

(b) $z = -1$. In a small punctured neighborhood K_2: $0 < |z +$

$+1| < r$ the function $F(z)$ splits into three analytic branches, for each of which point $z = -1$ is a branch point of multiplicity 2. Indeed, by the monodromy theorem, $G(z)$ splits in K_2 into three regular branches, $G_1(z)$, $G_2(z) \equiv e^{2\pi i/3}G_1(z)$, and $G_3(z) \equiv e^{4\pi i/3}G_1(z)$, so that $F(z)$ splits into three regular branches $G_j(z) + H(z)$, $j = 1, 2, 3$.

(c) $z = \infty$. Let us show that $z = \infty$ is a branch point of multiplicity 6 for $F(z)$. Suppose γ is a simple closed curve whose initial and terminal points are at $z = 1$ and whose interior contains points $z = 0$ and $z = -1$. As γ is traversed in the positive sense,

$$\Delta_\gamma \arg z = \Delta_\gamma \arg (z + 1) = 2\pi.$$

The initial value of $F(z)$ at point $z = 1$ is $F(1) = 1 + \sqrt{2}$. After N traversals of γ we have the following values of $F(1)$:

$$N=1: \quad F(1) = e^{2\pi i/3} - \sqrt{2},$$

$$N=2: \quad F(1) = e^{4\pi i/3} + \sqrt{2},$$

$$N=3: \quad F(1) = 1 - \sqrt{2},$$

$$N=4: \quad F(1) = e^{2\pi i/3} + \sqrt{2},$$

$$N=5: \quad F(1) = e^{4\pi i/3} - \sqrt{2}.$$

At $N = 6$ we again have $F(1) = 1 + \sqrt{2}$.

Hence, $F(z)$ is a six-valued function. This fact can be established in a different manner, namely, we will show that the function $w = F(z)$ satisfies an algebraic equation of the sixth degree. Raising both sides of the identity $w - \sqrt{z+1} = \sqrt[3]{z}$ to the third power, we find that

$$w^3 + 3w(z+1) - z = \sqrt{z+1}\,(3w^2 + z + 1).$$

After we square both sides of the latter identity, we find that

$$P(w, z) \equiv w^6 - 3(z+1)w^4 - 2zw^3 + 3(z+1)^2 w^2$$
$$- 6z(z+1)w - z^3 - 2z^2 - 3z - 1 = 0.$$

For each fixed value of z this equation has six roots; it can also be shown that the roots are different if $z \neq 0$ and $z \neq -1$. □

26.3 The structure of an analytic function in the neighborhood of an algebraic branch point In any neighborhood of an algebraic branch point $a \neq \infty$ we can always expand an analytic function in a series in fractional powers of $z - a$.

Theorem 3 *Let a function $F(z)$ be analytic in an annulus K: $0 < |z - a| < r$ and let point a be a branch point of multiplicity*

$n < \infty$. *Then*

$$F(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^{k/n}, \tag{26.7}$$

where the series is convergent in K.

Proof. Take the function $\Phi(\zeta) = F(a + \zeta^n)$. It is analytic in the annulus $\tilde{K}$: $0 < |\zeta| < \sqrt[n]{r}$. If we show that $\Phi(\zeta)$ is single-valued in $\tilde{K}$, we will have proved that $\Phi(\zeta)$ is regular in $\tilde{K}$ and, by Theorem 1 of Sec. 17, can be expanded in a Laurent series,

$$\Phi(\zeta) = \sum_{k=-\infty}^{\infty} c_k \zeta^k, \tag{26.8}$$

that is convergent in K.

Let us take the circle γ: $|\zeta| = \rho$, where $0 < \rho < \sqrt[n]{r}$, that starts at point ρ. When point ζ traverses γ in the positive sense one full circuit, point $z = a + \zeta^n$ traverses the circle $\tilde{\gamma}$: $|z - a| = \rho^n$ in the positive sense n times. Let $f_0(z)$ be the element of $F(z)$ at point $z_0 = a + \rho^n$. Since point a is branch point of multiplicity n for $F(z)$, we conclude that $f_0(z) \to f_0(z)$ after n traversals of $\tilde{\gamma}$. Hence $\varphi_0(\zeta) \to \varphi_0(\zeta)$ after one traversal of γ, where $\varphi_0(\zeta) = f_0(a + \zeta^n)$ is the element of $\Phi(\zeta)$ at point $\zeta_0 = \rho$, and $\Phi(\zeta)$ is regular in $\tilde{K}$. Since $\zeta^n = z - a$, we find that (26.8) leads to (26.7). The proof of the theorem is complete.

Corollary *Suppose point* $z = \infty$ *is a branch point of multiplicity* $n < \infty$ *for an analytic function* $F(z)$. *Then*

$$F(z) = \sum_{k=-\infty}^{\infty} c_k z^{k/n}, \tag{26.9}$$

where the series is convergent in an annulus $R < |z| < \infty$.

To prove this proposition we only need to note that point $\zeta = 0$ is a branch point of multiplicity n for the function $F(1/\zeta)$ and then employ Theorem 3.

Series of the type (26.7) and (26.9) are called *Puiseux' series.*

26.4 The concept of a Riemann surface An analytic function is not a function in the common sense, since one value of the independent variable z may have several (or even a countable number of) values corresponding to it. To be able to interpret an analytic function $F(z)$ as a function in the ordinary sense, we relate $F(z)$ to a surface R on which $F(z)$ is single-valued. The surface is called the *Riemann surface* for the analytic function $F(z)$. Here is the definition of a Riemann surface.

We study an analytic function $F(z)$. A pair $P_0 = (z_0, f_0(z))$, where $f_0(z)$ is an element of $F(z)$ at point z_0, is called a *point on the Riemann surface* R (for function $F(z)$). We will assume that two pairs, $(z_0, f_0(z))$ and $(z_0, f_1(z))$, define the same point on a Riemann surface if the elements $f_0(z)$ and $f_1(z)$ are equivalent. The point z_0 is said to be the *projection* of point $P_0 = (z_0, f_0(z))$ on the Riemann surface onto the complex z plane:

$$(z_0, f_0(z)) \to z_0.$$

The set of the points $P = (\zeta, f_0(z))$ for which $|\zeta - z_0| < \varepsilon$, with $\varepsilon > 0$ such that $f_0(z)$ is regular in the circle $|z - z_0| < \varepsilon$, is called a *U_ε-neighborhood* of point $P_0(z_0, f_0(z))$. The circle $|z - z_0| < \varepsilon$ is the projection of the U_ε-neighborhood onto the complex z plane.

Remark 1. The reader can use Evgrafov [1], Springer [1], and Shabat [1] to find a comprehensive coverage of the theory of Riemann surfaces and to attain a deeper understanding of this concept. A Riemann surface is connected, since any two elements of an analytic function can be obtained by continuing one element analytically into the other.

A Riemann surface can be graphically illustrated by a surface in three-dimensional space. In Secs. 21 and 22 we constructed the Riemann surfaces for $\ln z$ and $\sqrt[n]{z}$.

Remark 2. Using the concept of a Riemann surface, we can graphically interpret the concept of a branch of an analytic function. Precisellly, a branch corresponds to a connected part of the Riemann surface (the converse is also true). The proposition that a function $F(z)$ splits in D into m analytic branches means, in terms of a Riemann surface, that the part of R that is projected onto D consists of m connected parts.

Another concept can be related to an analytic function, the *graph* of this function. Since a function $w = F(z)$ assumes complex values, its graph lies in a four-dimensional space (z, w), with z and w complex numbers. The graph of an analytic function $F(z)$ is the set of all pairs $(z, F(z))$, where $F(z)$ stands for all the values of $F(z)$ at point z. Generally speaking, this graph is a two-dimensional surface in four-dimensional space (z, w). There can be no self-intersections of this surface along "curves", according to the uniqueness theorem. However, at isolated points the various parts of the surface may get pasted together. For instance, if $F(z) = (z - 1) \ln z$, then at the point with coordinates $z = 1$ and $w = 0$ there is an infinite number of parts of the graph of this function that are pasted together.

27 Analytic Theory of Linear Second-Order Ordinary Differential Equations

27.1 Equations with regular coefficients Many problems of mathematical physics lead to linear second-order ordinary differential equations of the type

$$y'' + p(x) y' + q(x) y = 0.$$

The coefficients $p(x)$ and $q(x)$ may be analytic functions, and in the majority of cases they are even rational functions. It is therefore natural to study the solutions of the equation of the type

$$w''(z) + p(z)\,w'(z) + q(z)\,w(z) = 0 \tag{27.1}$$

from the viewpoint of the theory of analytic functions. Such an approach or, as it is usually called, the complex variable approach, enables us to employ the powerful tools of the theory of analytic functions and obtain important results concerning the structure of the solutions of Eq. (27.1). Precisely, it has been found that if $p(z)$ and $q(z)$ are rational functions, then any solution of Eq. (27.1) is a function that is analytic in the entire complex plane except, perhaps, at the poles of the coefficients $p(z)$ and $q(z)$. These poles, as a rule, are the singularities of all the solutions of Eq. (27.1), namely, branch points. Moreover, it is possible to study the structure of the solutions in the neighborhood of these singular points.

This section is devoted to the study of Eq. (27.1) with coefficients $p(z)$ and $q(z)$ that are regular in a domain D of the complex z plane. The Cauchy problem is formulated in the following terms: Given two complex numbers w_0 and w_1, find the solution of Eq. (27.1) for which

$$w(z_0) = w_0, \quad w'(z_0) = w_1, \tag{27.2}$$

where $z_0 \in D$.

Theorem 1 *Suppose the coefficients $p(z)$ and $q(z)$ of Eq.* (27.1) *are regular in the circle K: $|z - z_0| < R$. Then the Cauchy problem* (27.1), (27.2) *has a solution $w(z)$ that is regular in K.*

The proof can be found in Smirnov [1] and Whittaker and Watson [1].

Let us see how one can find the solution $w(z)$ of the Cauchy problem (27.1), (27.2) by employing Theorem 1. The solution is expanded in a power series that is convergent in K:

$$w(z) = \sum_{n=0}^{\infty} w_n (z - z_0)^n. \tag{27.3}$$

The Cauchy data give the first two expansion coefficients:

$$w_0 = w(z_0), \quad w_1 = w'(z_0).$$

By the hypothesis of Theorem 1, the functions $p(z)$ and $q(z)$ can also be expanded in Taylor series:

$$p(z) = \sum_{n=0}^{\infty} p_n (z - z_0)^n, \quad q(z) = \sum_{n=0}^{\infty} q_n (z - z_0)^n, \tag{27.4}$$

which converge in K, too.

Then we substitute (27.3) and (27.4) and the series for w' and w''

into Eq. (27.1). This yields a Taylor series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ that is identically equal to zero in K. Whence $c_n = 0$ for all $n = 0, 1, 2, \ldots$. This gives us the recurrence relations for $w_2, w_3, \ldots$.

Here are the actual formulas, with $z_0 = 0$ (for the sake of simplicity). We have

$$w'' + p(z) w' + q(z) w$$

$$= \sum_{n=2}^{\infty} n(n-1) w_n z^{n-2} + \sum_{n=1}^{\infty} n w_n z^{n-1} \sum_{m=0}^{\infty} p_m z^m + \sum_{n=0}^{\infty} w_n z^n \sum_{m=0}^{\infty} q_m z^m$$

$$= \sum_{n=0}^{\infty} c_n z^n = 0.$$

The equation $c_n = 0$ has the form

$$(n+2)(n+1) w_{n+2}$$

$$= -\sum_{k=1}^{n+1} k p_{n-k+1} w_k - \sum_{k=0}^{n} q_{n-k} w_k, \quad n = 0, 1, 2, \ldots \tag{27.5}$$

The coefficient w_{n+2} is expressed in terms of $w_0, w_1, \ldots, w_{n+1}$. Since w_0 and w_1 are known, Eq. (27.5) makes it possible to find successively $w_2, w_3, \ldots$.

Example 1. Consider the equation

$$w'' - zw = 0. \tag{27.6}$$

This is *Airy's differential equation.* Here $p(z) \equiv 0$ and $q(z) = -z$, so that the coefficients of the equation are regular in the entire complex plane. By Theorem 1, every solution of Airy's differential equation is an entire function.

Let us solve Eq. (27.6). Equation (27.5) in the case at hand has the form

$$(n+2)(n+1) w_{n+2} = w_{n-1}; \quad n = 1, 2, \ldots.$$

In fact, $w_2 = 0$, which implies that $w_5 = w_8 = \ldots = w_{2+3k} = \ldots = 0$. We also find that

$$w_{3n} = \frac{w_0}{(2\times 3)(5\times 6)\ldots[(3n-1)\times 3n]},$$

$$w_{3n+1} = \frac{w_1}{(3\times 4)(6\times 7)\ldots[3n(3n+1)]}.$$

Suppose $w_1(z)$ is the solution with the Cauchy data $w_1(0) = 1$ and $w_1'(0) = 0$, and $w_2(z)$ is the solution with the Cauchy data $w_2(0) =$

0 and $w_2'(0) = 1$. Then

$$w_1(z) = 1 + \frac{z^3}{2\times 3} + \frac{z^6}{(2\times 3)(5\times 6)} + \ldots + \frac{z^{3n}}{(2\times 3)(5\times 6)\ldots[(3n-1)\times 3n]} + \ldots,$$

$$w_2(z) = z + \frac{z^4}{3\times 4} + \frac{z^7}{(3\times 4)(6\times 7)} + \ldots + \frac{z^{3n+1}}{(3\times 4)(6\times 7)\ldots[3n(3n+1)]} + \ldots.$$

Each solution of Airy's equation is a linear combination of solutions $w_1(z)$ and $w_2(z)$. For instance, the solution

$$\mathrm{Ai}(z) = \frac{w_1(z)}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} - \frac{w_2(z)}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}$$

is called the *Airy function*. □

Example 2. The equation

$$w'' + (a + b\cos z)\, w = 0,$$

where a and b are constants, is known as *Mathieu's differential equation*. According to Theorem 1, each solution of Mathieu's differential equation is an entire function. □

Example 3. Every solution of *Weber's differential equation*

$$w''(z) - (z^2 - a^2)\, w(z) = 0$$

(a is constant) is an entire function. □

In Theorem 1 we assumed that the coefficients of the equation are regular in the circle K. Using the concept of analytic continuation, we can prove the analog of Theorem 1 for the case where the coefficients of Eq. (27.1) are regular in a simply connected domain.

Theorem 2 (the theorem of existence and uniqueness) *Let the coefficients $p(z)$ and $q(z)$ of Eq.* (27.1) *be regular in a simply connected domain D and let point z_0 belong to D. Then*

(1) *there exists a solution $w(z)$ of the Cauchy problem* (27.1), (27.2) *that is regular in D;*

(2) *this solution is unique, i.e. if $w_1(z)$ and $w_2(z)$ are the solutions of the Cauchy problem* (27.1), (27.2) *that are regular in D, then $w_1(z) \equiv w_2(z)$ in D.*

Proof. We start with the uniqueness of the solution $w(z)$ of the Cauchy problem (27.1), (27.2) that is regular in D. By hypothesis,

$$w(z_0) = w_0, \quad w'(z_0) = w_1.$$

Equation (27.1) then yields the value $w''(z_0) = -p(z_0)\, w_1 -$

$-q(z_0)w_0$. Differentiating Eq. (27.1) once, we obtain

$$w''' = -(p(z)w')' - (q(z)w)'$$

which yields $w'''(z_0)$. Differentiating this equation over and over, we can find $w^{(4)}(z_0)$, $w^{(5)}(z_0)$, etc. Hence, the Cauchy data uniquely determine all the derivatives $w^{(n)}(z_0)$, which proves the uniqueness of the solution.

Let us now prove that such a solution indeed exists. By Theorem 1, there exists a solution $w_0(z)$ of the Cauchy problem (27.1), (27.2) that is regular in the circle K: $|z-z_0| < R$, where R is the distance between point z_0 and the boundary of D. We have thus fixed an element $w_0(z)$ at point z_0. Let us show that this element allows for an analytic continuation along any curve γ that lies in D and starts at z_0. Suppose ρ is the distance between γ and the boundary of D, $\rho > 0$. We cover γ by a finite sequence of circles $K_0, K_1, \ldots, K_n$ of radius ρ. The centers of the circles K_j form an ordered sequence of points $z_0, \ldots, z_n$ on γ (z_n is the terminal point of γ), and the center z_j of K_j lies in the circle K_{j-1}, $j = 1, 2, \ldots, n$.

At point z_1 we fix the Cauchy data, which coincide with the value $w_0(z)$ of the solution and its derivative, i.e.

$$w(z_1) = \tilde{w}_0, \quad w'(z_1) = \tilde{w}_1,$$

where $\tilde{w}_0 = w_0(z_1)$ and $\tilde{w}_1 = w_0'(z_1)$. By Theorem 1, this problem has a solution $w_1(z)$ that is regular in K_1. The above-proved then yields

$$w_1(z) \equiv w_0(z), \quad z \in K_0 \cap K_1.$$

Similarly, there is a solution $w_2(z)$ of the Cauchy problem

$$w(z_2) = w_1(z_2), \quad w(z_2) = w_1'(z_2),$$

that is regular in the circle K_2, and this solution coincides with $w_1(z)$ for $z \in K_1 \cap K_2$. This process can be continued, and we find that the element $w_0(z)$ is continued analytically along the sequence of circles $K_0, \ldots, K_n$; all the elements $w_0(z), \ldots, w_n(z)$ are solutions of Eq. (27.1).

Thus, the element $w_0(z)$ generates a function $w(z)$ that is analytic in D and all elements of which satisfy Eq. (27.1). Since D is a simply connected domain, we can conclude from the monodromy theorem that $w(z)$ is regular in D. The proof of the theorem is complete.

Theorem 2 yields

Theorem 3 *Suppose the coefficients $p(z)$ and $q(z)$ of Eq.* (27.1) *are regular in a domain D. Then every solution of Eq.* (27.1) *is an analytic function in D.*

If D is a multiply connected domain, the solution of Eq. (27.1) may be a multiple-valued analytic function.

Example 4. Consider *Euler's differential equation*

$$w'' + \frac{a}{z} w' + \frac{b}{z^2} w = 0, \tag{27.7}$$

where a and b are constants. The coefficients of Euler's equation are regular in the complex z plane with point $z = 0$ deleted.

We wish to find the particular solution of Eq. (27.7) in the form $w = z^{\lambda}$. Substituting this solution into Eq. (27.7) and dividing it by $z^{\lambda-2}$, we arrive at a solution for λ:

$$\lambda (\lambda - 1) + a\lambda + b = 0. \tag{27.8}$$

If the roots λ_1 and λ_2 of this equation are different, the functions $w_1 (z) = z^{\lambda_1}$ and $w_2 (z) = z^{\lambda_2}$ form a fundamental system of solutions for Euler's equation. But if the roots coincide, i.e. $\lambda_1 = \lambda_2 = \lambda$, or $(a - 1)^2 - 4b = 0$, Eq. (27.7) has a solution $w_1 (z) = z^{\lambda}$ and, in addition, a solution $w_2 (z) = z^{\lambda} \ln z$; this pair of solutions forms the fundamental system of solutions of Eq. (27.7).

Let us study the behavior of Eq. (27.1) in the neighborhood of the point at infinity. Introducing the change of the variable $z = 1/\zeta$ and assuming that $\varphi (\zeta) = w (1/\zeta)$, we arrive at the equation

$$\varphi''_{\zeta\zeta} + \left(\frac{2}{\zeta} - \frac{1}{\zeta^2} p\left(\frac{1}{\zeta}\right)\right) \varphi'_{\zeta} + \frac{1}{\zeta^4} q\left(\frac{1}{\zeta}\right) \varphi = 0. \tag{27.9}$$

Every solution of Eq. (27.9) is regular at point $\zeta = 0$ if the functions $2\zeta^{-1} - \zeta^{-2} p (1/\zeta)$ and $\zeta^{-4} q (1/\zeta)$ are regular at point $\zeta = 0$. Hence, every solution of Eq. (27.1) is regular at point $z = \infty$ if the functions $2z - z^2 p (z)$ and $z^4 q (z)$ are regular at this point.

27.2 Singular points of an equation The singular points of the coefficients of Eq. (27.1) are called the *singular points* of this equation. As a rule, the singular points of an equation are singular points for all the solutions of this equation. For instance, the point $z = 0$ is a singular point for Euler's equation (27.7). If the roots λ_1 and λ_2 of Eq. (27.8) are different, then a solution of Euler's equation has the form $w (z) = C_1 z^{\lambda_1} + C_2 z^{\lambda_2}$, where C_1 and C_2 are constants. If the numbers λ_1 and λ_2 are real and nonintegers or if $\operatorname{Im} \lambda_1 \neq 0$ and $\operatorname{Im} \lambda_2 \neq 0$, then point $z = 0$ is a singular point for every solution of Euler's equation (the only exception is the trivial solution $w (z) \equiv 0$).

We will now study the behavior of solutions of Eq. (27.1) in a neighborhood of a point z_0 that is a pole for at least one coefficient of the equation. We take an annulus K: $0 < | z - z_0 | < r$ where the functions $p (z)$ and $q (z)$ are regular and a point $\tilde{z} \in K$. We study the solutions $w_1 (z)$ and $w_2 (z)$ with the Cauchy data

$$w_1 (\tilde{z}) = a_0, \quad w'_1 (\tilde{z}) = a_1, \quad w_2 (\tilde{z}) = b_0, \quad w'_2(\tilde{z}) = b_1, \tag{27.10}$$

and select the numbers a_j and b_j such that

$$\Delta = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \neq 0.$$

Let U be a circle centered at point $\tilde{z}$ and lying in K. By Theorem 1, there are solutions in this circle, $w_1^0(z), w_2^0(z)$, of the Cauchy problem (27.10) that are regular in U. The solutions are linearly independent since $\Delta \neq 0$. Suppose $w_1(z)$ and $w_2(z)$ are the analytic functions generated by the elements $w_1^0(z)$ and $w_2^0(z)$ given at point $\tilde{z}$. By Theorem 3, the functions $w_1(z)$ and $w_2(z)$ are analytic in K and are solutions of Eq. (27.1), i.e. each of their elements satisfies Eq. (27.1).

Let us take a closed curve γ with the initial point at z and lying entirely in K and continue the elements $w_1^0(z)$ and $w_2^0(z)$ analytically along γ. Then

$$w_1^0(z) \to w_1^1(z), \quad w_2^0(z) \to w_2^1(z),$$

where $w_1^1(z)$ and $w_2^1(z)$ are regular functions in U. In addition, these functions are solutions of Eq. (27.1) for $z \in U$. Since the solutions $w_1^0(z)$ and $w_2^0(z)$ are, by construction, linearly independent for $z \in U$, there are constants c_{jk} such that

$$w_1^1(z) = c_{11}w_1^0(z) + c_{12}w_2^0(z), \qquad w_2^1(z) = c_{21}w_1^0(z) + c_{22}w_2^0(z). \quad (27.11)$$

Let us consider the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

and assume that λ_1 and λ_2 are its eigenvalues, i.e. the roots of the equation

$$\begin{vmatrix} c_{11}-\lambda & c_{12} \\ c_{21} & c_{22}-\lambda \end{vmatrix} = 0. \quad (27.12)$$

Theorem 4 (1) *If the roots λ_1 and λ_2 of Eq.* (27.12) *are different, then Eq.* (27.1) *has two solutions of the form*

$$w_1(z) = (z-z_0)^{\rho_1}\varphi_1(z), \qquad w_2(z) = (z-z_0)^{\rho_2}\varphi_2(z), \quad (27.13)$$

where $\lambda_j = e^{2\pi i\rho_j}$ ($j = 1, 2$), and the functions $\varphi_1(z)$ and $\varphi_2(z)$ are regular in the annulus K.

(2) *If the roots of Eq.* (27.12) *are equal, i.e. $\lambda_1 = \lambda_2 = \lambda$, then Eq.* (27.1) *has two solutions of the form*

$$\begin{aligned} w_1(z) &= (z-z_0)^{\rho}\varphi_1(z), \\ w_2(z) &= (z-z_0)^{\rho}\varphi_2(z) + a(z-z_0)^{\rho}\varphi_1(z)\ln(z-z_0). \end{aligned} \quad (27.14)$$

Here $\lambda = e^{2\pi i\rho}$, ρ is a constant, and the functions $\varphi_1(z)$ and $\varphi_2(z)$ are regular in K.

We recall that K is an annulus of the form $0 < |z - z_0| < r$ and that the coefficients of Eq. (27.1) are regular in K.

Proof. We introduce the following vector function in the form of a column vector:

$$w^j(z) = \begin{pmatrix} w_1^j(z) \\ w_2^j(z) \end{pmatrix}, \quad j = 0, 1.$$

We can now write (27.11) in the form

$$w^1(z) = Cw^0(z). \tag{27.15}$$

Suppose T is a nonsingular 2-by-2 matrix such that

$$\tilde{w}^0(z) = Tw^0(z). \tag{27.16}$$

Then after traversal of γ we have

$$\tilde{w}^0(z) \to \tilde{w}^1(z) = Tw^1(z),$$

and (27.15) becomes

$$\tilde{w}^1(z) = TCT^{-1}\tilde{w}^0(z). \tag{27.17}$$

(1) Suppose the eigenvalues λ_1 and λ_2 of matrix C are different. Then there is a matrix T that diagonalizes C:

$$TCT^{-1} = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

With such a choice of T we arrive at the following expression for (27.17):

$$\tilde{w}_1^1(z) = \lambda_1\tilde{w}_1^0(z), \quad \tilde{w}_2^1(z) = \lambda_2\tilde{w}_2^0(z). \tag{27.18}$$

We select ρ_1 in such a way that $e^{2\pi i\rho_1} = \lambda_1$ and consider the function

$$\varphi_1(z) = (z - z_0)^{-\rho_1}\tilde{w}_1^0(z).$$

After we have traversed γ, we have

$$(z - z_0)^{-\rho_1} \to e^{-2\pi i\rho_1}(z - z_0)^{-\rho_1},$$

so that

$$\varphi_1(z) \to e^{2\pi i\rho_1}(z - z_0)^{-\rho_1}\lambda_1\tilde{w}_1^0(z)$$
$$= \lambda_1^{-1}(z - z_0)^{-\rho_1}\lambda_1(z - z_0)^{\rho_1}\varphi_1(z) = \varphi_1(z).$$

Hence, the analytic function $\varphi_1(z)$ is single-valued and therefore is regular in K. This means that Eq. (27.1) has a solution

$$w_1(z) = (z - z_0)^{\rho_1}\varphi_1(z)$$

and, similarly, a solution

$$w_2(z) = (z - z_0)^{\rho_2} \varphi_2(z).$$

This proves proposition (1) of Theorem 4.

(2) Suppose $\lambda_1 = \lambda_2 = \lambda$. Then there is a matrix T that reduces C either (a) to the Jordan normal form or (b) to the diagonal form.

In the case (a) we have

$$TCT^{-1} = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix},$$

so that instead of (27.18) we have

$$\tilde{w}_1^1(z) = \lambda \tilde{w}_1^0(z), \quad \tilde{w}_2^1(z) = \lambda \tilde{w}_2^0(z) + \tilde{w}_1^0(z). \qquad (27.19)$$

For the solution $w_1(z)$ generated by the element $w_1^0(z)$ we once more have the representation (27.13). Dividing the second equation in (22.19) by the first, we find that

$$\frac{\tilde{w}_2^1(z)}{\tilde{w}_1^1(z)} = \frac{\tilde{w}_2^0(z)}{\tilde{w}_1^0(z)} + \frac{1}{\lambda},$$

so that the function $\psi(z) = \tilde{w}_2^0(z)/\tilde{w}_1^0(z)$ possesses the followin property:

$$\psi(z) \to \psi(z) + \frac{1}{\lambda}$$

as γ is traversed. Hence (see Remark 2 in Sec. 21), the function

$$\psi(z) - \frac{1}{2\pi i \lambda} \ln(z - z_0) = \varphi_2(z)$$

is regular in annulus K, and for the solution $w_2(z)$ generated by the element $\tilde{w}_2^0(z)$ we have the representat on (27.14).

In the case (b) we have

$$TCT^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},$$

so that (27.18) has the form

$$\tilde{w}_1^1(z) = \lambda \tilde{w}_1^0(z), \quad \tilde{w}_2^1(z) = \lambda \tilde{w}_2^0(z),$$

while the solutions $w_j(z)$ have the form $w_j(z) = (z - z_0)^{\rho} \varphi_j(z)$, $j = 1, 2$, with $\varphi_j(z)$ regular in K. The proof of the theorem is complete.

Corollary 1 *Suppose the coefficients of Eq.* (27.1) *are regular or have a pole at* $z = \infty$. *Then Eq.* (27.1) *has either two solutions*

$$w_1(z) = z^{\rho_1} \varphi_1(z), \quad w_2(z) = z^{\rho_2} \varphi_2(z) \qquad (27.20)$$

or two solutions

$$w_1(z) = z^\rho \varphi_1(z), \quad w_2(z) = z^\rho \varphi_2(z) + a z^\rho \varphi_1(z) \ln z, \quad (27.21)$$

where the functions $\varphi_1(z)$ *and* $\varphi_2(z)$ *are regular in a punctured neighborhood of point* $z = \infty$.

Indeed, the change of variable $z = 1/\zeta$ reduces Eq. (27.1) to Eq. (27.9); the coefficients of the latter are either regular or have a pole at $\zeta = 0$.

27.3 Regular singular points Suppose z_0 is a pole or a point of regularity for the coefficients of Eq. (27.1). Then there are two possibilities:

(a) point z_0 is a pole or a point of regularity for both functions $\varphi_1(z)$ and $\varphi_2(z)$ in (27.13) and (27.14);

(b) point z_0 is an essential singularity for at least one function $\varphi_1(z)$ or $\varphi_2(z)$.

In the case (a) point z_0 is said to be a *regular singular point* for Eq. (27.1), while in the case (b) it is said to be an *irregular singular point* for Eq. (27.1). These definitions remain valid for point $z_0 = \infty$.

Regular singular points are the more simple singularities and are well documented, while the structure of solutions in a neighborhood of an irregular singular point is very complex and for this reason we will not consider such points here (the interested reader can refer to Smirnov [1]).

Remark 1. Suppose z_0 is a regular singular point and $w_1(z)$ is a solution of the type (27.13). Then $\varphi_1(z) = (z - z_0)^m \tilde{\varphi}_1(z)$, where m is an integer, and $\tilde{\varphi}_1(z)$ is regular and nonzero at point z_0. Substituting $\tilde{\rho}_1 = \rho_1 + m$ for ρ_1, we obtain

$$w_1(z) = (z - z_0)^{\tilde{\rho}_1} \tilde{\varphi}_1(z).$$

Note that $\lambda_1 = e^{2\pi i \tilde{\rho}_1}$.

Therefore, in the case o a regular singular point we can assume that the functions $\varphi_1(z)$ and $\varphi_2(z)$ are regular at point z_0 and that $\varphi_1(z_0) \neq 0$ and $\varphi_2(z_0) \neq 0$.

The definition of a regular singular point we just gave is of an inderect nature, since it was formulated in terms of the properties of solutions rather than in terms of the properties of the coefficients. Let us show that knowing the properties of the coefficients enables us to determine whether a singular point of an equation is regular or irregular. For the sake of simplicity we assume the singular point to be $z = 0$.

Lemma *For point* $z = 0$ *to be a regular singular point or a point of regularity for Eq.* (27.1) *it is necessary that at this point*

(1) *the function* $p(z)$ *have a first order pole or be regular, and*

(2) *the function* $q(z)$ *have a pole of an order no higher than* 2 *or be regular.*

Proof. For the sake of simplicity we will assume that the roots of Eq. (27.12) are different. Then

$$\frac{w''}{w}+p(z)\frac{w'}{w}+q(z)=0. \tag{27.22}$$

Moreover, $w_1(z) = z^{\rho_1}\varphi_1(z)$, with $\lambda_1 = e^{2\pi i\rho_1}$ and $\varphi_1(0) \neq 0$ (see Remark 1). Substituting $w_1(z)$ into Eq. (27.22), we obtain

$$\frac{\rho_1(\rho_1-1)}{z^2}+\frac{2\rho_1}{z}\frac{\varphi_1'(z)}{\varphi_1(z)}+\frac{\varphi_1''(z)}{\varphi_1(z)}+\frac{\rho_1}{z}p(z)+\frac{\varphi_1'(z)}{\varphi_1(z)}p(z)+q(z)=0. \tag{27.23}$$

Similarly, the solution $w_2(z)$ can be written in the form (27.13), where $\varphi_2(z)$ is regular and nonzero at point $z = 0$, so that

$$\frac{\rho_2(\rho_2-1)}{z^2}+\frac{2\rho_2}{z}\frac{\varphi_2'(z)}{\varphi_2(z)}+\frac{\varphi_2''(z)}{\varphi_2(z)}+\frac{\rho_2}{z}p(z)+\frac{\varphi_2'(z)}{\varphi_2(z)}p(z)+q(z)=0. \tag{27.24}$$

The functions $\varphi_j'(z)/\varphi_j(z)$ and $\varphi_j''(z)/\varphi_j(z)$, $j = 1, 2$, are regular at point $z = 0$ since $\varphi_{1,2}(0) \neq 0$. Subtracting (27.24) from (27.23), we find that

$$[\rho_1-\rho_2+za(z)]\,p(z) = z^{-1}b(z), \tag{27.25}$$

where $a(z)$ and $b(z)$ are regular at $z = 0$. Since $\lambda_1 \neq \lambda_2$, we conclude that $\rho_1 \neq \rho_2$, and from (27.25) it follows that the function $p(z)$ is either regular at $z = 0$ or has a first order pole at this point. But then (27.23) implies that $q(z)$ at point $z = 0$ is either regular or has a pole of an order no higher than 2.

Thus, if $z = 0$ is a regular singular point for Eq. (27.1), this equation is of the form

$$w''+\frac{a(z)}{z}w'+\frac{b(z)}{z^2}w=0, \tag{27.26}$$

where $a(z)$ and $b(z)$ are regular at $z = 0$.

It has been proved by L. Fuchs (see Smirnov [1]) that condition (27.26) is sufficient, i.e. we have

Theorem 5 *The point $z = 0$ is a regular singular point for Eq.* (27.1) *if and only if this equation has the form* (27.26), *with $a(z)$ and $b(z)$ regular functions at point $z = 0$.*

Corollary 2 *The point $z = \infty$ is a regular singular point for Eq.* (27.1) *if and only if this equation has the form* (27.26), *with $a(z)$ and $b(z)$ regular functions at point $z = \infty$.*

Example 5. Euler's differential equation (27.7) has two singular points, 0 and ∞. Both are regular singular points. □

Example 6. Bessel's differential equation

$$z^2w'' + zw' + (z^2 - \nu^2)\, w = 0 \tag{27.27}$$

(ν is a constant) has two singular points, 0 and ∞. The first is a regular singular point and the second an irregular singular point. □

Example 7. The hypergeometric differential equation

$$z\,(1 - z)\, w'' + [\gamma - (\alpha + \beta + 1)\, z]\, w' - \alpha\beta w = 0 \tag{27.28}$$

(α, β, and γ are constants) has three singular points, 0, 1, and ∞. All three are regular singular points.

27.4 Construction of solutions in the neighborhood of a regular singular point Let us take Eq. (27.26), for which point $z = 0$ is a regular singular point. In this case we can construct the solution explicitly. We look for the solution in the form of a power series,

$$w\,(z) = z^{\rho} \sum_{n=0}^{\infty} w_n z^n, \tag{28.29}$$

where $w_0 \neq 0$, ρ, and w_n are unknown numbers.

By Theorem 5, such solution indeed exists and the series (27.29) is convergent in a punctured neighborhood of point $z = 0$. We have

$$w'\,(z) = \sum_{n=0}^{\infty} (n+\rho)\, w_n z^{n+\rho-1},$$

$$w''\,(z) = \sum_{n=0}^{\infty} (n+\rho)(n+\rho-1)\, w_n z^{n+\rho-2}.$$

We expand the coefficients $a\,(z)$ and $b\,(z)$ in Taylor series:

$$a\,(z) = \sum_{n=0}^{\infty} a_n z^n, \quad b\,(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Substituting into Eq. (27.29), we find that

$$z\, w_0\, [\rho\,(\rho - 1) + a_0\rho + b_0] + z^{\rho+1} \{w_1\, [\rho\,(\rho + 1) + a_0\,(\rho + 1) + b_0] + w_0\,(\rho a_1 + b_1)\} + \ldots + z^{\rho+n} \{w_n\, [(\rho + n)\,(\rho + n - 1) + a_0\,(\rho + n) + b_0] + \ldots + w_0\,(\rho a_n + b_n)\} + \ldots = 0.$$

Nullifying the coefficients of $z^{\rho+n}$, $n = 1, 2, \ldots$, we arrive at a recurrence system of equations:

$$\begin{gathered} w_0 f_0(\rho) = 0, \\ w_1 f_0(\rho+1) + w_0 f_1(\rho) = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ w_n f_0(\rho+n) + w_{n-1} f_1(\rho+n-1) \\ + w_{n-2} f_2(\rho+n-2) + \ldots + w_0 f_n(\rho) = 0, \end{gathered} \tag{27.30}$$

where

$$\begin{gathered} f_0(\rho) = \rho(\rho-1) + a_0\rho + b_0, \\ f_k(\rho) = \rho a_k + b_k, \quad k \geqslant 1. \end{gathered} \tag{27.31}$$

Since $w_0 \neq 0$, we can write $f_0(\rho) = 0$, i.e.

$$\rho(\rho-1) + a_0\rho + b_0 = 0. \tag{27.32}$$

This equation is known as the *indicial equation*. Let ρ_1 and ρ_2 be the roots of the indicial equation. There are two possibilities here.

(1) If this equation has two distinct roots, not differing by an integer, then $f_0(\rho_1 + n) \neq 0$ and $f_0(\rho_2 + n) \neq 0$ for a single integer $n \geqslant 1$. By selecting either root and solving Eq. (27.30) sequentially we can find $w_1, w_2, \ldots$. In this case Eq. (27.26) has two linearly independent solutions $w_1(z)$ and $w_2(z)$ of the type (27.13).

(2) If this equation has two roots that differ by an integer, $\rho_1 - \rho_2 = m \geqslant 0$, then $f_0(\rho_1) = f_0(\rho_2 + m) = 0$, but $f_0(\rho_1 + n) \neq 0$ for a single integer $n \geqslant 1$. In this case there is only one solution $w_1(z)$ of the type (27.13). The second linearly independent solution can be found by employing Liouville's formula (see Fedoryuk [1]):

$$\begin{vmatrix} w_1(z) & w_2(z) \\ w_1'(z) & w_2'(z) \end{vmatrix} = Ce^{-\int_{z_0}^{z} p(z)dz} \qquad C = \text{const}.$$

We can rewrite this as

$$\left(\frac{w_2(z)}{w_1(z)}\right)' = \frac{C}{w_1^2(z)} e^{-\int_{z_0}^{z} p(z)d},$$

which yields

$$w_2(z) = w_1(z) \int_{z_1}^{z} \frac{Ce^{-\int_{z_0}^{z} p(\zeta)d\zeta}}{w_1^2(z)} dz \tag{27.33}$$

Since $p(z) = a(z)/z = a_0/z + a_1 + \ldots$, we conclude that

$$\int_{z_0}^{z} p(\zeta)\, d\zeta = a_0 \ln z + \psi(z),$$

where $\psi(z)$ is regular at point $z = 0$. Hence, the integrand in (27.33) has the form $z^{-a_0-2\rho_1}\chi(z)$, where $\chi(z)$ is regular and nonzero at point $z = 0$. The indicial equation (27.32) implies that $\rho_1 + \rho_2 = -a_0 + 1$, and since $\rho_1 = \rho_2 + m$, we conclude that $-a_0 - 2\rho_1 = -(m+1)$. We have

$$\chi(z) = \sum_{n=0}^{\infty} \chi_n z^n,$$

whence

$$z^{-a-2\rho_1}\chi(z) = \sum_{n=0}^{\infty} \chi_n z^{-m+n-1}.$$

Integrating (27.33), we find that

$$w_2(z) = w_1(z)\,(\chi_m \ln z + z^{-m} h(z)),$$

where $h(z)$ is regular at point $z = 0$. We finally find that the second linearly independent solut f Eq. (27.26) has the form

$$w_2(z) = z^{\rho_1 - m}\varphi_2(z) + w_1(z)\,\chi_m \ln z, \tag{27.34}$$

where $\varphi_2(z)$ is regular at point $z = 0$

We will now investigate the structure of the solutions of some differential equations in the neighborhood of a regular singular point.

Example 8. The point $z = 0$ is a regular singular point for Bessel's differential equation (27.27). The indicial equation (27.32) in this case is

$$\rho(\rho - 1) + \rho - \nu^2 = 0,$$

and its roots are $\rho_{1,2} = \pm\nu$.

(1) ν is not an integer. Then Bessel's equation has two linearly independent solutions of the form

$$w_1(z) = z^{\nu}\varphi_1(z), \quad w_2(z) = z^{-\nu}\varphi_2(z),$$

where the $\varphi_{1,2}(z)$ are regular and nonzero at point $z = 0$.

(2) ν is an integer. Suppose, for the sake of definiteness, that ν is nonnegative. Then Bessel's equation has a solution of the form $w_1(z) = z^{\nu}\varphi_1(z)$, where the function $\varphi_1(z)$ is regular and nonzero at point $z = 0$. The second linearly independent solution can be found

via (27.33):

$$w_2(z) = w_1(z) \int_{z_1}^{z} \frac{d\zeta}{\zeta w_1^2(\zeta)}.$$

Let ν be nonnegative or not a real number. We wish to find the solution of Bessel's equation. Equations (27.30) are (at $\rho = \nu$)

$$w_1 f_0(\nu + 1) = 0, \quad w_2 f_0(\nu + 2) - w_0 = 0, \ldots,$$
$$w_n f_0(\nu + n) - w_{n-2} = 0,$$

with $f_0(\rho) = \rho - \nu^2$. From this we find that $w_1 = w_3 = \ldots = w_{2n+1} = \ldots = 0$ and $w_{2n} = \frac{w_0(-1)^n}{4^n n!(\nu + 1)\ldots(\nu + n)}$. Hence, the function

$$w_1(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+\nu}}{4^n n!(\nu+1)\ldots(\nu+n)}$$

is a solution of Bessel's equation. Note that

$$(\nu + 1)\ldots(\nu + n) = \Gamma(\nu + n + 1)/\Gamma(\nu).$$

The solution

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{\Gamma(n+1)\Gamma(n + \nu + 1)},$$

which differs from $w_1(z)$ only by a numerical factor, is called the *Bessel function of the first kind of order* ν. If ν is not an integer, then the solutions $J_\nu(z)$ and $J_{-\nu}(z)$ form the fundamental system of solutions of Bessel's equation. □

Example 9. The points $z = \pm 1$ are egular singular points for *Legendre's differential equation*

$$(1 - z^2) w'' - 2zw' + \lambda w = 0$$

(λ is a constant). Let us study the structure of the solutions of this equation in the neighborhood of point $z = 1$. The indicial equation is $\rho(\rho - 1) + \rho = 0$, whence $\rho_1 = \rho_2 = 0$. Hence, Legendre's equation has a solution $w_1(z)$ that is regular and nonzero at point $z = 1$. The second linearly independent solution can be found via (27.33). In the case at hand,

$$p(z) = -\frac{2z}{1-z^2} \quad \frac{d}{dz}\ln(z^2 - 1),$$

so that

$$w_2(z) = w_1(z) \int\limits_{z_1}^{z} \frac{d\zeta}{(\zeta^2 - 1)\, w_1^2(\zeta)}.$$

The integrand in a neighborhood of point $\zeta = 1$ can be expanded in the following series:

$$\frac{1}{w_0(\zeta - 1)} + \sum_{n=0}^{\infty} a_n (\zeta - 1)^n.$$

Integrating this series termwise, we find that

$$w_2(z) = w_1(z) \left(\frac{1}{w_0} \ln (z - 1) + \sum_{n=1}^{\infty} b_n (z - 1)^n \right),$$

whence $w_2(z) = w_1(z) \ln (z - 1) + \varphi(z)$, where $\varphi(z)$ is regular at point $z = 1$, $\varphi(1) = 0$. Thus, the solution $w_2(z)$ has a logarithmic singularity at point $z = 1$. □

Chapter V

Residues and Their Applications

28 Residue Theorems

28.1 Residues at finite points We start with a function $f(z)$ that is regular in a punctured neighborhood of a point a ($a \neq \infty$), i.e. in the annulus K: $0 < |z - a| < \rho_0$. Then point a is either an isolated singular point (a pole or an essential singularity) for $f(z)$ or a point of regularity, and $f(z)$ is represented in K by a convergent Laurent series, $f(z) = \sum\limits_{n=-\infty}^{\infty} c_n (z-a)^n$.

Definition 1. The *residue* of a function $f(z)$ at a point a (denoted by $\operatorname*{Res}\limits_{z=a} f(z)$) is the coefficient c_{-1} of the Laurent expansion of $f(z)$ about point a, i.e.

$$\operatorname*{Res}_{z=a} f(z) = c_{-1}. \tag{28.1}$$

According to (17.7),

$$c_{-1} = \frac{1}{2\pi i} \int\limits_{\gamma_\rho} f(\zeta)\, d\zeta,$$

where the circle γ_ρ: $|z-a| = \rho$ $(0 < \rho < \rho_0)$ is oriented in the positive sense. This yields the following formula:

$$\int\limits_{\gamma_\rho} f(z)\, dz = 2\pi i \operatorname*{Res}_{z=a} f(z). \tag{28.2}$$

Thus, if $z = a$ is an isolated singular point for $f(z)$, the integral of $f(z)$ along the boundary of a small neighborhood of point a is equal to the residue at this point times $2\pi i$. It is obvious then that if a is a point of regularity for $f(z)$, $\operatorname*{Res}\limits_{z=a} f(z) = 0$.

In all the examples in this chapter the path of integration is oriented in the positive sense (if the contrary is not stated explicitly).

Example 1. Find the residue of $e^{1/z}$ at point $z=0$. Since $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \ldots$, we can write $c_{-1} = 1$ and $\operatorname*{Res}\limits_{z=0} e^{1/z} = 1$.

This also means that

$$\int_{|z|=1} e^{1/z}\, dz = 2\pi i \operatorname*{Res}_{z=0} e^{1/z} = 2\pi i. \ \square$$

Example 2. Suppose $f(z) = (\sin z)/z^6$. Then $\operatorname*{Res}_{z=0} f(z) = 1/5!$, since $f(z) = \frac{1}{z^6}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right)$ and $c_{-1} = 1/5!$ Whence,

$$\int_{|z|=2} \frac{\sin z}{z^6}\, dz = \frac{2\pi i}{5!}. \ \square$$

Example 3. If $f(z) = z \cos \frac{1}{z+1}$, then $\operatorname*{Res}_{z=-1} f(z) = -\frac{1}{2}$, since

$$f(z)\ [(z+1)-1]\left[1 - \frac{1}{2(z+1)^2} + \ldots\right] \text{and } c_{-1} = -\frac{1}{2}. \ \square$$

28.2 Calculating residues at poles (a $\neq \infty$) (1) We start with the case of a simple pole. If a is a simple pole for $f(z)$, the Laurent expansion of $f(z)$ about point a has the form

$$f(z) = c_{-1}(z-a)^{-1} + \sum_{n=0}^{\infty} c_n (z-a)^n,$$

with $c_{-1} = \lim_{z \to a} (z-a) f(z)$; whence,

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} (z-a) f(z). \tag{28.3}$$

For instance, if $f(z) = \varphi(z)/\psi(z)$, where $\varphi(z)$ and $\psi(z)$ are regular functions at point a, with $\varphi(a) \neq 0$, $\psi(a) = 0$, and $\psi'(a) \neq 0$, then point a is a simple pole for $f(z)$, and (28.3) yields $\operatorname*{Res}_{z=a} f(z) =$ $\lim_{z\to a} \frac{(z-a)\varphi(z)}{\psi(z)} = \lim_{z \to a} \frac{\varphi(z)}{\frac{\psi(z)-\psi(a)}{z-a}} = \frac{\varphi(a)}{\psi'(a)}$, i.e.

$$\operatorname*{Res}_{z=a} \frac{\varphi(z)}{\psi(z)} = \frac{\varphi(a)}{\psi'(a)}. \tag{28.4}$$

(2) Now we go over to the case where the order of the pole is greater than unity. If point a is a pole of order m for $f(z)$, the Laurent expansion of $f(z)$ about point a has the form

$$f(z) = \frac{c_{-m}}{(z-a)^m} + \ldots + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \ldots. \tag{28.5}$$

Multiplying both sides of (27.5) by $(z-a)^m$, we find that

$$(z-a)^m f(z) = c_{-m} + \ldots + c_{-1}(z-a)^{m-1} + c_0(z-a)^m + \ldots. \tag{28.6}$$

Differentiating (28.6) $m-1$ times and going over to the limit as $z \to a$, we find that $(m-1)!\,c_{-1} = \lim\limits_{z\to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$, which yields a formula for calculating the residue at an mth order pole:

$$\operatorname*{Res}_{z=a} f(z) = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]. \tag{28.7}$$

For instance, if $f(z) = h(z)/(z-a)^m$, with $h(z)$ regular at point a, $h(a) \neq 0$, then (28.7) yields the following formula:

$$\operatorname*{Res}_{z=a} \frac{h(z)}{(z-a)^m} = \frac{1}{(m-1)!} h^{(m-1)}(a) \tag{28.8}$$

Example 4. Let us consider the function $f(z) = \frac{z}{(z-1)(z-2)^2}$, which has a first order pole at point $z=1$ and a second order pole at $z=2$. Formula (28.3) yields $\operatorname*{Res}\limits_{z=1} f(z) = \left[\frac{z}{(z-2)^2}\right]_{z=1} = 1$, while formula (28.8) yields $\operatorname*{Res}\limits_{z=2} f(z) = \left(\frac{z}{z-1}\right)'_{z=2} = -1$. □

Example 5. The points $z = k\pi$ (where k is an integer) are simple poles for $\cot z = \frac{\cos z}{\sin z}$; whence (28.4) yields

$$\operatorname*{Res}_{z=k\pi} \cot z = \left[\frac{\cos z}{(\sin z)'}\right]_{z=k\pi} = 1.$$

From this it follows, for instance, that the principal part of the Laurent expansion of $\cot z$ about $k\pi$ is $1/(z-k\pi)$. □

28.3 Residues at the point at infinity Suppose a function $f(z)$ is regular in a punctured neighborhood of point $z = \infty$, i.e. in the domain $\rho_0 < |z| < \infty$. Then point $z = \infty$ is either an isolated singular point for $f(z)$ (a pole or an essential singularity) or a point of regularity, and $f(z)$ can be represented in $\rho_0 < |z| < \infty$ by a convergent Laurent series,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^n + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \dots. \tag{28.9}$$

Definition 2. The *residue* of a function $f(z)$ at point $z = \infty$ (denoted by $\operatorname*{Res}\limits_{z=\infty} f(z)$) is the number $-c_{-1}$, where c_{-1} is the coefficient of $1/z$ in the Laurent expansion of $f(z)$ about the point at infinity, i.e.

$$\operatorname*{Res}_{z=\infty} f(z) = -c_{-1}. \tag{28.10}$$

According to (17.7) $c_{-1}=\frac{1}{2\pi i}\int\limits_{|z|=\rho} f(z)\,dz$, where the circle $|z|=\rho$ $(\rho>\rho_0)$ is oriented counterclockwise. Combining this with (28.10), we find that

$$\int\limits_{\gamma_\rho} f(z)\,dz=2\pi i \operatorname*{Res}_{z=\infty} f(z), \tag{28.11}$$

where γ_ρ is the circle $|z|=\rho$ oriented clockwise.

Remark 1. We can combine formulas (28.2) and (28.11). Indeed, if $f(z)$ is regular in a punctured neighborhood U of a finite point or the point at infinity, both denoted by a, then the integral of $f(z)$ along the boundary γ_ρ of this neighborhood is equal to the residue at point a times $2\pi i$. (The neighborhood U is to the left as γ_ρ is traversed in (28.2) and (28.11).)

Suppose $z=\infty$ is a kth order zero for $f(z)$. Then in a neighborhood of the point at infinity we can represent $f(z)$ by the Laurent expansion $f(z)=\frac{c_{-k}}{z^k}+\frac{c_{-(k+1)}}{z^{k+1}}+\ldots$, where $c_{-k}\neq 0$. As $z\to\infty$, we arrive at the following asymptotic formula

$$f(z)\sim\frac{A}{z^k}\quad (A=c_{-k}\neq 0).$$

If $k=1$, then $\operatorname*{Res}_{z=\infty} f(z)=-c_1=-A$, while if $k\geqslant 2$, $\operatorname*{Res}_{z=\infty} f(z)=0$. Thus,

$$f(z)\sim\frac{A}{z}\quad (z\to\infty)\qquad\Rightarrow \operatorname*{Res}_{z=\infty} f(z)=-A, \tag{28.12}$$

$$f(z)\sim\frac{A}{z^k}\quad (z\to\infty,\quad k\geqslant 2)\Rightarrow \operatorname*{Res}_{z=\infty} f(z)=0. \tag{28.13}$$

Example 6. For the function $e^{1/z}=1+\frac{1}{z}+\frac{1}{2!z^2}+\ldots$ the coefficient c_{-1} is equal to 1, so that $\operatorname*{Res}_{z=\infty} e^{1/z}=-1$. Note that this function is regular at point $z=\infty$, and nevertheless the residue at this point is nonzero. □

Example 7. For the function $f(z)=\frac{1}{z+2}\cos\frac{1}{z}$ the point $z=\infty$ is a first order zero: $f(z)\sim 1/z$ $(z\to\infty)$. From (28.12) we find that $\operatorname*{Res}_{z=\infty} f(z)=-1$. □

Example 8. For the function $f(z)=\frac{z}{z^3+1}\sin\frac{1}{z}$ the point $z=\infty$ is a third order zero: $f(z)\sim 1/z^3$ $(z\to\infty)$. From (28.13) we find that $\operatorname*{Res}_{z=\infty} f(z)=0$. □

Example 9. Suppose $f(z)$ is a regular branch of the function

$\left(\frac{1-z}{1+z}\right)^\alpha$ in the plane with a cut along $[-1, 1]$ on whose upper bank the function assumes the value 1 at point $z=0$ (see Example 17 in Sec. 24). Then the Laurent expansion of $f(z)$ about point $z=\infty$ has the form $f(z)=e^{-i\alpha\pi}\left(1-\frac{2\alpha}{z}+\dots\right)$. This yields $\operatorname*{Res}_{z=\infty} f(z)=2\alpha e^{-i\alpha\pi}$. □

28.4 The fundamental residue theorem

Theorem 1 (the fundamental residue theorem) *Suppose a function $f(z)$ is regular in a simply connected domain D except at a finite number of singular points $z_1, z_2, \dots, z_n$ and suppose γ is a simple closed curve lying in D and containing $z_1, z_2, \dots, z_n$ in its interior. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z), \tag{28.14}$$

where γ is oriented in the positive sense.

Proof. Let the γ_k ($k=1, 2, \dots, n$) be small circles centered at the z_k and oriented counterclockwise. In view of Corollary 2 of Sec. 9 we have

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz,$$

which, when combined with (28.2), yields (28.14).

Corollary *Suppose $f(z)$ is regular in the entire extended complex z plane except at a finite number of singular points. Then the sum of all the residues of $f(z)$, including the residue at point $z=\infty$, is zero, i.e.*

$$\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0. \tag{28.15}$$

Here the z_k ($k=1, 2, \dots, n$) are the finite singular points of $f(z)$, and point $z=\infty$ is either a singular point for $f(z)$ or a point of regularity.

Proof. Suppose γ is the circle $|z|=R$ oriented in the positive sense, with R so chosen that all the points z_k ($k=1, 2, \dots, n$) lie in the interior of γ. By Theorem 1,

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z). \tag{28.16}$$

On the other hand, (28.11) implies that

$$\int_{\gamma} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z). \tag{28.17}$$

Formula (28.15) follows from (28.16) and (28.17).

The following theorem is a generalization of Theorem 1:

Theorem 2 *Suppose a function $f(z)$ is regular in a domain D in the extended complex plane except at a finite number of singular points and is continuous up to the boundary Γ of D. Suppose Γ consists of a finite number of limited piecewise smooth curves. Then*

(a) $$\int_{\Gamma} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \tag{28.18}$$

if D does not contain point $z=\infty$, and

(b) $$\int_{\Gamma} f(z)\,dz = 2\pi i\Big(\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z)\Big) \tag{28.19}$$

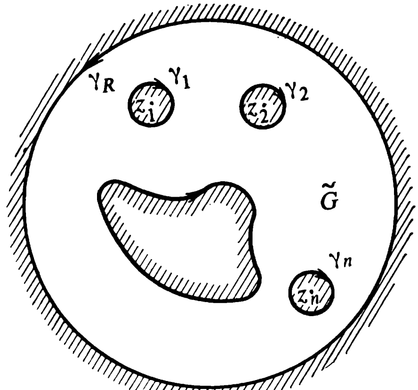


Fig. 68

if D contains point $z=\infty$. Here $z_1, z_2, \ldots, z_n$ are all the finite singular points of $f(z)$ lying in D.

Proof. (a) Let D be a bounded domain. Consider the multiply connected domain D obtained as a result of deleting D of small circles K centered at the respective points z_j ($j = 1, 2, \ldots, n$). By Theorem 4 of Sec. 9, the integral of $f(z)$ along the boundary $\tilde{\Gamma}$ of $\tilde{D}$ is equal to zero, i.e.

$$\int_{\tilde{\Gamma}} f(z)\,dz = \int_{\Gamma} f(z)\,dz + \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,dz = 0, \tag{28.20}$$

where the boundary γ_j of circle K_j is oriented clockwise. Since

$$\int_{\gamma_j} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=z_j} f(z)$$

(see (28.2)), we see that (28.18) follows from (28.20).

(b) Let K be a circle $|z| < R$ that contains the boundary Γ of D and all the finite singular points of $f(z)$ (Fig. 68). Consider the domain $\widetilde{G}$ obtained from $G = D \cap K$ by deleting the circles K_j mentioned earlier. The boundary $\widetilde{\Gamma}$ of $\widetilde{G}$ consists of Γ, the circles γ_j (the boundaries of the K_j), and the circle γ_R : $|z| = R$. We have

$$\int_{\widetilde{\Gamma}} f(z)\,dz = \int_{\Gamma} f(z)\,dz + \sum_{j=1}^{n} \int_{\gamma_j} f(z)\,dz + \int_{\gamma_R} f(z)\,dz = 0, \qquad (28.21)$$

where γ_R is oriented in the positive sense. Since

$$\int_{\gamma_R} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z),$$

we see that (28.19) follows from (28.21). The proof of the theorem is complete.

The residue theorems we have just proved are most important to the theory of functions of a complex variable. They can be effectively used when evaluating many definite integrals.

28.5 Evaluating integrals along closed curves Here we give some examples concerned with evaluating integrals along closed curves via the theory of residues. In all examples the traversal of the integration path γ is in the positive sense, i.e. the interior of γ remains to the left.

Example 10. Let $f(z) = (\cos z)/z^3$. Then (28.14) yields

$$\int_{|z|=2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z).$$

Since in the circle $|z| < 2$ the function $f(z)$ has one singular point at $z = 0$ (a pole) and $f(z) = \frac{1}{z^3} - \frac{1}{2!z} + \frac{1}{4!}z + \ldots$, we conclude that $\operatorname*{Res}_{z=0} f(z) = c_{-1} = -1/2$. Hence, $\int_{|z|=2} \frac{\cos z}{z^3}\,dz = -\pi i$. □

Example 11. Let $f(z) = 1/(e^z + 1)$. Then $I = \int_{|z-2i|=2} f(z)\,dz = 2\pi i \times \operatorname*{Res}_{z=\pi i} f(z)$, since $f(z)$ has one singular point in the circle $|z - 2i| < 2$, namely, a first order pole at $z = \pi i$. Formula (28.4) then yields

$$\operatorname*{Res}_{z=\pi i} f(z) = \frac{1}{(e^z + 1)'_{z=\pi i}} = -1,$$

whence $I = -2\pi i$. □

Example 12. If $f(z) = (2z - 1)\cos\frac{z}{z-1}$, then $I = \int_{|z|=2} f(z)\,dz =$

$2\pi i \operatorname*{Res}_{z=1} f(z)$, since $f(z)$ is regular in the circle $|z| < 2$ except at point $z = 1$, which is an essential singularity for $f(z)$. We have

$$\cos \frac{z}{z-1} = \cos\left(1 + \frac{1}{z-1}\right)$$
$$= \cos 1 \times \cos \frac{1}{z-1} - \sin 1 \times \sin \frac{1}{z-1}$$
$$= \cos 1 \left(1 - \frac{1}{2(z-1)^2} + \ldots\right) - \sin 1 \left(\frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \ldots\right),$$
$$2z - 1 = 2(z-1) + 1,$$

which shows that the coefficient c_{-1} of the term $(z-1)^{-1}$ in the Laurent expansion of $f(z)$ is

$$c_{-1} = -(\cos 1 + \sin 1).$$

Hence, $I = -2\pi i (\cos 1 + \sin 1)$. □

Example 13. Let us evaluate the integral $I = \int\limits_{|z|=4} \frac{e^{1/(z-1)}}{z-2}\,dz$. There are two ways in which we can do this.

(1) The function $f(z) = \frac{1}{z-2} e^{1/(z-1)}$ has in the circle $|z| < 4$ two singular points, $z = 1$ and $z = 2$. Hence,

$$I = 2\pi i \left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z)\right).$$

Since

$$e^{1/(z-1)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!(z-1)^n}, \quad \frac{1}{z-2} = -\frac{1}{1-(z-1)} = -\sum_{n=0}^{\infty} (z-1)^n,$$

we conclude that $\operatorname*{Res}_{z=1} f(z) = -\sum_{n=1}^{\infty} \frac{1}{n!} = 1 - e$. We can also write

$$\operatorname*{Res}_{z=2} f(z) = (e^{1/(z-1)})_{z=2} = e.$$

The final result is $I = 2\pi i$.

(2) $I = -2\pi i \operatorname*{Res}_{z=\infty} f(z)$. The point $z = \infty$ is a first order zero for $f(z)$:

$$\frac{1}{z-2} \sim \frac{1}{z}, \quad e^{1/(z-1)} \sim 1, \quad f(z) \sim \frac{1}{z} \quad (z \to \infty).$$

Formula (28.12) yields $\operatorname*{Res}_{z=\infty} f(z) = -1$ and, hence, $I = 2\pi i$. □

Example 14. We take $P(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$, a polynomial of a degree not lower than 2, and γ, a circle whose interior contains all the zeros of the polynomial. Let us show that the function

$$w(z)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{e^{z\zeta}}{P(\zeta)}\,d\zeta \tag{28.22}$$

satisfies the equation

$$P\left(\frac{d}{dz}\right)w(z)\equiv w^{(n)}(z)+a_1w^{(n-1)}(z)+\ldots+a_{n-1}w'(z)+a_nw(z)=0 \tag{28.23}$$

and the following initial conditions:

$$w(0)=0,\quad w'(0)=0,\ \ldots,\ w^{(n-2)}(0),\quad w^{(n-1)}(0)=1. \tag{28.24}$$

The definition (28.22) gives us the derivatives of $w^{(k)}(z)$, i.e. $w^{(k)}(z)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\zeta^k e^{z\zeta}}{P(\zeta)}\,d\zeta$; hence,

$$P\left(\frac{d}{dz}\right)w(z)=\frac{1}{2\pi i}\int\limits_{\gamma}e^{z\zeta}\,d\zeta=0,$$

which proves (28.23). Let us see whether this function satisfies the initial conditions (28.24). We have

$$w^{(k)}(0)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\zeta^k}{P(\zeta)}\,d\zeta=-\operatorname*{Res}_{\zeta=\infty}\frac{\zeta^k}{P(\zeta)}. \tag{28.25}$$

If $k<n$, the function $\zeta^k/P(\zeta)$ has an $(n-k)$ th zero at point $\zeta=\infty$, which means that $\operatorname*{Res}_{\zeta=\infty}\zeta^k/P(\zeta)=0$ for $k\leqslant n-2$. Thus, $w^{(k)}(0)=0$ at $k=0, 1, \ldots, n-2$. Now let $k=n-1$. Then $\zeta^{n-1}/P(\zeta)\sim 1/\zeta(\zeta\to\infty)$, so that $\operatorname*{Res}_{\zeta=\infty}(\zeta^{n-1}/P(\zeta))=-1$, and from (28.25) it follows that $w^{(n-1)}(0)=1$. □

28.6 Integrals of multiple-valued functions Here are some examples concerned with the evaluation of integrals of regular branches of multiple-valued analytic functions. In Examples 15 to 18 the integrals of all the branches of the multiple-valued analytic function under the integral sign will be evaluated.

Example 15. Let us evaluate $\int\limits_{|z-1|=1/2}\frac{\sqrt{z}}{z-1}\,dz$. The function $\sqrt{z}$ splits in the circle K: $|z-1|<1/2$ into two regular branches, $g_1(z)$ and $g_2(z)\equiv-g_1(z)$, which means that the integrand splits into two regular branches, $f_1(z)=g_1(z)/(z-1)$ and $f_2(z)=g_2(z)/(z-1)$. Let $g_1(z)$ be the branch of the root on which

$g_1(1) = 1$. Then $g_2(1) = -1$. Each function $f_{1,2}(z)$ is regular in K except at point $z = 1$, which is a simple pole. By the residue theorem (Theorem 1),

$$\int_{|z-1|=1/2} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=1} f_1(z) = 2\pi i g_1(1) = 2\pi i.$$

Similarly, $\displaystyle\int_{|z-1|=1/2} f_2(z)\,dz = -2\pi i.$ □

Example 16. Let us calculate $\displaystyle\int_{|z|=2} \frac{dz}{z\sqrt{z^2-1}}$. The integrand splits in the domain $|z| > 2$ into two regular branches, $f_1(z)$ and $f_2(z)$. Suppose $f_1(z)$ is the branch on which $z^2 f_1(z) \to 1$ as $z \to \infty$. Then for branch $f_2(z)$ we have $z^2 f_2(z) \to -1$ as $z \to \infty$. Since for $f_1(z)$ and $f_2(z)$ point $z = \infty$ is a second order zero, we have

$$\int_{|z|=2} f_1(z)\,dz = \int_{|z|=2} f_2(z)\,dz = 0. \quad \square$$

Example 17. Let us calculate $\displaystyle\int_{|z+1|=1/2} \frac{z^2+1}{\ln z - \pi i}dz$. The function $\ln z$ splits in the circle K_0: $|z+1| < 1/2$ into an infinite number of regular branches $g_k(z)$ defined by the condition $g_k(-1) = (2k + 1)\pi i$. We introduce the notation $f_k(z) = \dfrac{z^2+1}{g_k(z) - \pi i}$. Since $g_k(z) \neq \pi i$ in the circle K_0 if $k \neq 0$, we find that each function $f_k(z)$ for $k \neq 0$ is regular and, hence,

$$\int_{|z+1|=1/2} f_k(z)\,dz = 0 \quad (k \neq 0).$$

For the branch $f_0(z)$ the point $z = -1$ is a first order pole. For this reason $\operatorname*{Res}_{z=-1} f_0(z) = \left[\dfrac{z^2+1}{(\ln z - \pi i)'}\right]_{z=-1} = \dfrac{2}{(1/z)_{z=-1}} = -2$ and

$$\int_{|z+1|=1/2} f_0(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_0(z) = -4\pi i. \quad \square$$

Example 18. Suppose $f(z)$ is the branch of the analytic function $\sqrt[3]{z/(1-z)}$ in the plane with the cut along $[0, 1]$ on whose upper bank the function assumes positive values. Let us calculate the integral $\displaystyle\int_{|z|=2} f(z)\,dz$. Since $f(z)$ is regular in the domain $|z| > 2$, we can write $\displaystyle\int_{|z|=2} f(z)\,dz = -2\pi i \operatorname*{Res}_{z=\infty} f(z)$. If we now employ the Laurent expansion of $f(z)$ about point $z=\infty$ from Example 18 in Sec. 24,

$$f(z)=e^{i\pi/3}\sum_{n=0}^{\infty}C^{n}_{-1/3}(-1)^{n}z^{-n}=e^{i\pi/3}\left(1+\frac{1}{3z}+\ldots\right),$$

we find that $\operatorname*{Res}_{z=\infty} f(z)=-\frac{1}{3}e^{i\pi/3}$. Hence,

$$\int\limits_{|z|=2} f(z)\,dz=-2\pi i\operatorname*{Res}_{z=\infty} f(z)=\frac{2\pi i}{3}e^{i\pi/3}. \quad \square$$

29 Use of Residues for Evaluating Definite Integrals

The residue theorems (Theorems 1 and 2 of Sec. 28) make it possible to reduce the evaluation of integrals of complex valued functions along closed curves to finding the residues of the integrand in the interior of the path of integration. The same method can be used to evaluate many integrals of functions of a real variable. In many cases it is possible to evaluate definite integrals via residues quite simply even when other integration methods fail. For instance, if all the singular points of the integrand lying inside the integration contour are poles, calculating the residues is reduced to finding derivatives. Hence, in this case the evaluation of integrals is reduced to finding derivatives.

29.1 Integrals of the type $I=\int\limits_0^{2\pi} R(\cos\varphi,\ \sin\varphi)\,d\varphi$ Integrals of the type

$$I=\int\limits_0^{2\pi} R(\cos\varphi,\ \sin\varphi)\,d\varphi \tag{29.1}$$

are reduced to integrals along closed curves (here $R(u, v)$ is a rational function of u and v). Let $z=e^{i\varphi}$. Then

$$\sin\varphi=\frac{1}{2i}\left(z-\frac{1}{z}\right),\quad \cos\varphi=\frac{1}{2}\left(z+\frac{1}{z}\right),\quad d\varphi=-i\frac{dz}{z}.$$

If we vary φ form 0 to 2π, the variable z runs along the circle $|z|=1$ in the positive direction. The integral (29.1) can be reduced to an integral along a closed curve, $I=\int\limits_{|z|=1} R_1(z)\,dz$, where $R_1(z)=-\frac{i}{z}R\left[\frac{1}{2}\left(z+\frac{1}{z}\right),\ \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]$ is a rational function of z. By the residue theorem (Theorem 1 of Sec. 28),

$$I = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} R_1(z),$$

where $z_1, z_2, \ldots, z_n$ are the poles of the rational function $R_1(z)$ that lie in the circle $|z| < 1$.

Example 1. We wish to evaluate the integral

$$I = \int_0^{2\pi} \frac{d\varphi}{1-2a\cos\varphi + a^2} \qquad |a| < 1.$$

To this end we introduce a new variable, $z = e^{i\varphi}$. This yields

$$I = \int_{|z|=1} \frac{i\,dz}{az^2-(a^2+1)z+a}.$$

The denominator $az^2 - (a^2+1)z + a$ has its zeros at $z_1 = a$ and $z_2 = 1/a$. Since $|a| < 1$, only one of these points lies in the circle $|z| < 1$, i.e. $z_1 = a$, which is a first order pole of the integrand $f(z)$. Formula (28.4) then yields

$$\operatorname*{Res}_{z=a} f(z) = \frac{i}{[2az-(a^2+1)]_{z=a}} = \frac{i}{a^2-1},$$

whence $I = 2\pi/(1-a^2)$. □

29.2 Integrals of rational functions Consider the integral

$$I = \int_{-\infty}^{\infty} R(x)\,dx, \qquad (29.2)$$

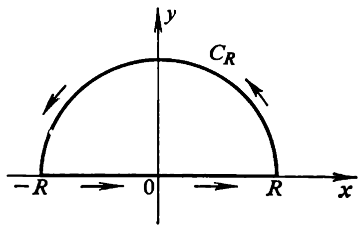


Fig. 69

where $R(x)$ is a rational function. We assume that the integral (29.2) has a finite value.

Here we cannot apply directly the residue theorems since the path of integration is an infinite open curve. To be able to employ these theorems, we introduce an auxiliary closed curve Γ_R (Fig. 69) consisting of the segment $[-R, R]$ and the semicircle C_R ($|z| = R$, $0 \leqslant \arg z \leqslant \pi$) and consider the integral

$$\int_{\Gamma_R} R(z)\,dz.$$

But first let us prove

Lemma 1 *Let a function $f(z)$ be regular in the domain* $\operatorname{Im} z > 0$ *except at a finite number of singular points and continuous up to the* *boundary of this domain. If the integral*

$$\int_{-\infty}^{\infty} f(x)\,dx \tag{29.3}$$

has a finite value and

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0, \tag{29.4}$$

where C_R is the semicircle $|z| = R$, $\operatorname{Im} z \geqslant 0$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\operatorname{Im} z_k>0} \operatorname*{Res}_{z=z_k} f(z). \tag{29.5}$$

In the last formula the residues are calculated at all singular points of the function $f(z)$ that lie in the upper half-plane.

Proof. We take the contour Γ_R (Fig. 69) and select R so large that all the singular points of $f(z)$ that lie in the upper half-plane are inside Γ_R. The residue theorem (Theorem 1 of Sec. 28) then yields

$$\int_{\Gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \sum_{\operatorname{Im} z_k>0} \operatorname*{Res}_{z=z_k} f(z).$$

Now we go over to the limit as $R \to \infty$. Since the integral (29.3) has a finite value, we have

$$\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Moreover, $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$, in view of (29.4). This proves the validity of (29.5).

We now turn to the integral (29.2). Suppose $R(z) = P_n(z)/Q_m(z)$, where $P_n(z)$ and $Q_m(z)$ are polynomials of degrees n and m, respectively. The fact that the integral (29.2) is finite (by hypothesis) implies that $k = m - n \geqslant 2$ and $R(z)$ has no poles on the real axis. Hence,

$$R(z) \sim A/z^k \quad (z \to \infty,\ k \geqslant 2 \text{ is an integer}),$$

so that $|R(z)| \leqslant c\,|z|^{-2}$ when $|z|$ is large. Then $|R(z)| \leqslant cR^{-2}$ on C_R and, hence,

$$\left| \int_{C_R} R(z)\,dz \right| \leqslant cR^{-2}\pi R \to 0 \quad (R \to \infty).$$

We have thus proved that condition (29.4) is satisfied and, by Lemma 1,

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname*{Res}_{z=z_k} R(z). \tag{29.6}$$

Here the residues are calculated at all the poles of $R(z)$ that lie in the upper half-plane.

Similarly, we can write

$$\int_{-\infty}^{\infty} R(x)\,dx = -2\pi i \sum_{\operatorname{Im} z_k < 0} \operatorname*{Res}_{z=z_k} R(z).$$

Example 2. Let us evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^4}.$$

Since the function $R(z) = \frac{1}{(z^2+1)^4} = \frac{1}{(z-i)^4(z+i)^4}$ has only one (fourth order) pole in the upper half-plane, at $z = i$, formula (28.8) yields

$$\operatorname*{Res}_{z=i} R(z) = \frac{1}{3!}\left[\frac{1}{(z+i)^4}\right]^{(3)}_{z=i} = -\frac{5i}{32}.$$

and, by (29.6), $I = 2\pi i \operatorname*{Res}_{z=i} R(z) = 5\pi/16$. □

Example 3. Let us evaluate the integral $I = \int_0^{\infty} \frac{dx}{1+x^{2n}}$ where n is a positive integer.

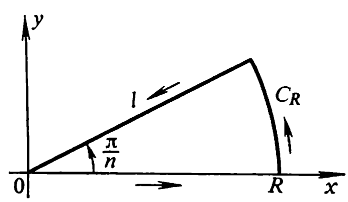


Fig. 70

The equation $z^{2n} + 1 = 0$ has the following roots: $z_k = e^{i(2k+1)\pi/2n}$, $k = 0, 1, \ldots, 2n-1$. Let us employ the following property of the integrand, i.e. $R(e^{i\pi/n} z) \equiv R(z)$. We take the integration path in the form of the curve Γ_R depicted in Fig. 70; this curve consists of the segment $[0, R]$, the arc C_R: $z = Re^{i\varphi}$, $0 \leqslant \varphi \leqslant \pi/n$, and the segment $l: z = re^{i\pi/n}$, $0 \leqslant r \leqslant R$. By the residue theorem (Theorem 1 of Sec. 28),

$$\int_{\Gamma_R} R(z)\,dz = \int_0^R R(x)\,dx + \int_{C_R} R(z)\,dz + \int_l R(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_0} R(z), \tag{29.7}$$

since $R(z)$ has only one pole at $z_0 = e^{i\pi/2n}$ in the interior of Γ_R.

The residue at this point is

$$\operatorname*{Res}_{z=z_0} R(z) = \frac{1}{2nz_0^{2n-1}} = -\frac{z_0}{2n}.$$

The integral along l can be reduced to the integral along $[0, R]$:

$$\int_l R(z)\,dz = -\int_0^R R(re^{i\pi/n})\,e^{i\pi/n}\,dr = -e^{i\pi/n}\int_0^R \frac{dr}{1+r^{2n}}.$$

Finally, we estimate the integral along C_R. Since $|R(z)| \sim 1/|z|^{2n}$ $(z \to \infty)$, we can write

$$\int_{C_R} R(z)\,dz \to 0 \quad (R \to \infty).$$

Taking the limit of (29.7) as $R \to \infty$, we find that

$$(1-e^{i\pi/n})\,I = -\frac{\pi i}{n}\,e^{i\pi/(2n)};\ I = \frac{\pi}{2n\sin\frac{\pi}{2n}}. \quad \square$$

29.3 Integrals of the type $I = \int_{-\infty}^{\infty} e^{i\alpha x} R(x)\,dx$ Here $R(x)$ is a rational function of x. The integral

$$I = \int_{-\infty}^{\infty} e^{i\alpha x} R(x)\,dx \tag{29.8}$$

is the Fourier transform of $R(x)$. To evaluate integrals of the type (29.8), we employ

Lemma 2 (Jordan's lemma) *Suppose α is positive and the following conditions are met:*

(1) *a function $g(z)$ is continuous in the domain* $\operatorname{Im} z \geqslant 0$, $|z| \geqslant R_0 > 0$;

(2) $$M(R) = \max_{z \in C_R} |g(z)| \to 0 \text{ as } R \to \infty, \tag{29.9}$$

where C_R is the semicircle $|z| = R$, $\operatorname{Im} z \geqslant 0$.

Then

$$\lim_{R\to\infty} \int_{C_R} g(z)\,e^{i\alpha z}\,dz = 0. \tag{29.10}$$

Proof. Suppose $z \in C_R$ and $R > R_0$. Then $z = Re^{i\varphi}$, $0 \leqslant \varphi \leqslant \pi$, $dz = iRe^{i\varphi}d\varphi$, and

$$|e^{i\alpha z}| = |e^{i\alpha(R\cos\varphi + iR\sin\varphi)}| = e^{-\alpha R\sin\varphi}. \tag{29.11}$$

Since α is positive, we can write $|e^{i\alpha z}| \leqslant 1$ $(z \in C_R)$. But this estimate is not sufficient for proving the validity of (29.10). To estimate $|e^{i\alpha z}|$ on C_R more precisely, we employ the inequality

$$\sin\varphi \geqslant \frac{2}{\pi}\varphi, \quad 0 \leqslant \varphi \leqslant \frac{\pi}{2}, \tag{29.12}$$

which is valid due to the convexity of $\sin\varphi$ on $[0, \pi/2]$.

Let us estimate the integral $I_1 = \int\limits_{C_R} e^{i\alpha z} g(z)\, dz$. Using (29.11), we find that

$$|I_1| \leqslant \max_{z \in C_R} |g(z)| \int\limits_0^{\pi} e^{-\alpha R \sin\varphi} R\, d\varphi = 2RM(R) \int\limits_0^{\pi/2} e^{-\alpha R \sin\varphi}\, d\varphi,$$

whence by virtue of (29.12) we find that

$$|I_1| \leqslant 2RM(R) \int\limits_0^{\pi/2} e^{-2R\frac{\alpha}{\pi}\varphi}\, d\varphi = M(R)\left(-\frac{\pi}{\alpha}\right) e^{-2R\frac{\alpha}{\pi}\varphi}\Big|_0^{\pi/2}$$

$$= M(R)\frac{\pi}{\alpha}(1 - e^{-\alpha R}) \leqslant \frac{\pi}{\alpha} M(R).$$

This estimate combined with (29.9) yields (29.10). The proof of Jordan's lemma is complete.

Now let us study the integral (29.8). The integral has a finite value if and only if $R(z)$ has no poles on the real axis and $R(x) \sim c/x^k$ $(x \to \infty)$, $k \geqslant 1$. This means that condition (29.9) is met and, by Jordan's lemma,

$$\int\limits_{C_R} e^{i\alpha z} R(z)\, dz \to 0 \quad (R \to \infty,\ \alpha > 0).$$

Formula (29.5) then yields

$$\int\limits_{-\infty}^{\infty} e^{i\alpha x} R(x)\, dx = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname*{Res}_{z=z_k} (e^{i\alpha z} R(z)). \tag{29.13}$$

Remark 1. If α is negative, then, replacing the contour Γ_R in Fig. 69 with its counterpart symmetric with respect to the real axis, we have

$$\int\limits_{-\infty}^{\infty} R(x) e^{i\alpha x}\, dx = -2\pi i \sum_{\operatorname{Im} z_k < 0} \operatorname*{Res}_{z=z_k} (e^{i\alpha z} R(z)).$$

Remark 2. If $R(x)$ is real for real x's and if $\alpha > 0$, then we can separate in (29.13) the real and imaginary parts, which yields

$$\begin{aligned}\int_{-\infty}^{\infty} R(x)\cos\alpha x\,dx &= -2\pi\,\mathrm{Im}\left[\sum_{\mathrm{Im}\,z_k>0} \operatorname*{Res}_{z=z_k}(e^{i\alpha z}R(z))\right],\\ \int_{-\infty}^{\infty} R(x)\sin\alpha x\,dx &= 2\pi\,\mathrm{Re}\left[\sum_{\mathrm{Im}\,z_k>0} \operatorname*{Res}_{z=z_k}(e^{i\alpha z}R(z))\right].\end{aligned} \tag{29.14}$$

There is no need in memorizing these formulas. It is much more important to understand the methods that led us to these formulas.

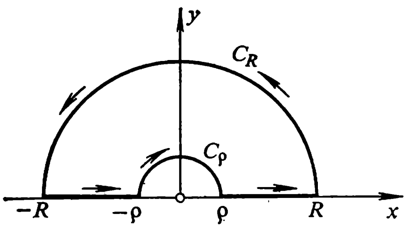


Fig. 71

Example 4. We wish to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{(x-1)\cos 5x}{x^2-2x+5}\,dx.$$

Formula (29.14) yields

$$I = -2\pi\,\mathrm{Im}\left[\operatorname*{Res}_{z=1+2i}\left(e^{i5z}\frac{z-1}{z^2-2z+5}\right)\right],$$

since the integrand $f(z)$ has one (first order) pole in the upper half-plane. Formula (28.4) then yields

$$\operatorname*{Res}_{z=1+2i} f(z) = \left[\frac{e^{i5z}(z-1)}{(z^2-2z+5)'}\right]_{z=1+2i} = \frac{e^{-10}}{2}(\cos 5 + i\sin 5).$$

Whence $I = -\pi e^{-10}\sin 5$. □

Example 5. Let us evaluate the integral $I = \int_0^{\infty} \frac{\sin x}{x}\,dx$. Let $\Gamma_{\rho,R}$ be the contour depicted in Fig. 71. We consider the integral

$$I_{\rho,R} = \int_{\Gamma_{\rho,R}} \frac{e^{iz}}{z}\,dz.$$

It is equal to zero because e^{iz}/z is regular inside $\Gamma_{\rho,R}$, but, on the other hand, it is equal to the sum of the integrals taken along C_ρ and C_R and the segments $[-R, -\rho]$ and $[\rho, R]$. We have

$$\frac{e^{iz}}{z} = \frac{1}{z} + h(z),$$

where $h(z)$ is regular at point $z = 0$. If $z \in C_\rho$, then $z = \rho e^{i\varphi}$, $0 \leqslant \varphi \leqslant \pi$, $dz = i\rho e^{i\varphi}\,d\varphi$, and

$$\int_{C_\rho} \frac{1}{z}\,dz = i\int_{\pi}^{0} d\varphi = -i\pi.$$

The function $h(z)$ is bounded in a neighborhood of point $z=0$ and, hence, $\int\limits_{C_\rho} h(z)\,dz \to 0$ as $\rho\to 0$. This yields

$$\int\limits_{C_\rho} \frac{e^{iz}}{z}\,dz \to -i\pi \text{ as } \rho\to 0.$$

The integral along C_R tends to zero as $R\to\infty$ (Jordan's lemma). The sum of the integrals along $[-R, -\rho]$ and $[\rho, R]$ is

$$\int\limits_{-R}^{-\rho} \frac{e^{ix}}{x}\,dx + \int\limits_{\rho}^{R} \frac{e^{ix}}{x}\,dx = \int\limits_{\rho}^{R} \frac{e^{ix}-e^{-ix}}{x}\,dx = 2i\int\limits_{\rho}^{R} \frac{\sin x}{x}\,dx.$$

Hence,

$$0 = I_{\rho,R} = 2i\int\limits_{0}^{R} \frac{\sin x}{x}\,dx - i\pi + \varepsilon_1(\rho) + \varepsilon_2(R), \qquad (29.15)$$

where $\varepsilon_1(\rho)\to 0$ as $\rho\to 0$ and $\varepsilon_2(R)\to 0$ as $R\to\infty$. Since I has a finite value,

$$\lim_{\substack{\rho\to 0\\ R\to\infty}} \int\limits_{\rho}^{R} \frac{\sin x}{x}\,dx = I.$$

Going over to the limit in (29.15) as $\rho\to 0$ and $R\to\infty$, we obtain $2iI - i\pi = 0$, whence $I = \pi/2$. □

Example 6. Evaluate the integral

$$I = \int\limits_{0}^{\infty} \frac{\cos\alpha x - \cos\beta x}{x^2}\,dx \quad (\alpha\geqslant 0,\ \beta\geqslant 0),$$

We consider the integral $I_{\rho,R} = \int\limits_{\Gamma_{\rho,R}} \frac{e^{i\alpha z} - e^{i\beta z}}{z^2}\,dz$, where $\Gamma_{\rho,R}$ is the contour depicted in Fig. 71. On the one hand, this integral is equal to zero, by Cauchy's integral theorem. On the other hand, it is equal to the sum of the integrals taken along C_ρ, C_R, $[-R, -\rho]$, and $[\rho, R]$. Point $z=0$ is a simple pole for the function $f(z) = (e^{i\alpha z} - e^{i\beta z})/z^2$, and $\operatorname*{Res}_{z=0} f(z) = i(\alpha-\beta)$. Just as we did in Example 5, we can show that the integral along C_ρ tends to $\pi(\alpha-\beta)$ as $\rho\to 0$, while the integral along C_h tends to zero as $R\to\infty$. The sum of the integrals along the two segments is

$$\int\limits_{\rho}^{R} \left(\frac{e^{i\alpha x}+e^{-i\alpha x}}{x^2} - \frac{e^{i\beta x}+e^{-i\beta x}}{x^2}\right) dx = 2\int\limits_{\rho}^{R} \frac{\cos\alpha x - \cos\beta x}{x^2}\,dx.$$

Sending

$$2\int_{\rho}^{R}\frac{\cos\alpha x-\cos\beta x}{x^2}\,dx+\pi(\alpha-\beta)+\varepsilon_1(\rho)+\varepsilon_2(R)=0$$

$$(\varepsilon_1\to 0 \text{ as } \rho\to 0 \text{ and } \varepsilon_2\to 0 \text{ as } R\to\infty)$$

to the limit as $\rho\to 0$ and $R\to\infty$, we obtain $2I+\pi(\alpha-\beta)=0$, whence

$$I=\frac{\pi}{2}(\beta-\alpha).\quad\square$$

Example 7. Let us calculate the Fresnel integrals

$$I_1=\int_0^{\infty}\cos x^2\,dx,\qquad I_2=\int_0^{\infty}\sin x^2\,dx.$$

We take the contour Γ_R depicted in Fig. 70 ($n=4$). Since e^{iz^2} is regular in the interior of Γ_R, we conclude that

$$\int_{\Gamma_R}e^{iz^2}\,dz=\int_0^R e^{ix^2}\,dx+\int_{C_R}e^{iz^2}\,dz+\int_l e^{iz^2}\,dz=0.\qquad(29.16)$$

We estimate the integral $\int_{C_R}e^{iz^2}\,dz$. For $z\in C_R$ we have $z=Re^{i\varphi}$, $0\leqslant\varphi\leqslant\pi/4$, so that

$$|e^{iz^2}|=e^{-R^2\sin 2\varphi}\leqslant e^{-(4R^2/\pi)\varphi},$$

by virtue of the inequality $\sin 2\varphi\geqslant 4\varphi/\pi$ $(0\leqslant\varphi\leqslant\pi/4)$. Hence,

$$\left|\int_{C_R}e^{iz^2}\,dz\right|\leqslant R\int_0^{\pi/4}e^{-(4R^2/\pi)\varphi}\,d\varphi=\frac{\pi}{4R}(1-e^{-R^2})\to 0\quad(R\to\infty).$$

Moreover, if $z\in l$, then $z=re^{i\pi/4}$, so that $e^{iz^2}=e^{-r^2}$. Whence

$$\int_l e^{iz^2}\,dz=-e^{i\pi/4}\int_0^R e^{-r^2}\,dr.$$

The reader must know from the course of analysis (e.g. see Kudryavtsev [1]) that

$$\int_0^{\infty}e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}.$$

If in (29.16) we go over to the limit as $R \to \infty$, we obtain

$$\int_0^\infty e^{ix^2}\,dx = e^{i\pi/4}\frac{\sqrt{\pi}}{2}. \tag{29.17}$$

Separating in (29.17) the real and imaginary parts, we find that

$$\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{\sqrt{2\pi}}{4}. \quad \square$$

29.4 Integrals of the type $I = \int_0^\infty x^{\alpha-1}R(x)\,dx$ We will consider integrals of the type

$$I = \int_0^\infty x^{\alpha-1}R(x)\,dx, \tag{29.18}$$

with α a noninteger and $R(x)$ a rational function. (The case where α is an integer is considered in Sec. 29.6) The integral (29.18) is the Mellin transform of $R(x)$. This transform is widely used in mathematical physics and analytic number theory.

Integral (29.18) has a finite value if and only if $R(z)$ has no poles on the semiaxis $(0, +\infty)$ and

$$\lim_{z\to 0} |z|^\alpha R(z) = 0, \quad \lim_{z\to\infty} |z|^\alpha R(z) = 0. \tag{29.19}$$

We can assume that point $z = 0$ is neither a pole nor a zero for $R(z)$.

Under this assumption concerning the behavior of $R(z)$ at zero the first condition in (29.19) is met if and only if α is positive. Let us now turn to the second condition in (29.19). Note that for $R(z)$ the following asymptotic formula is valid:

$$R(z) \sim A/z^k \quad (z\to\infty,\ A\neq 0,\ k \text{ is an integer}), \tag{29.20}$$

and therefore the second condition in (29.19) is met if and only if $k - \alpha$ is positive. Thus, the integral (29.18), where $R(z)$ is a rational function of z without any poles on the real semiaxis $[0, +\infty)$ and such that $R(0) \neq 0$, has a finite value if and only if $0 \leqslant \alpha \leqslant k$, where k is determined by the asymptotic formula (29.20). All this implies that $R(z) \to 0$ as $z \to \infty$.

To employ the theory of residues in calculating (29.18), we continue the integrand analytically into the complex plane. Suppose D is the complex z plane with a cut along $[0, +\infty)$. In D we isolate the regular branch $h(z)$ of $z^{\alpha-1}$ that is positive on the upper bank of the cut; we denote this branch by $z^{\alpha-1}$, so that $h(z) = z^{\alpha-1}$.

In this domain we have $z = re^{i\varphi}$, where $r = |z|$, $\varphi = \arg z$, $0 < \varphi < 2\pi$, and hence,

$$h(z) = z^{\alpha-1} = (re^{i\varphi})^{\alpha-1} = r^{\alpha-1}e^{i(\alpha-1)\varphi}, \quad 0 < \varphi < 2\pi.$$

On the upper bank $\varphi = 0$, so that

$$h(x + i0) = h(x) = x^{\alpha-1} > 0 \quad (x > 0).$$

But if point x lies on the lower bank, i.e. $z = \tilde{x} = x - i0$ $(x > 0)$, then $\varphi = 2\pi$ and $h(x - i0) = h(\tilde{x}) = x^{\alpha-1}e^{i2\pi(\alpha-1)}$, or

$$h(\tilde{x}) = h(x)e^{i2\pi\alpha}, \quad h(x) > 0 \quad (x > 0).$$

We introduce the notation $f(z) = h(z)R(z) = z^{\alpha-1}R(z)$. Then $f(\tilde{x}) = h(\tilde{x})R(x)$ and

$$f(\tilde{x}) = e^{i2\pi\alpha} f(x). \qquad (29.21)$$

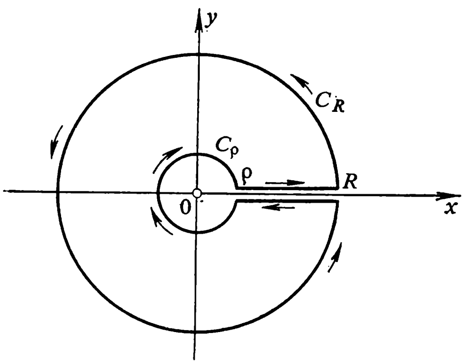


Fig. 72

Let us show that (29.18) obeys the following formula:

$$I = \frac{2\pi i}{1 - e^{i2\pi\alpha}} \sum_{z=z_k} \operatorname{Res}\,(z^{\alpha-1}R(z)), \qquad (29.22)$$

where the sum is taken over all the poles of $R(z)$.

Let us consider the contour $\Gamma_{\rho,R}$ (Fig. 72) consisting of the circles C_ρ: $|z| = \rho$ and C_R: $|z| = R$ and the segments $[\rho, R]$ and $[R, \rho]$, which lie on the upper and lower banks, respectively. Suppose $R > 0$ is so large and $\rho > 0$ so small that all the poles of $R(z)$ lie inside $\Gamma_{\rho,R}$. By the residue theorem (Theorem 1 of Sec. 28),

$$I_{\rho,R} = \int_{\Gamma_{\rho,R}} f(z)\,dz = 2\pi i \sum_{z=z_k} \operatorname{Res}\,(z^{\alpha-1}R(z)), \qquad (29.23)$$

where the sum is taken over all the poles of $R(z)$.

On the other hand, $I_{\rho,R}$ is equal to the sum of four integrals:

$$I_{\rho,R} = \int_\rho^R f(x)\,dx + \int_R^\rho f(\tilde{x})\,dx + \int_{C_\rho} f(z)\,dz + \int_{C_R} f(z)\,dz. \qquad (29.24)$$

Let us show that the integrals along C_ρ and C_R tend to zero as $\rho \to 0$ and $R \to \infty$. This follows from

Lemma 3 *Suppose* $M(\rho) = \max\limits_{z \in C_\rho} |f(z)|$, *where* C_ρ *is the circle* $|z| = \rho$. *If* $\rho M(\rho) \to 0$ *as* $\rho \to 0$ *and* $RM(R) \to 0$ *as* $R \to \infty$, *then*

$$\int\limits_{C_\rho} f(z)\,dz \to 0 \quad (\rho \to 0),$$

$$\int\limits_{C_R} f(z)\,dz \to 0 \quad (R \to \infty).$$

Proof. These relationships follow from the estimate

$$\left| \int\limits_{C_\rho} f(z)\,dz \right| \leqslant M(\rho)\, 2\pi\rho.$$

The hypothesis holds for the integrals in C_ρ and C_R in (29.24). Indeed,

$$M(\rho) = \max_{z \in C_\rho} |z^{\alpha-1} R(z)| = \rho^{\alpha-1} \max_{z \in C_\rho} |R(z)|,$$

and from (29.19) it follows that $\rho^\alpha \max\limits_{z \in C_\rho} |R(z)| \to 0$ as $\rho \to 0$ and, hence, $\rho M(\rho) \to 0$ as $\rho \to 0$. The fact that $RM(R) \to 0$ as $R \to \infty$ can be proved, similarly.

Now, if in (29.23) we go over to the limit as $\rho \to 0$ and $R \to \infty$ and then use (29.21), we obtain

$$I - e^{i2\pi\alpha} I = 2\pi i \sum \operatorname*{Res}_{z=z_k} (z^{\alpha-1} R(z)),$$

which leads to (29.22).

Example 8. Let us evaluate the integral

$$I = \int\limits_0^\infty \frac{x^{\alpha-1}}{x+1}\,dx, \quad 0 < \alpha < 1. \tag{29.25}$$

Here $R(z) = 1/(z+1)$, $|z|^\alpha \; |R(z)| \sim 1/|z|^{1-\alpha} \to 0$ as $z \to \infty$, since $\alpha < 1$. Moreover, $|z|^\alpha \, |R(z)| \sim |z|^\alpha \to 0$ as $z \to 0$, since $\alpha > 0$. Thus, conditions (29.19) are met, and (29.22) yields

$$I = \frac{2\pi i}{1 - e^{i2\pi\alpha}} \operatorname*{Res}_{z=-1} f(z),$$

where $f(z) = z^{\alpha-1}/(z+1)$. Moreover, $\operatorname*{Res}\limits_{z=-1} f(z) = z^{\alpha-1}|_{z=-1} = e^{i(\alpha-1)\pi}$, since $\varphi = (\arg z)_{z=-1} = \pi$. Thus, $\operatorname*{Res}\limits_{z=-1} f(z) = -e^{i\alpha\pi}$, whence

$$I = \frac{2\pi i}{1 - e^{i2\pi\alpha}} (-e^{i\alpha\pi}) = \pi \left(\frac{e^{i\alpha\pi} - e^{-i\alpha\pi}}{2i} \right)^{-1},$$

i.e.

$$I=\frac{\pi}{\sin\alpha\pi}. \quad \square \tag{29.26}$$

Remark 3. Let us take the integral (29.25) as a function of parameter α, i.e. $I = I(\alpha)$. This integral is uniformly convergent at $\varepsilon < \alpha < 1 - \varepsilon$, where ε is positive, so that $I(\alpha)$ is continuous for $0 < \alpha < 1$ and formula (29.26) is valid. The right-hand side of (29.26) is regular in the entire complex α plane with points $0, \pm 1, \pm 2, \ldots$ deleted. Hence, (29.26) gives the analytic continuation of (29.25) from the $(0, 1)$ interval into the complex α plane with points $0, \pm 1, \pm 2, \ldots$ deleted.

Example 9. Let us calculate

$$I=\int_0^\infty \frac{x^{\alpha-1}}{(1+x^2)^2}\,dx, \quad 0<\alpha<4.$$

Here $R(z)=\frac{1}{(1+z^2)^2}$, $|z|^\alpha |R(z)| \sim 1/|z|^{4-\alpha}\to 0$ as $z\to\infty$

since $\alpha<4$, and $|z|^\alpha |R(z)| \sim |z|^\alpha \to 0$ as $z\to 0$ since $\alpha>0$. The conditions in (29.19) are met, and formula (29.22) yields

$$I=\frac{2\pi i}{1-e^{i2\pi\alpha}}\left[\operatorname*{Res}_{z=i} f(z)+\operatorname*{Res}_{z=-i} f(z)\right],$$

where $f(z) = z^{\alpha-1} R(z)$. The function $R(z)$ has second order poles at points i and $-i$. According to (28.8), we have

$$\operatorname*{Res}_{z=i} f(z)=\left[\frac{z^{\alpha-1}}{(z+i)^2}\right]'_{z=i}.$$

At this point the reader will recall the formula for the derivative of a power function (Sec. 22): $(z^\beta)' = \beta z^\beta/z$. Using this formula, we can write

$$\operatorname*{Res}_{z=i} f(z)=\left\{z^{\alpha-1}(z+i)^{-2}\left[\frac{\alpha-1}{z}-2(z+i)^{-1}\right]\right\}_{z=i},$$

where $(z^{\alpha-1})_{z=i}=e^{i(\alpha-1)\pi/2}$. Hence

$$\operatorname*{Res}_{z=i} f(z)=\frac{i(\alpha-2)}{4}e^{i(\alpha-1)\pi/2}=\frac{\alpha-2}{4}e^{i\alpha\pi/2}.$$

Similarly, we find that

$$\operatorname*{Res}_{z=-i} f(z)=\frac{\alpha-2}{4}e^{i(3/2)\alpha\pi}.$$

The final result is

$$I=\frac{2\pi i}{1-e^{i2\pi\alpha}}\,\frac{\alpha-2}{4}\,\left(e^{i(\pi/2)\alpha}+e^{i(3/2)\pi\alpha}\right)$$

$$=\pi\left(\frac{e^{i\alpha\pi}-e^{-i\alpha\pi}}{2i}\right)^{-1}\frac{2-\alpha}{2}\,\frac{e^{i(\pi/2)\alpha}+e^{-i(\pi/2)\alpha}}{2}$$

$$=\frac{\pi(2-\alpha)\cos(\alpha\pi/2)}{2\sin\alpha\pi},$$

or

$$I=\frac{\pi(2-\alpha)}{4\sin(\alpha\pi/2)}. \quad \square$$

29.5 Integrals of the beta-function type $I=\int_0^1\left(\frac{x}{1-x}\right)^\alpha R(x)\,dx$

Here we will consider integrals of the beta-function type

$$I=\int_0^1\left(\frac{x}{1-x}\right)^\alpha R(x)\,dx, \tag{29.27}$$

where α is a noninteger, and $R(x)$ is a rational function. (The case where α is an integer will be considered in Sec. 29.6.) We will assume that $R(z)$ has no poles in the segment [0, 1] and that $-1<\alpha<1$. Then the integral (29.27) has a finite value.

Note that we can reduce (29.17) to (29.18) by introducing a new variable, $y=x/(1-x)$. However, in many cases it is more convenient to employ the theory of residues when calculating (29.27). To this end we continue the integrand analytically into the complex plane.

Suppose D is the complex z plane with a cut along the segment [0, 1] (Fig. 73). In this domain we isolate the regular branch $h(z)$ of $\left(\frac{z}{1-z}\right)^\alpha$ that is positive on the upper bank of the cut. If $z=x+i0$ $(0<x<1)$ is a point on the upper bank, then

$$h(x+i0)=h(x)=\left(\frac{x}{1-x}\right)^\alpha>0 \quad (0<x<1).$$

Let us find $h(\tilde{x})=h(x-i0)$, $0<x<1$, where $\tilde{x}=x-i0$ is a point on the lower bank of the cut. We have (see Example 16 in Sec. 24)

$$h(z)=\left|\frac{z}{1-z}\right|^\alpha e^{i\alpha(\varphi_1-\varphi_2)},$$

where $\varphi_1=\Delta_\gamma\arg z$ and $\varphi_2=\Delta_\gamma\arg(z-1)$, with γ a curve that connects a point on the upper bank of the cut with a point $z\in D$

(Fig. 73). If $z = \tilde{x} = x - i0$ $(0 < x < 1)$, then $\varphi_1 = 2\pi$, $\varphi_2 = 0$, and

$$h(\tilde{x}) = e^{i2\pi\alpha}\left(\frac{x}{1-x}\right)^{\alpha} = e^{i2\pi\alpha}h(x).$$

We introduce the notation $f(z) = h(z)R(z)$. Then $f(z - i0) = e^{i2\pi\alpha} f(x + i0)$, or

$$f(\tilde{x}) = e^{i2\pi\alpha} f(x),$$

where $f(x)$ coincides with the integrand in (29.17).

Let us prove that the integral (29.27) can be expressed thus:

$$I = \frac{2\pi i}{1 - e^{i2\pi\alpha}}\left(\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z)\right), \tag{29.28}$$

where $z_1, z_2, \ldots, z_n$ are all the finite poles of $R(z)$.

To prove the validity of (29.28), we take a contour Γ_ρ in the form

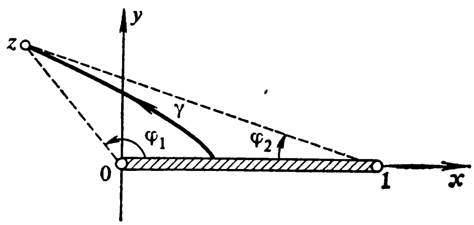


Fig. 73

of a dumbbell (Fig. 74). This contour consists of the circles C_ρ: $|z| = p$ and C'_ρ: $|z - 1| = \rho$ and the segments l_1: $\rho \leqslant x \leqslant 1 - \rho$ and l_2: $\rho \leqslant x \leqslant 1 - \rho$ lying, respectively, on the upper and lower banks of the cut. By the residue theorem (Theorem 2 of Sec. 28),

$$I_\rho = \int_{\Gamma_\rho} f(z)\,dz = 2\pi i\left(\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z)\right), \tag{29.29}$$

where ρ is so small that all the poles of $R(z)$ lie outside of the dumbbell.

On the other hand,

$$I_\rho = \int_{C_\rho} f(z)\,dz + \int_{\rho}^{1-\rho} f(x)\,dx + \int_{C'_\rho} f(z)\,dz + \int_{1-\rho}^{\rho} f(\tilde{x})\,dx. \tag{29.30}$$

We estimate the integrals along C_ρ and C'_ρ. By hypothesis $R(z)$ is regular at point $z = 0$ and, hence, $|R(z)| \leqslant M$ for small values of

$|z|$. Moreover, $|h(z)| = |z|^{\alpha}/|1-z|^{\alpha} \leqslant M_1 |z|^{\alpha}$ for small values of $|z|$. Whence, if $z \in C_{\rho}$, then $|f(z)| = |h(z)|\,|R(z)| \leqslant MM_1\rho^{\alpha}$, so that $M(\rho) = \max\limits_{z \in C_{\rho}} |f(z)| \leqslant M_2\rho^{\alpha}$. Consequently,

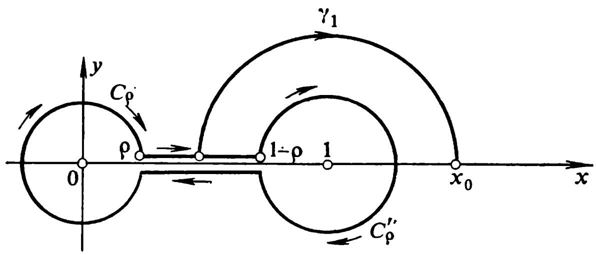


Fig. 74

$\rho M(\rho) \leqslant M_2\rho^{1+\alpha} \to 0$ as $\rho \to 0$, since $1+\alpha > 0$. By Lemma 3, $\int\limits_{C_\rho} f(z)\,dz \to 0$ $(\rho \to 0)$. We can similarly prove that the integral along C_ρ' tends to zero as $\rho \to 0$. Going over in (29.29) to the limit as $\rho \to 0$, we obtain

$$(1-e^{i2\pi\alpha})\, I = 2\pi i \left(\sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) \right),$$

which yields (29.28).

Now we have to find the residue at point $z=\infty$, assuming that $R(z)$ is regular at this point, i.e.

$$R(z) = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \ldots, \qquad |z| > R.$$

We expand $h(z) = \left(\frac{z}{1-z}\right)^{\alpha}$ in a Laurent series about $z=\infty$. We have (see Example 17 in Sec. 24) $h(z) = h(\infty)\, g(z)$, where

$$g(z) = \left(1 - \frac{1}{z}\right)^{-\alpha} = 1 + \frac{\alpha}{z} + \ldots$$

is regular at point $z=\infty$, $g(\infty)=1$. Here $h(\infty) = e^{i\alpha(\varphi_1-\varphi_2)}$, $\varphi_1 = \Delta_{\gamma_1} \arg z$, $\varphi_2 = \Delta_{\gamma_1} \arg(z-1)$, and γ_1 is a curve that connects a point on the upper bank of the cut with a point $x_0 > 1$ on the real axis (Fig. 74). Since $\varphi_1 = 0$ and $\varphi_2 = -\pi$, we can write $h(\infty) = e^{i\alpha\pi}$. Hence,

$$f(z) = h(z)\,R(z) = h(\infty)\,g(z)\,R(z)$$
$$= e^{i\alpha\pi}\left(c_0 + \frac{\alpha c_0 + c_{-1}}{z} + \ldots\right), \qquad |z| > R,$$

whence

$$\operatorname*{Res}_{z=\infty} f(z) = -e^{i\pi\alpha}(\alpha c_0 + c_{-1}).$$

In particular, if point $z = \infty$ is a first order zero for $R(z)$, i.e. $c_0 = 0$ and $c_{-1} \neq 0$, then

$$\operatorname*{Res}_{z=\infty} f(z) = -e^{i\pi\alpha} c_{-1}. \tag{29.31}$$

Remark 4. The above method of evaluating integrals of the type (29.27) can be applied without any modifications to integrals of the type

$$I = \int_a^b \left(\frac{x-a}{b-x}\right)^\alpha R(x)\,dx,$$

where $R(x)$ is a rational function, and $-1 < \alpha < 1$, $\alpha \neq 0$.

In this case we have

$$\operatorname*{Res}_{z=\infty} f(z) = -e^{i\alpha\pi}[\alpha c_0(b-a) + c_{-1}],$$

since

$$\left(\frac{z-a}{b-z}\right)^\alpha = e^{i\alpha\pi}\left(1-\frac{a}{z}\right)^\alpha\left(1-\frac{b}{z}\right)^{-\alpha} = e^{i\alpha\pi}\left(1+\frac{\alpha(b-a)}{z}+\dots\right).$$

Example 10. Let us evaluate the integral

$$I = \int_0^1 \left(\frac{x}{1-x}\right)^\alpha \frac{dx}{x+1}, \quad -1 < \alpha < 1.$$

The function $R(z) = 1/(z+1)$ has only one (simple) pole, at point $z = -1$. Formula (29.28) yields

$$I = \frac{2\pi i}{1-e^{i2\pi\alpha}}\left(\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=\infty} f(z)\right),$$

where $f(z) = h(z)R(z) = \left(\frac{z}{1-z}\right)^\alpha \frac{1}{z+1}$. We have

$$\operatorname*{Res}_{z=-1} f(z) = h(-1) = \left(\frac{z}{1-z}\right)^\alpha_{z=-1},$$

where $h(-1) = 2^{-\alpha}e^{i\alpha(\varphi_1-\varphi_2)}$, $\varphi_1 = \pi$ and $\varphi_2 = 0$. Hence,

$$\operatorname*{Res}_{z=-1} f(z) = 2^{-\alpha}e^{i\pi\alpha}.$$

Since $R(z) = \frac{1}{z} - \frac{1}{z^2} + \dots$, we obtain, via (29.31),

$$\operatorname*{Res}_{z=\infty} f(z) = -e^{i\alpha\pi}.$$

This yields

$$I=\frac{2\pi i e^{i\alpha\pi}}{1-e^{i2\alpha\pi}}(2^{-\alpha}-1)=\frac{\pi(1-2^{-\alpha})}{\sin\alpha\pi}. \quad \square$$

Example 11. We wish to evaluate the integral

$$I=\int_0^1 \sqrt{\frac{1-x}{x}}\,\frac{dx}{(x+2)^2}.$$

Point $z=\infty$ is a second order zero for $R(z)=1/(z+2)^2$, while $h(z)=(z/(1-z))^{-1/2}$ is regular at this point. Hence, point $z=\infty$ is a second order zero for $f(z)=h(z)R(z)$ and therefore $\operatorname*{Res}_{z=\infty} f(z)=0$ (see Sec. 28.3).

Formula (29.28) yields $I=\pi i \operatorname*{Res}_{z=-2} f(z)$, since $\alpha=-1/2$. We have $\operatorname*{Res}_{z=-2} f(z)=h'(-2)$, where $h(z)=(1/z-1)^{1/2}$. Using Example 25 in Sec. 24, we find that

$$h'(-2)=\left[\frac{1}{2h(z)}\left(\frac{1}{z}-1\right)'\right]_{z=-2}=-\frac{1}{8h(-2)},$$

where $h(-2)=\sqrt{3/2}\,e^{i(-1/2)\varphi}$, $\varphi=\pi$. Hence, $h(-2)=-i\sqrt{3/2}$ and $\operatorname*{Res}_{z=-2} f(z)=-i/4\sqrt{6}$. From this we find that $I=\pi/4\sqrt{6}$. $\square$

Example 12. Let us evaluate the integral

$$I=\int_{-1}^{1}\frac{\sqrt[4]{(1-x)(1+x)^3}}{1+x^2}\,dx.$$

In the complex z plane with a cut along $[-1, 1]$ we isolate the regular branch of $\sqrt[4]{(1-z)(1+z)^3}$ that is positive on the upper bank of the cut and denote it by $h(z)$, so that $h(x+i0)=h(x)=\sqrt[4]{(1-x)(1+x)^3}>0$, $-1<x<1$. Then $h(\tilde{x})=h(x-i0)=e^{3\pi i/2}h(x)=-ih(x)$ is the value of $h(z)$ on the lower bank of the cut (see Example 16 of Sec. 24).

We introduce the notation $R(z)=1/(1+z^2)$, $f(z)=h(z)R(z)$, and consider the contour Γ (Fig. 75) consisting of the segments $[-1, 1]$ and $[1, -1]$ lying on the upper and lower banks of the cut, respectively. By the residue theorem (Theorem 2 of Sec. 28),

$$\int_{-1}^{1} f(x)\,dx+\int_{1}^{-1} f(\tilde{x})\,dx=2\pi i\left(\operatorname*{Res}_{z=i} f(z)+\operatorname*{Res}_{z=-i} f(z)+\operatorname*{Res}_{z=\infty} f(z)\right),$$

where $f(\tilde{x})=-if(x)$. Let us find these residues. Here $\operatorname*{Res}_{z=\pm i} f(z)=(h(z)/2z)_{z=\pm i}$. To find the values of the power function

$h(z)$ at points i and $-i$, we must first calculate $\varphi_1^{(k)} = \Delta_{\gamma_k} \arg(z-1)$ and $\varphi_2^{(k)} = \Delta_{\gamma_k} \arg(z+1)$, where the γ_k $(k = 1, 2)$ are curves

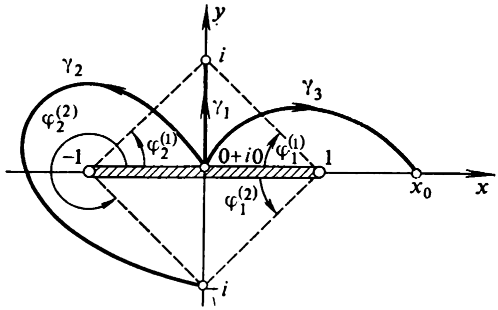


Fig. 75

that connect point $0+i0$ with points i and $-i$, respectively (Fig. 75). We have

$$h(i) = |h(i)|\, e^{(i/4)(\varphi_1^{(1)}+3\varphi_2^{(1)})},$$

where $h(i) = \sqrt{2}$, $\varphi_1^{(1)} = -\pi/4$, $\varphi_2^{(1)} = \pi/4$ (Fig. 75), $h(i) = \sqrt{2}\, e^{i\pi/8}$, and

$$\operatorname*{Res}_{z=i} f(z) = -\frac{i}{\sqrt{2}} e^{i\pi/8}.$$

Similarly, $h(-i) = \sqrt{2} e^{(i/4)(\varphi_1^{(2)}+3\varphi_2^{(2)})} = \sqrt{2} e^{i(11/8)\pi}$ (Fig. 75) and

$$\operatorname*{Res}_{z=-i} f(z) = -\frac{i}{\sqrt{2}} e^{i(3/8)\pi}.$$

Now let us find the residue at $z=\infty$. We have

$$h_1(z) = \frac{h(z)}{z} = h_1(\infty)\left(1-\frac{1}{z}\right)^{1/4}\left(1+\frac{1}{z}\right)^{3/4}$$
$$= h_1(\infty)\left(1+\frac{1}{2z}+\dots\right),$$

where $h_1(\infty) = e^{i(1/4)(\varphi_1^{(3)}+3\varphi_2^{(3)})}$, $\varphi_1^{(3)} = \Delta_{\gamma_3} \arg(z-1)$, $\varphi_2^{(3)} = \Delta_{\gamma_3} \arg(z+1)$, and γ_3 is a curve connecting point $0+i0$ with a point $x_0 > 1$ (Fig. 75). Here $\varphi_1^{(3)} = -\pi$ and $\varphi_2^{(3)} = 0$. Consequently, $h_1(\infty) = e^{-i(\pi/4)}$ and

$$h(z) = e^{-i(\pi/4)}\left(z+\frac{1}{2}+\dots\right).$$

Moreover

$$R(z) = \frac{1}{z^2} - \frac{1}{z^4} + \dots, \quad f(z) = h(z) R(z)$$
$$= e^{-i\pi/4}\left(\frac{1}{z} + \frac{1}{2z^2} + \dots\right),$$

whence

$$\operatorname*{Res}_{z=\infty} f(z) = -e^{-i\pi/4}.$$

The final result is

$$(1+i) I = \sqrt{2}\pi\,(e^{i\pi/8} + e^{i3\pi/8} - \sqrt{2}\, e^{i\pi/4}),$$

which yields $I = \pi\sqrt{2}\left(\sqrt{2}\cos\frac{\pi}{8} - 1\right)$. □

Example 13. Let us evaluate the integral

$$I = \int_0^1 \frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3}\, dx.$$

Suppose $h(z)$ is the regular branch of $\sqrt[4]{z(1-z)^3}$ that in the complex z plane with a cut along $[0, 1]$ assumes positive values on the upper bank of the cut, i.e. $h(x + i0) = h(x) = \sqrt[4]{x(1-x)^3} > 0$, $0 < x < 1$. Then $h(x - i0) = h(\tilde{x}) = ih(x)$. We write $f(z) = h(z) R(z)$, with $R(z) = 1/(1 + z)^3$. We have

$$f(\tilde{x}) = f(x - i0) = if(x).$$

By the residue theorem (Theorem 2 of Sec. 28),

$$\int_0^1 f(x)\, dx + \int_1^0 f(\tilde{x})\, dx = (1-i) I = 2\pi i \left(\operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=\infty} f(z)\right).$$

Let us calculate the two residues. Since point $z = \infty$ is a second order zero for $f(z)$ ($R(z) \sim 1/z^3$ and $h(z) \sim Az$ as $z \to \infty$), we can write

$$\operatorname*{Res}_{z=\infty} f(z) = 0.$$

Point $z = -1$ is a third order pole for $f(z)$ and, by virtue of (28.8), we have $\operatorname*{Res}_{z=-1} f(z) = \frac{1}{2}h''(-1)$, with $h''(-1) = -3 \times 8^{-7/4}\, e^{i\pi/4}$ (see Example 25 in Sec. 24), i.e.

$$\operatorname*{Res}_{z=-1} f(z) = -\frac{3}{2}\, 8^{-7/4} e^{i\pi/4}.$$

The final result is $e^{-i\pi/4} I = -3\pi 8^{-7/4} ie^{i\pi/4}$, which yields

$$I = 3\pi \sqrt[4]{2}/64. \quad \square$$

29.6 Integrals of the type $I = \int\limits_0^\infty x^{\alpha-1} (\ln x)^m R(x)\, dx$ Let us consider integrals of the type

$$I = \int\limits_0^\infty x^{\alpha-1} (\ln x)^m R(x)\, dx. \tag{29.32}$$

Here α is a real number, m a positive integer, and $R(x)$ a rational function of x. We will assume that $R(x)$ satisfies the same conditions as specified in Sec. 29.4. Then the integral (29.32) has a finite value if and only if conditions (29.19) are met, since the factor $(\ln x)^m$ does not change the convergence of (29.32).

Note that the integral (29.32) can be obtained from (29.18) by differentiating with respect to parameter α. Indeed,

$$\frac{d}{d\alpha} \int\limits_0^\infty x^{\alpha-1} R(x)\, dx = \int\limits_0^\infty x^{\alpha-1} \ln x R(x)\, dx.$$

Differentiating (29.18) m times with respect to α, we arrive at (29.32).

Another way to evaluate (29.32) is to apply the residue theory directly. Suppose D is the complex z plane with a cut along $[0, +\infty)$ and $h(z) = z^{\alpha-1}$ is the regular branch of $z^{\alpha-1}$ in D (Sec. 29.4) that is positive on the upper bank of the cut. We fix the logarithm's regular branch that assumes real values on the upper bank of the cut and denote it by $\ln z$. Then in D we have

$$\ln z = \ln |z| + i \arg z, \quad 0 < \arg z < 2\pi.$$

On the upper bank, $z = x + i0$ $(x > 0)$, $\arg z = 0$, and

$$\ln(x + i0) = \ln x.$$

On the lower bank, $z = x - i0 = \tilde{x}$ $(x > 0)$, $\arg z = 2\pi$, and

$$\ln(x - i0) = \ln \tilde{x} = \ln x + i2\pi.$$

We introduce the notation $f(z) = h(z)(\ln z)^m R(z) = z^{\alpha-}$ $(\ln z)^m R(z)$. Then $f(x + i0) = f(x) = x^{\alpha-1} (\ln x)^m R(x)$ is the integrand in (29.32), while

$$f(x - i0) = f(\tilde{x}) = e^{i2\pi\alpha} x^{\alpha-1} (\ln x + 2\pi i)^m R(x)$$

is the value of $f(z)$ on the lower bank of the cut.

Consider the integral

$$I_{\rho, R} = \int_{\Gamma_{\rho, R}} f(z)\, dz, \tag{29.33}$$

where $\Gamma_{\rho,R}$ is the contour depicted in Fig. 72. Then, by Lemma 3, the integrals along C_R and C_ρ tend to zero as $R \to \infty$ and $\rho \to 0$.
Just as in Sec. 29.4, we have

$$\int_0^\infty [f(x) - f(\tilde{x})]\, dx = 2\pi i \sum \operatorname*{Res}_{z=z_k} f(z), \tag{29.34}$$

where the residues are taken at all the poles of $R(z)$.
Let us consider two cases.
(1) α is a noninteger. Then the left-hand side of (29.34) contains $I(1 - e^{i2\pi\alpha})$, as well as (at $m > 1$) integrals of the type $\int_0^\infty x^{\alpha-1} \times (\ln x)^s R(x)\, dx$, where $0 \leqslant s \leqslant m - 1$. For instance, at $m = 1$ (29.34) yields

$$(1 - e^{i2\pi\alpha}) I - 2\pi i e^{i2\pi\alpha} \int_0^\infty x^{\alpha-1} R(x)\, dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} (z^{\alpha-1} R(z) \ln z), \tag{29.35}$$

where $z_1, z_2, \ldots, z_n$ are all the poles of $R(z)$. From (29.35) we can go over to I and $\int_0^\infty x^{\alpha-1} R(x)\, dx$.

(2) α is an integer. Then (29.32) has the form

$$I = \int_0^\infty (\ln x)^m R(x)\, dx, \tag{29.36}$$

with $R(x)$ a rational function. In this case the integrand in (29.33) is $(\ln z)^{m+1} R(z)$ instead of $(\ln z)^m R(z)$. Indeed, if $f(z) = (\ln z)^m R(z)$, then $f(x) = (\ln x)^m R(x)$ and $f(\tilde{x}) = (\ln x + 2\pi i)^m R(x)$, and formula (29.34) does yield the sought-for integral.
But if $R(z)$ is an even function, then we can take $f(z) = (\ln z)^m R(z)$ as the integrand when calculation (29.36) via residues; the integration path in this case is the contour $\Gamma_{\rho,R}$ depicted in Fig. 71 (see Example 15 below).

Example 14. Let us evaluate the integral

$$I=\int_0^\infty x^{-1/2}\ln x\frac{dx}{(x+1)^2}.$$

Conditions (29.19) are met ($\alpha=1/2$ and $k=2$), and (29.35) yields

$$2I+2\pi i\int_0^\infty \frac{x^{-1/2}}{(x+1)^2}\,dx=2\pi i\operatorname*{Res}_{z=-1} f(z). \tag{29.37}$$

Since point $z=-1$ is a second order pole for $f(z)$, we can write

$$\operatorname*{Res}_{z=-1} f(z)=(z^{-1/2}\ln z)'_{z=-1}=\left[\left(-\frac{1}{2}\right)\frac{z^{-1/2}}{z}\ln z+\frac{1}{z}z^{-1/2}\right]_{z=-1},$$

where $(z^{-1/2})_{z=-1}=e^{-i\pi/2}=-i$, and $(\ln z)_{z=-1}=i\pi$. Hence,

$$\operatorname*{Res}_{z=-1} f(z)=\frac{\pi}{2}+i.$$

Equating in (29.37) the real and imaginary parts in the left- and right-hand sides, we find that

$$I=-\pi,\quad \int_0^\infty \frac{x^{-1/2}}{(x+1)^2}\,dx=\frac{\pi}{2}.\quad\square$$

Example 15. Let us evaluate the integral

$$I=\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx,\quad a>0.$$

We take the contour $\Gamma_{\rho,R}$ depicted in Fig. 71. Consider the integral

$$I_{\rho,R}=\int_{\Gamma_{\rho,R}} f(z)\,dz=2\pi i\operatorname*{Res}_{z=ia} f(z),$$

where $f(z)=\frac{\ln z}{z^2+a^2}$ is the regular branch of the logarithm that assumes positive values at $z=x>0$. The integrals along the semicircles C_ρ and C_R tend to zero as $\rho\to 0$ and $R\to\infty$, since $|f(z)|<M_1|\ln\rho|$ $(z\in C_\rho)$ and $|f(z)|<M_2\frac{\ln R}{R^2}$ $(z\in C_R)$. Here $\operatorname*{Res}_{z=ia} f(z)=\left(\frac{\ln z}{2z}\right)_{z=ia}=\frac{\ln a+i\pi/2}{2ai}$. Going over to the limit as $\rho\to 0$ and $R\to\infty$ and bearing in mind that

$$\int_{-\infty}^0 f(x)\,dx=\int_0^\infty \frac{\ln x+i\pi}{x^2+a^2}\,dx=I+i\pi\int_0^\infty \frac{dx}{x^2+a^2},$$

we obtain

$$2I+i\pi\int_0^\infty \frac{dx}{x^2+a^2}=\frac{\pi}{a}\left(\ln a+i\frac{\pi}{2}\right),$$

whence

$$I=\frac{\pi}{2a}\ln a. \quad \square$$

The method we have just discussed makes it possible to evaluate integrals of the type

$$I=\int_a^b \left[\ln\left(\frac{x-a}{b-x}\right)\right]^m R(x)\,dx, \tag{29.38}$$

where a and b are real numbers ($a<b$), and m is a nonnegative integer.

Suppose $m=0$, $a=0$, and $b=\infty$. Then (29.38) takes the form

$$I=\int_0^\infty R(x)\,dx,$$

where $R(x)$ is a rational function that satisfies the conditions stated in Sec. 29.4. If $R(x)$ is an even function, then

$$I=\frac{1}{2}\int_{-\infty}^{\infty} R(x)\,dx,$$

and we can evaluate the integral by the method described in Sec. 29.2.

But what happens if $R(x)$ is not an even function? Then we must consider the integral

$$I_{\rho,R}=\int_{\Gamma_{\rho,R}} \ln zR(z)\,dz,$$

where $\Gamma_{\rho,R}$ is the contour depicted in Fig. 72, and $\ln z$ is a regular branch of the logarithm.

Example 16. Let us evaluate the integral

$$I=\int_0^\infty \frac{dx}{(x^3+1)^2}$$

Here $R(z)=1/(z^3+1)^2$, $f(z)=R(z)\ln z$, $f(x)=(\ln x)/(x^3+1)^2$, and $\tilde{f}(x)=f(x)+2\pi i/(x^3+1)^2$. In view of (29.34) we have

$$I=-\sum_{k=1}^{3}\operatorname*{Res}_{z=z_k} f(z),$$

where $z_k = e^{(2k+1)i\pi/3}$, $k = 1, 2, 3$. Since point z_1 is a second order pole for $f(z) = \frac{\ln z}{(z-z_1)^2 (z-z_2)^2 (z-z_3)^2}$, we can write

$$\operatorname*{Res}_{z=z_1} f(z) = \left[\frac{\ln z}{(z-z_2)^2 (z-z_3)^2}\right]'_{z=z_1}$$

$$= \frac{1}{z_1 (z_1-z_2)^2 (z_1-z_3)^2} - \frac{2\,[2z_1-(z_2+z_3)]\ln z_1}{(z_1-z_2)^3 (z_1-z_3)^3}.$$

Using the formulas $\ln z_1 = i\pi$, $(z_1 - z_2)(z_1 - z_3) = (z^3 + 1)'_{z=z_1} = 3z_1^2$, $z_1^3 = -1$, and $z_1 + z_2 + z_3 = 0$, we find that

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{z_1}{9}(1-2\pi i).$$

Similarly, if we take into account that $\ln z_2 = (5/3)\,\pi i$ and $\ln z_3 = (\pi/3)\,i$, we find that

$$\operatorname*{Res}_{z=z_2} f(z) = \frac{z_2}{9}\left(1-\frac{10}{3}\pi i\right), \quad \operatorname*{Res}_{z=z_3} f(z) = \frac{z_3}{9}\left(1-\frac{2}{3}\pi i\right).$$

The final result is

$$I = \frac{\pi i}{9}\left(2z_1 + \frac{10}{3} z_2 + \frac{2}{3} z_3\right) = \frac{4\pi\sqrt{3}}{27}. \ \square$$

Now let us turn to (29.38). By introducing a new variable, $y = (x-a)/(b-x)$, we can rewrite (29.38)

$$I = \int_0^\infty R(x)(\ln x)^m\,dx.$$

In conclusion we note that other types of integrals that can be evaluated by using the theory of residues can be found in Evgrafov *et al.* [1] and Lavrent'ev and Shabat [1].

Here is another example.

Example 17. Let us evaluate the integral

$$\int_0^\infty e^{-ax^2}\cos bx\,dx, \quad a > 0.$$

Suppose Γ_R is the boundary of a rectangle with its vertices at

$$z_1 = -R, \quad z_2 = -R + \frac{b}{2a} i, \quad z_3 = R + \frac{b}{2a} i, \quad z_4 = R,$$

with $R > 0$. Consider the function

$$f(z) = e^{-az^2}, \quad z = x + iy.$$

By Cauchy's integral theorem

$$\int_{\Gamma_R} e^{-az^2}\,dz = 0. \tag{29.39}$$

On the segments $[z_1, z_2]$ and $[z_4, z_3]$ we have

$$|f(z)| = e^{-a(R^2-y^2)} \leqslant e^{-aR^2+\frac{b^2}{4a^2}}.$$

Hence, the integrals of $f(z)$ along these segments tend to zero as $R \to \infty$. If $z \in [z_2, z_3]$, then

$$f(z) = e^{-a\left(x+i\frac{b}{2a}\right)^2} = f(x)\, e^{\frac{b^2}{4a}-ibx},$$

and we can rewrite Eq. (29.39) as

$$\int_{-R}^{R} e^{-ax^2}\,dx - e^{\frac{b^2}{4a}} \int_{-R}^{R} e^{-ax^2-ibx}\,dx + \alpha(R) = 0, \tag{29.40}$$

where $\alpha(R) \to 0$ as $R \to \infty$. Since

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \frac{2}{\sqrt{a}} \int_{0}^{\infty} e^{-t^2}\,dl = \sqrt{\frac{\pi}{a}},$$

we find that, going over to the limit in (29.40) and isolating the real parts, we obtain

$$\int_{0}^{\infty} e^{-ax^2}\cos bx\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\, e^{-\frac{b^2}{4a}}. \ \square$$

30 The Argument Principle and Rouche's Theorem

30.1 The argument principle

Theorem 1 *Suppose a function $f(z)$ is regular in a domain G except, perhaps, at its poles and suppose a domain D is a simply connected, bounded domain that lies together with its boundary Γ in G.*

If $f(z)$ has not a single zero or pole on Γ, then

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f'(z)}{f(z)}\,dz = N - P, \tag{30.1}$$

where N is the number of zeros and P is the number of poles of $f(z)$ in D, each counted according to their order.

Proof. Note that $f(z)$ may have only a finite number of poles in D, since otherwise there would be a limit or accumulation point of poles (a nonisolated singular point) in G. The number of zeros of

$f(z)$ in D must also be finite, since otherwise there is a limit or accumulation point of zeros of $f(z)$ in G and, hence, $f(z) \equiv 0$ in D, by the uniqueness theorem.

Only the zeros and poles of $f(z)$ can be the singular points of the integrand $F(z) = f'(z)/f(z)$, and, according to the residue theorem (Sec. 28), the left-hand side of (30.1) is equal to the sum of the residues at all the zeros and poles of $f(z)$ that lie in D.

Suppose point $z = a$ is a zero for $f(z)$ of order n. Then

$$f(z) = (z-a)^n g(z),$$

where $g(z)$ is a function regular at point a, $g(a) \neq 0$. Hence,

$$F(z) = \frac{f'(z)}{f(z)} = \frac{n}{z-a} + \frac{g'(z)}{g(z)},$$

which yields $\operatorname*{Res}_{z=a} F(z) = n$, i.e. the residue of $F(z)$ at a point $z = a$ that is a zero for $f(z)$ is equal to the order of the zero.

Similarly, if $z = b$ is a pole of $f(z)$ of order p, then

$$f(z) = (z-b)^{-p} h(z),$$

where $h(z)$ is a function regular at point b, $h(b) = \neq 0$. Hence,

$$F(z) = \frac{-p}{z-b} + \frac{h'(z)}{h(z)},$$

which yields $\operatorname*{Res}_{z=b} F(z) = -p$, i.e. the residue of $F(z)$ at a point $z = b$ that is a pole for $f(z)$ is equal to the order of the pole taken with the minus sum.

Thus, the left-hand side of (30.1) is equal to the difference between the sum of the orders of the zeros of $f(z)$ and the sum of the orders of the poles of $f(z)$. The proof of the theorem is complete.

Remark 1. Formula (30.1) remains valid for a multiply connected domain.

Corollary *Under the conditions of Theorem 1 we can rewrite* (30.1) *thus*:

$$\frac{1}{2\pi} \Delta_\Gamma \arg f(z) = N - P. \tag{30.2}$$

Here $\Delta_\Gamma \arg f(z)$ *is the variation of the argument of* $f(z)$ *when* Γ *is traversed once in the positive sense.*

Proof. By hypothesis, $f(z)$ is regular in a neighborhood of Γ and $f(z) \neq 0$ on Γ. Hence, $f(z) \neq 0$ in a neighborhood of Γ, and in this neighborhood we can isolate an analytic branch of $\ln f(z)$. Since $[\ln f(z)]' = f'(z)/f(z)$, we can write

$$\frac{1}{2\pi i} \int\limits_\Gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int\limits_\Gamma d(\ln f(z)) = \frac{1}{2\pi i} \Delta_\Gamma \ln f(z), \tag{30.3}$$

where $\Delta_\Gamma \ln f(z)$ is the variation of $\ln f(z)$ when Γ is traversed once in the positive sense. But $\ln f(z) = \ln |f(z)| + i \arg f(z)$, where $\ln |f(z)|$ is a single-valued function and hence $\Delta_\Gamma \ln |f(z)| = 0$. This means that

$$\Delta_\Gamma \ln f(z) = i\Delta_\Gamma \arg f(z),$$

and (30.3) yields

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\Delta_\Gamma \arg f(z),$$

which, by virtue of (30.1), leads to (30.2).

Equation (30.2) is known as the argument principle. According

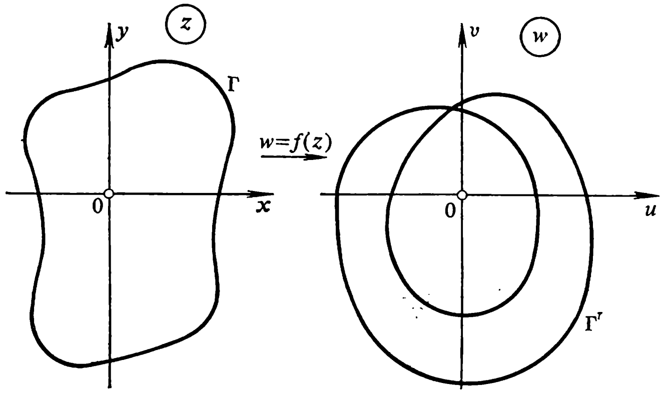


Fig. 76

to (30.2), the difference between the number of zeros and the number of poles of $f(z)$ inside Γ is equal to the variation of the argument of $f(z)$ as Γ is traversed divided by 2π. (This is true if $f(z)$ is regular inside Γ and on Γ, except at a finite number of poles, and does not vanish on Γ.)

Remark 2. Equation (30.2) remains valid for the case where $f(z)$ is regular in D, except at finite number of poles, and is continuous up to the boundary Γ of D.

For instance, if $f(z)$ has no poles in D (i.e. $P = 0$), Eq. (30.2) assumes the form

$$\frac{1}{2\pi}\Delta_\Gamma \arg f(z) = N. \tag{30.4}$$

What is the geometrical meaning of $\Delta_\Gamma \arg f(z)$? Suppose Γ' is the image of Γ (Fig. 76) obtained as a result of the mapping $w = f(z)$. As point z traverses the closed contour Γ completely, the corresponding point in the w plane traverses Γ'. The variation of the argument of $f(z)$ on Γ is determined by the number of complete revolutions that vector w does when point w traverses Γ'. If vector w performs not a single complete revolution about point $w = 0$, the variation $\Delta_\Gamma \arg f(z) = 0$.

30.2 Rouchc's theorem To find the number of zeros of a rational function in a given domain we often use the following

Theorem 2 (Rouche's theorem) *Let $f(z)$ and $g(z)$ be regular in a simply connected, bounded domain D and on its boundary Γ, and suppose that*

$$|f(z)| > |g(z)| \tag{30.5}$$

on Γ. Then $f(z)$ and $F(z) = f(z) + g(z)$ in D have the same number of zeros, counted according to their order.

Proof. In view of (30.5), $f(z) \neq 0$ for all $z \in \Gamma$. In addition, $f(z) \neq 0$ on Γ, since $|F(z)| \geqslant |f(z)| - |g(z)| > 0$. Suppose N_F and N_f is the number of zeros in D of $F(z)$ and $f(z)$, respectively. By (30.4),

$$N_F = \frac{1}{2\pi} \Delta_\Gamma \arg F(z). \tag{30.6}$$

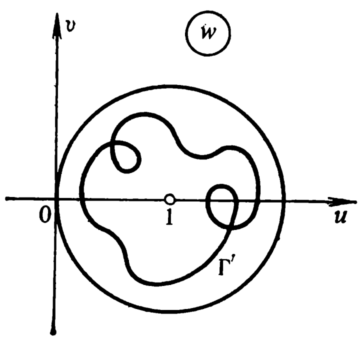


Fig. 77

Since $f(z) \neq 0$ on Γ, for $z \in \Gamma$ the relation

$$F(z) = f(z) + g(z) = f(z)\left[1 + \frac{g(z)}{f(z)}\right]$$

implies

$$\Delta_\Gamma \arg F(z) = \Delta_\Gamma \arg f(z) + \Delta_\Gamma \arg \left(1 + \frac{g(z)}{f(z)}\right). \tag{30.7}$$

Let us show that the second term on the right-hand side of (30.7) is zero. Indeed, when point z traverses Γ, point $w = 1 + g(z)/f(z)$ traverses the closed curve Γ' (Fig. 77) lying in the circle $|w - 1| < 1$, since by virtue of (30.5) we can write $|w - 1| = |g(z)/f(z)| < 1$ for $z \in \Gamma$. Consequently, vector w, whose terminal point traverses Γ', performs not a single complete revolution about point $w = 0$ and, whence, $\Delta_\Gamma \arg (1 + g(z)/f(z)) = 0$. Thus, (30.6) and (30.7) imply that $N_F = N_f$.

Example 1. Let us find the number of roots of the equation

$$z^9 - 6z^4 + 3z - 1 = 0$$

in the circle $|z| < 1$. We introduce the notations $f(z) = -6z^4$ and $g(z) = z^9 + 3z - 1$. If $z \in \Gamma$: $|z| = 1$, then

$$|f(z)| = 6, \quad |g(z)| \leqslant |z|^9 + 3|z| + 1 = 5,$$

whence $|f(z)| > |g(z)|$ for $z \in \Gamma$. By Rouche's theorem, the number of roots of the initial equation in the circle $|z| < 1$ coincides with the number of roots of the equation $f(z) = -6z^4$ in this circle, i.e. is equal to four. □

Example 2. Let us prove that the equation

$$z + \lambda - e^z = 0, \quad \lambda > 1, \tag{30.8}$$

has only one (and real) root in the left half-plane, Re $z < 0$.

Let us take a closed contour consisting of the semicircle C_R: $|z| = R$, Re $z \leqslant 0$ and the segment $l : [-iR, iR]$. We put $f(z) = z + \lambda$ and $g(z) = -e^z$. On l we have $|f(z)| = |\lambda + iy| \geqslant \lambda > 1$ and $|g(z)| = |e^{iy}| = 1$. On the semicircle C_R at $R > \lambda + 1$ we have

$$|f(z)| \geqslant |z| - \lambda = R - \lambda > 1, \quad |g(z)| = |e^{x+iy}| = e^x \leqslant 1,$$

since $x \leqslant 0$. By Rouche's theorem, the number of roots of Eq. (30.8) in $|z| < R$, Re $z < 0$ for any $R > \lambda + 1$ is equal to the number of roots of the equation $z + \lambda = 0$, or 1. This means that in the entire left half-plane the equation has only one root. This root is real because the left-hand side of Eq. (30.8) is positive (equal to $\lambda - 1$) for $z = x = 0$ and tends to $-\infty$ as $x \to -\infty$. □

Remark 3. Rouche's theorem remains valid if we replace the condition that $f(z)$ and $g(z)$ be regular on the boundary Γ of D by the condition that these functions be continuous up to Γ, while all other conditions remain the same.

Rouche's theorem provides an easy method of proving the fundamental theorem of algebra.

Theorem 3 (the fundamental theorem of algebra) *Every polynomial of degree n with complex valued coefficients has exactly n roots.*

Proof. Suppose

$$P_n(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$$

is an arbitrary polynomial of degree n ($a_0 \neq 0$). We introduce the notations $f(z) = a_0 z^n$ and $g(z) = a_1 z^{n-1} + \ldots + a_{n-1} z + a_n$. Then

$$P_n(z) = F(z) = f(z) + g(z).$$

Since $\lim\limits_{z \to \infty} \frac{g(z)}{f(z)} = 0$, there is a positive R such that

$$\left| \frac{g(z)}{f(z)} \right| < 1 \tag{30.9}$$

for all z: $|z| \geqslant R$. Let Γ be the circle $|z| = R$. Since (30.9) is valid on Γ, we can write $N_F = N_f$ (according to Rouche's theorem). But $N_f = n$, since $f(z) = a_0 z^n$ has n zeros in the circle $|z| < R$ (point $z = 0$ is an nth order zero for $f(z)$). Thus, the number of zeros of $F(z) = P_n(z)$ in the circle $|z| < R$ is n, i.e. the polynomial

$P_n(z)$ has n zeros in this circle. Since in view of (30.9) $F(z)$ has no zeros at $|z| \geqslant R$, we conclude that the theorem is indeed true.

31 The Partial-Fraction Expansion of Meromorphic Functions

Here we apply the theory of residues to the problem of partial-fraction expansion of meromorphic functions. We start by recalling the notion of a meromorphic function (Sec. 19). A function is said to be *meromorphic* if it is regular in every finite part of the complex plane except at a finite number of poles.

In Sec. 19 it was shown that a meromorphic function $f(z)$ that has a finite number of poles in the entire extended complex plane (a rational function) can be represented in the form of the sum of a polynomial (the principal part of the Laurent expansion of $f(z)$ about point $z = \infty$) and partial fractions (the principal parts of the Laurent expansions of $f(z)$ about the poles of $f(z)$). This proposition can be generalized to include the case of a meromorphic function that has an infinite (countable) number of poles in the entire extended complex plane.

31.1 The meromorphic-function expansion theorem We start with

Definition 1. Suppose we have a sequence $\{\Gamma_n\}$ of nested closed contours Γ_n (Γ_n lies inside Γ_{n+1}, $n = 1, 2, \ldots$), all containing point $z = 0$ and such that

$$\frac{S_n}{d_n} \leqslant C \quad (n = 1, 2, \ldots), \tag{31.1}$$

where S_n is the length of Γ_n, and d_n is the distance between the origin of coordinates and curve Γ_n ($d_n = \inf\limits_{z \in \Gamma_n} |z|$), with

$$d_n \to \infty \quad (n \to \infty). \tag{31.2}$$

We call such a system of contours *regular.*

Theorem 1 *Suppose all the poles z_k ($k = 1, 2, \ldots$) of a meromorphic function $f(z)$ that is regular at point $z = 0$, are simple and numbered according to the order in which their absolute values do not decrease: $|z_1| \leqslant |z_2| \leqslant \ldots$ If $f(z)$ is bounded on a regular system of contours $\{\Gamma_n\}$, i.e. if*

$$|f(z)| \leqslant M, \quad z \in \Gamma_n \quad (n = 1, 2, \ldots), \tag{31.3}$$

then

$$f(z) = f(0) + \sum_{k=1}^{\infty} A_k \left(\frac{1}{z - z_k} + \frac{1}{z_k} \right), \tag{31.4}$$

with $A_k = \operatorname*{Res}_{z=z_k} f(z)$. *The series* (31.4) *converges uniformly in every bounded domain with the poles of* $f(z)$ *deleted.*

Proof. Consider the integral

$$I_n(z) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{zf(\zeta)}{\zeta(\zeta - z)} d\zeta, \tag{31.5}$$

where $z \in G_n$ (G_n is the interior of Γ_n) and $z \neq z_k$ ($k = 1, 2, \ldots$). We denote the integrand in (31.5) by $F(\zeta)$.

The function $F(\zeta)$ has in G_n only simple poles: at $\zeta = z$ and $\zeta = z_k \in G_n$, point $\zeta = 0$ is either a simple pole or a regularity point (if $f(0) = 0$) for $F(\zeta)$. By the residue theorem

$$I_n(z) = \operatorname*{Res}_{\zeta=0} F(\zeta) + \operatorname*{Res}_{\zeta=z} F(\zeta) \sum_{z_k \in G_n} \operatorname*{Res}_{\zeta=z_k} F(\zeta). \tag{31.6}$$

In view of (28.3) we have

$$\operatorname*{Res}_{\zeta=0} F(\zeta) = \left[\frac{zf(\zeta)}{\zeta - z}\right]_{\zeta=0} = -f(0), \tag{31.7}$$

$$\operatorname*{Res}_{\zeta=z} F(\zeta) = \left[\frac{zf(\zeta)}{\zeta}\right]_{\zeta=z} = f(z), \tag{31.8}$$

$$\operatorname*{Res}_{\zeta=z_k} F(\zeta) = \left[\frac{1}{\zeta(\zeta - z)}\right]_{\zeta=z_k} \operatorname*{Res}_{\zeta=z_k} f(\zeta) = \frac{A_k z}{z_k(z_k - z)}. \tag{31.9}$$

Substituting (31.7)-(31.9) into (31.6) we obtain

$$I_n(z) = -f(0) + f(z) + \sum_{z_k \in G_n} \frac{A_k z}{z_k(z_k - z)},$$

which in view of the fact that $\frac{z}{z_k(z_k - z)} = -\left(\frac{1}{z - z_k} + \frac{1}{z_k}\right)$ yields

$$f(z) = f(0) + \sum_{z_k \in G_n} A_k \left(\frac{1}{z - z_k} + \frac{1}{z_k}\right) + \frac{1}{2\pi i} \int_{\Gamma_n} \frac{zf(\zeta)}{\zeta(\zeta - z)} d\zeta. \tag{31.10}$$

We estimate $I_n(z)$. Suppose D is a bounded domain. Then there is a circle K: $|z| < R$ such that $D \subset K$. We have

$$|I_n(z)| < \frac{|z|}{2\pi} \int_{\Gamma_n} \frac{|f(\zeta)|}{|\zeta| \, |\zeta - z|} |d\zeta|.$$

Here $|z| < R$ ($z \in D \subset K$), $|\zeta| \geqslant d_n$ (d_n is the distance between the origin of coordinates and the contour Γ_n), $|\zeta - z| \geqslant |\xi| - |z| > d_n - R$, and $|f(\zeta)| \leqslant M$. Hence,

$$|I_n(z)| \leqslant \frac{MR}{2\pi} \frac{1}{d_n(d_n - R)} S_n \leqslant \frac{CMR}{2\pi(d_n - R)},$$

since $S_n \leqslant Cd_n$, in view of (31.1). This estimate and condition (31.2) imply that $I_n(z) \to 0$ as $n \to \infty$ uniformly in $z \in D$ ($z \neq z_k$, $k = 1, 2, \ldots$).

Sending (31.10) to the limit as $n \to \infty$ yields

$$f(z) = f(0) + \lim_{n\to\infty} \sum_{z_k \in G_n} A_k \left(\frac{1}{z - z_k} + \frac{1}{z_k}\right). \qquad (31.11)$$

We can write (31.11) in compact form (31.4) if we assume that summation in (31.4) is performed in the following order: first we take the terms with poles in the interior of Γ_1, then we add to them the terms with poles lying between Γ_1 and Γ_2, etc. The proof of the theorem is complete.

Remark 1. Theorem 1 can be generalized if we replace (31.3) by

$$|f(z)| \leqslant M |z|^p, \quad z \in \Gamma_n \quad (n = 1, 2, \ldots), \qquad (31.12)$$

where p is a nonnegative integer (the other conditions of Theorem 1 remaining the same). In this case we have

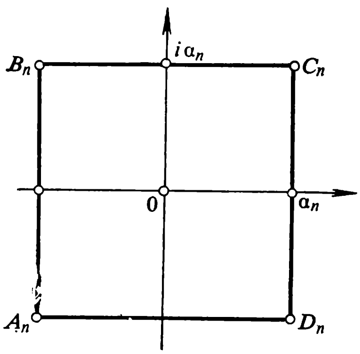


Fig. 78

$$f(z) = \sum_{k=0}^{p-1} \frac{f^{(k)}(0)}{k!} z^k + \sum_{k=1}^{\infty} A_k \left(\frac{1}{z - z_k} + \frac{1}{z_k} + \frac{z}{z_k^2} + \ldots + \frac{z^{p-1}}{z_k^p}\right). \qquad (31.13)$$

To prove the validity of (31.13) we need only apply the residue theorem to the integral

$$\frac{1}{2\pi i} \int_{\Gamma_n} \frac{z^p f(\zeta)}{\zeta^p (\zeta - z)} d\zeta.$$

31.2 The partial-fraction expansion of cot z Consider the function $\cot z - 1/z$. It is meromorphic since it has only simple poles at points $z_k = k\pi$ ($k = \pm 1, \pm 2, \ldots$) and no other finite singular points, and $\operatorname*{Res}_{z=k\pi} f(z) = 1$ (see Example 5 in Sec. 28). We will now show that $f(z)$ is bounded on the regular system of contours $\{\Gamma_n\}$, where Γ_n is the square $A_nB_nC_nD_n$ (Fig. 78) centered at point $z = 0$, with sides parallel to the coordinate axes (the length of each side is $2\alpha_n$, with $\alpha_n = \pi/2 + \pi n$ ($n = 0, 1, 2, \ldots$)).

We select a point $z \in C_nD_n$. Then $z = \alpha_n + iy$, where $-\alpha_n \leqslant y \leqslant \alpha_n$ and, hence,

$$|\cot z| = \left|\cot\left(\frac{\pi}{2} + \pi n + iy\right)\right| = |\tan iy| = \left|\frac{e^y - e^{-y}}{e^y + e^{-y}}\right|,$$

whence

$$|\cot z| \leqslant 1, \quad z \in C_nD_n \quad (n = 0, 1, 2, \ldots). \qquad (31.14).$$

Now let $z \in B_nC_n$. Then $z = x + i\alpha_n$, where $-\alpha_n \leqslant x \leqslant \alpha_n$, a

$$|\cot z| = \left|\frac{e^{2iz}+1}{e^{2iz}-1}\right| = \left|\left|\frac{1+e^{-2\alpha_n}e^{2ix}}{1-e^{-2\alpha_n}e^{2ix}}\right|\right| \leqslant \frac{1+e^{-2\alpha_n}}{1-e^{-2\alpha_n}},$$

whence

$$|\cot z| \leqslant \frac{1+e^{-\pi}}{1-e^{-\pi}}, \quad z \in B_nC_n \quad (n=0, 1, 2, \ldots). \qquad (31.15)$$

Since $|\cot(-z)| = |\cot z|$, we can write the inequalities (31.14) and (31.15), respectively, for the sides A_nB_n and D_nA_n on the square $A_nB_nC_nD_n$, i.e. on Γ_n. Thus,

$$|\cot z| \leqslant M, \quad z \in \Gamma_n, \quad n = 0, 1, 2, \ldots.$$

This implies that $f(z) = \cot z - 1/z$ is bounded on the system of contours $\{\Gamma_n\}$. Moreover, $f(0) = 0$, since $f(z)$ (regular at point $z = 0$) is odd. Thus, in (31.4) we can put $f(0) = 0$ and $A_k = 1$ $(k = 1, 2, \ldots)$; consequently,

$$\cot z = \frac{1}{z} + \sum_{k=-\infty}^{\infty}{}' \left(\frac{1}{1-k\pi} + \frac{1}{k\pi}\right), \qquad (31.16)$$

where the prime on the summation sign means that $k \neq 0$.

Note that there are exactly two poles in the region between Γ_{k-1} and Γ_k that belong to $f(z)$, namely, at $z_k = k\pi$ and at $\tilde{z}_k = -k\pi$. Combining in (31.16) the terms with these poles, we obtain

$$\frac{1}{z-k\pi} + \frac{1}{k\pi} + \frac{1}{z+k\pi} - \frac{1}{k\pi} = \frac{2z}{z^2-k^2\pi^2}.$$

Thus, we have arrived at the following formula:

$$\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2-k^2\pi^2}, \qquad (31.17)$$

Example 1. Let us find the partial-fraction expansions for the following meromorphic functions: (a) $\tan z$, (b) $1/\sin^2 z$, and (c) $1/(e^z - 1)$.

(a) Since $\tan z = -\cot(z - \pi/2)$, formula (31.16) yields

$$\tan z = -\frac{1}{z-(\pi/2)} - \sum_{k=1}^{\infty}\left(\frac{1}{z-(\pi/2)-k\pi} + \frac{1}{z-(\pi/2)-k\pi}\right)$$

$$= -\sum_{k=1}^{\infty}\left(\frac{1}{z-\frac{2k-1}{2}\pi} + \frac{1}{z+\frac{2k-1}{2}\pi}\right);$$

whence

$$\tan z = -\sum_{k=1}^{\infty} \frac{2z}{z^2-(2k-1)^2\pi^2/4}. \tag{31.18}$$

(b) Since $1/\sin^2 z = -(\cot z)'$, differentiation of the uniformly convergent series (31.16) results in

$$\frac{1}{\sin^2 z} = \frac{1}{z^2} + \sum_{k=-\infty}^{\infty}{}' \frac{1}{(z-k\pi)^2} = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k\pi)^2}. \tag{31.19}$$

(c) Since

$$\frac{1}{e^z-1} = \frac{e^{-z/2}}{e^{z/2}-e^{-z/2}} = \frac{1}{2}\,\frac{e^{-z/2}-e^{z/2}+e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}} = -\frac{1}{2}+\coth\frac{z}{2},$$

we can combine (31.17) and the fact that $\coth\zeta = i\cot(i\zeta)$ and obtain

$$\frac{1}{e^z-1} = -\frac{1}{2}+\frac{1}{z}+\sum_{k=1}^{\infty}\frac{2z}{z^2+4k^2\pi^2}. \ \square$$

Example 2. Let us show that

$$s_1 = \sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\left(\coth a\pi - \frac{1}{a\pi}\right), \tag{31.20}$$

$$s_2 = \sum_{n=1}^{\infty} \frac{1}{(n^2+a^2)^2} = \frac{1}{4a^4}\left(\frac{\pi^2a^2}{\sinh^2 a\pi} + \pi a\coth a\pi - 2\right), \tag{31.21}$$

assuming that the denominators in (31.20) and (31.21) are nonzero.

Putting $z = ia\pi$ in (31.17), we obtain

$$\cot ia\pi = -i\coth a\pi = -\frac{i}{a\pi} - \frac{2ai}{\pi}s_1,$$

from which (31.20) follows. Formula (31.21) can be found by differentiating (31.20) with respect to a. $\square$

31.3 Expansion of an entire function in an infinite product It is well known that every polynomial $P_n(z)$ of degree n can be represented in the form of the product

$$P_n(z) = A(z-z_1)(z-z_2)\dots(z-z_n) = A\prod_{k=1}^{n}(z-z_k), \tag{31.22}$$

where $z_1, z_2, \dots, z_n$ are the roots of $P_n(z)$ (there may be multiple roots among them). This formula can be generalized (under certain conditions) so that it incorporates entire functions. The only interesting cases arise when the number of roots of an entire function $f(z)$

is countable. Indeed, if $f(z)$ is nonzero in the entire complex plane, then the function $F(z) = \ln f(z)$, where we have taken one of the regular branches of the logarithm (see Example 6 in Sec. 24), is an entire function, with $F'(z) = f'(z)/f(z)$. Whence

$$f(z) = e^{F(z)}. \tag{31.23}$$

Moreover, if an entire function $f(z)$ has only a finite number of zeros a_k ($k = 1, 2, \ldots, s$) and p_k is the order of the zero at a_k, then $\Phi(z) = f(z)/\varphi(z)$, where $\varphi(z) = (z - a_1)^{p_1} \ldots (z - a_s)^{p_s}$, vanishes nowhere and, hence, can be represented in the form (31.23), from which we arrive at $f(z) = (z - a_1)^{p_1} \ldots (z - a_s)^{p_s} e^{F(z)}$, i.e.

$$f(z) = e^{F(z)} \prod_{k=1}^{s} (z - a_k)^{p_k}, \tag{31.24}$$

where $F(z)$ is an entire function.

Now suppose an entire function $f(z)$ has an infinite number of zeros. We wish to generalize (31.24) so that it incorporates this case. But instead of finite products we will have infinite products. For this reason some preliminary information about infinite products is in order (the interested reader can refer to Bitsadze [1] and Markushevich [1]).

Definition 2. The infinite product

$$\prod_{k=1}^{\infty} (1 + a_k) \tag{31.25}$$

is said to be *convergent* if all its factors are nonzero and there is a finite and nonzero limit A of the sequence of the finite products

$$A_n = \prod_{k=1}^{n} (1 + a_k).$$

Note that a necessary and sufficient condition for the convergence of the infinite product (31.25) is the convergence of the series

$$\sum_{k=1}^{\infty} \ln (1 + a_k), \tag{31.26}$$

where $-\pi < \arg (1 + a_k) \leqslant \pi$, $k = 1, 2, \ldots$

Definition 2. The infinite product (31.25) is said to be *absolutely convergent* if the series (31.26) is absolutely convergent.

It is possible to show that the absolute convergence of the infinite product (31.25) is equivalent to the absolute convergence of the series

$$\sum_{k=1}^{\infty} |a_k|.$$

The concept of convergence of an infinite product can be naturally generalized to the case where the factors of the product are functions of a complex variable. Consider the infinite product

$$\prod_{k=1}^{\infty} [1+f_k(z)], \tag{31.27}$$

where the $f_k(z)$ are functions regular in a domain D.

Definition 4. The infinite product (31.27) is said to be *convergent in domain D* if its factors (except perhaps a finite number of them) do not vanish in D and if the product of the nonzero factors is convergent at every point in D.

Definition 5. The infinite product (31.27) with its factors being nonzero in a domain D is said to be *uniformly convergent in domain D* if the sequence of the functions

$$F_n(z)=\prod_{k=1}^{n} [1+f_k(z)]$$

is uniformly convergent in D.

If the infinite product (31.27) is uniformly convergent in D, the function

$$F(z)=\lim_{n\to\infty} F_n(z)=\prod_{k=1}^{\infty} [1+f_k(z)]$$

is regular in D, by Weierstrass's first theorem (Sec. 12).

Theorem 1 on the partial-fraction expansion of meromorphic functions brings us to the following theorem on the infinite-product expansion of entire functions:

Theorem 2 *If an entire function $f(z)$ is such that the meromorphic function $F(z)=f'(z)/f(z)$ satisfies the hypothesis of Theorem 1, then*

$$f(z)=f(0)\,e^{Bz}\prod_{k=1}^{\infty}\left(1-\frac{z}{z_k}\right)e^{z/z_k},\quad B=\frac{f'(0)}{f(0)}. \tag{31.28}$$

The infinite product (31.28) *is uniformly convergent in each bounded part of the complex plane.*

In this formula each factor $(1-z/z_k)e^{z/z_k}$ is repeated the number of times equal to the order of the zero z_k.

Proof. The function $F(z)$ has simple poles at the points z_k that are the zeros of $f(z)$ and no other poles. Then $A_k=\operatorname*{Res}_{z=z_k} F(z)=n_k$, where n_k is the order of the zero z_k of $f(z)$ (see Sec. 30). By Theorem 1,

$$F(z)=F(0)+\sum_{k=1}^{\infty}\left(\frac{1}{z-z_k}+\frac{1}{z_k}\right). \tag{31.29}$$

Since $F(z) = \frac{d}{dz}[\ln f(z)]$, where we have taken an analytic branch of the logarithm, we can integrate (31.29) along a curve that connects points 0 and z and does not pass through the zeros of $f(z)$ and obtain

$$\ln f(z) - \ln f(0) = F(0)\, z + \sum_{k=1}^{\infty} \left[\ln\left(1 - \frac{z}{z_k}\right) + \frac{z}{z_k}\right]. \quad (31.30)$$

Finding the antilogarithm of (31.30), we obtain

$$f(z) = f(0)\, e^{F(0)z} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{z/z_k},$$

where $F(0) = f'(0)/f(0)$. The proof of the theorem is complete.

Remark 2. Under the conditions specified in Remark 1 formula (31.28) is replaced by

$$f(z) = e^{g(z)} \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) e^{h_k(z)},$$

where $h_k(z) = \frac{z}{z_k} + \frac{1}{2}\left(\frac{z}{z_k}\right)^2 + \ldots + \frac{1}{p}\left(\frac{z}{z_k}\right)^p$, and $g(z)$ is a polynomial whose degree is not higher than p.

31.4 The infinite-product expansion of sin z Let us take the entire function $f(z) = (\sin z)/z$. It has simple zeros at the points $z_k = k\pi$ ($k = \pm 1, \pm 2, \ldots$). The function $F(z) = f'(z)/f(z) = \cot z - 1/z$ satisfies the hypothesis of Theorem 2 and, hence, (31.28) is valid. Since $f(z) = (\sin z)/z = 1 - z^2/3 + \ldots$, we find that $f(0) = 1$ and $f'(0) = 0$. Then formula (31.28) yields

$$\frac{\sin z}{z} = \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{k\pi}\right) e^{\frac{z}{k\pi}} \left(1 + \frac{z}{k\pi}\right) e^{-\frac{z}{k\pi}}\right]. \quad (31.31)$$

Here we have collected the factors that belong to the zeros $k\pi$ and $-k\pi$ ($k = 1, 2, \ldots$) of $\sin z$. Multiplying the factors inside the square brackets, we obtain

$$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right). \quad (31.32)$$

Example 3. Let us expand the entire function $e^z - 1$ in an infinite product. We have

$$e^z - 1 = e^{z/2}\left(\frac{e^{z/2} - e^{-z/2}}{2}\right) = e^{z/2} \sinh \frac{z}{2}.$$

Employing (31.32) and the fact that $\sinh \zeta = -i \sin i\zeta$, we obtain

$$e^z - 1 = ze^{z/2} \prod_{k=1}^{\infty} \left(1 + \frac{z^2}{4k^2\pi^2}\right). \ \square$$

31.5 Inversion of power series We conclude this chapter by an example that illustrates another application of the theory of residues. We wish to find the inverse of a power series, i.e. find the coefficients in the series

$$z = h(w) = \sum_{n=0}^{\infty} b_n (w - w_0)^n, \tag{31.33}$$

where $z = h(w)$ is the inverse of the function that is regular at point z_0:

$$w = f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad f'(z_0) \neq 0, \quad w_0 = f(z_0).$$

Since $f'(z_0) \neq 0$, we can apply the inversion function theorem (Sec. 13). This theorem implies that there are two circles, K: $|z - z_0| < \rho$ and K_1: $|w - w_0| < \rho_1$, such that for each point $w \in K_1$ the equation $f(z) = w$ has a unique solution $z \in K$. This defines a single-valued function $z = h(w)$ that is regular in K_1. Let us find the coefficient in the expansion (31.33) of this function.

Consider the integral

$$I(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta, \tag{31.34}$$

where γ is the boundary of K, and $w \in K_1$. The integrand, which we denote by $F(\zeta)$, is regular in the interior of γ except at point $z = h(w)$, which is a simple pole for $F(\zeta)$. The residue theorem then yields

$$I(w) = \operatorname*{Res}_{\zeta = h(w)} F(\zeta) = \left[\frac{\zeta f'(\zeta)}{(f(\zeta) - w)'}\right]_{\zeta = h(w)} = h(w) = z,$$

i.e.

$$z = h(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta. \tag{31.35}$$

We have

$$\frac{1}{f(\zeta) - w} = \frac{1}{f(\zeta) - w_0} \frac{1}{1 - \frac{w - w_0}{f(\zeta)(w_0)}} = \sum_{n=0}^{\infty} \frac{(w - w_0)^n}{(f(\zeta) - w_0)^{n+1}}. \tag{31.36}$$

The series in (31.36) is uniformly convergent in ζ ($\zeta \in \gamma$), since $|w - w_0| < \rho_1$ ($w \in K_1$) and $|f(\zeta) - w_0| \geqslant \rho_1$ ($\zeta \in \gamma$). Multiplying (31.36) by $(1/2\pi i)\, \zeta f'(\zeta)$ and integrating termwise along γ, we obtain

$$z = h(w) = \sum_{n=0}^{\infty} b_n (w - w_0)^n,$$

where

$$b_n = \frac{1}{2\pi i} \int_\gamma \frac{\zeta f'(\zeta)}{[f(\zeta) - w_0]^{n+1}}\, d\zeta \quad (n = 0, 1, 2, \ldots). \tag{31.37}$$

Here $b_0 = z_0$. When n is nonzero ($n \geqslant 1$), integration by parts yields

$$b_n = \frac{1}{2\pi i n} \int_\gamma \frac{1}{[f(\zeta) - w_0]^n}\, d\zeta. \tag{31.38}$$

The integrand in (31.38) has only one singular point in the interior of γ, and nth order pole at $\zeta = z_0$. Calculating the residue via (28.7) yields

$$b_n = \frac{1}{n!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{z - z_0}{f(z) - f(z_0)} \right]^n; \quad n = 1, 2, \ldots. \tag{31.39}$$

The series (31.33) whose coefficients are calculated via (31.39) ($b_0 = z_0$) is known as the *Bürmann-Lagrange series.*

Here are the formulas that enable calculating the coefficients b_1, b_2, and b_3 in (31.33) in terms of the coefficients a_n in the series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

We have

$$b_1 = \frac{1}{a_1}, \quad b_2 = -\frac{a_2}{a_1^3}, \quad b_3 = \frac{1}{a_1^5}(2a_2^2 - a_1 a_3).$$

Example 4. Let us take $f(z) = ze^{-az}$ ($z_0 = 0$ and $w_0 = 0$). Then, employing (31.39), we find that

$$b_n = \frac{1}{n!} \lim_{z \to 0} \frac{d^{n-1}}{dz^{n-1}} (e^{anz}) = \frac{(an)^{n-1}}{n!}$$

and, hence,

$$z = h(w) = \sum_{n=1}^{\infty} \frac{(an)^{n-1}}{n!} w^n. \ \square$$

Chapter VI

Conformal Mapping

The concept of a conformal mapping was introduced in Sec. 8. This chapter considers the basis properties of conformal mappings and studies in detail the various mappings performed by elementary functions.

32 Local Properties of Mappings Performed by Regular Functions

32.1 The inverse function theorem In Sec. 13 we proved the inverse function theorem for a function $f(z)$ that is regular at a point z_0 when $f'(z_0) \neq 0$. But what happens when $f'(z_0) = 0$?

Theorem 1 *Suppose a function $w = f(z)$ is regular at a point $z_0 \neq \infty$ and*

$$f'(z_0) = f''(z_0) = \ldots = f^{(n-1)}(z_0) = 0, \quad f^{(n)}(z_0) \neq 0,$$

where $n \geqslant 2$. Then there are neighborhoods U and V of points z_0 and $w_0 = f(z_0)$, respectively, and a function $z = \psi(w)$ such that

(a) *the equation $f(z) = w$ (with respect to z) has for each point $w \in V$, $w \neq w_0$, exactly n different solutions $z = \psi(w)$ that belong to U,*

(b) *the function $z = \psi(w)$ is analytic in V, $w \neq w_0$, and*

$$f(\psi(w)) \equiv w, \quad w \in V. \tag{32.1}$$

From (32.1) it follows that $z = \psi(w)$ is the inverse of $w = f(z)$, $z \in U$. In view of (a), the inverse function is n-valued in V, $w \neq w_0$.

Proof. Point z_0, by hypothesis, is an nth order zero for the function $f(z) - f(z_0)$, i.e.

$$w - w_0 = f(z) - f(z_0) = (z - z_0)^n h(z),$$

where $h(z)$ is regular at point z_0, $h(z_0) \neq 0$ (see Sec. 12.5). Introducing the notation $w - w_0 = \zeta^n$, we obtain $\zeta^n = (z - z_0)^n h(z)$, whence $\zeta = (z - z_0)\sqrt[n]{h(z)}$. The function $\sqrt[n]{h(z)}$ splits in a neighborhood of point z_0 into regular branches, since $h(z_0) \neq 0$ (see Sec. 24.2). Suppose $h_1(z)$ is one such branch and $\zeta = (z - z_0) h_1(z)$.

Then we can write $w = f(z)$ as a sum of two regular functions:

$$w = w_0 + \zeta^n, \tag{32.2}$$

$$\zeta = \zeta(z) = (z - z_0)h_1(z), \tag{32.3}$$

where $h_1(z)$ is regular at point z_0, $h_1(z_0) \neq 0$.

The function specified by (32.3) satisfies the hypothesis of the theorem of Sec. 13, since $\zeta'(z_0) = h_1(z_0) \neq 0$. According to that theorem, there is a neighborhood U of point z_0 which the function $\zeta = \zeta(z)$ maps in a one-to-one manner onto a circle K: $|\zeta| < \rho$, $\rho > 0$ $(\zeta_0 = \zeta(z_0) = 0)$. The function that is the inverse of $\zeta = \zeta(z)$, $z \in U$, is $z = g(\zeta)$ regular in K.

The function $\zeta = \sqrt[n]{w - w_0}$ that is the inverse of (32.2) is n-valued and analytic in the annulus V: $0 < |w - w_0| < \rho^n$ (see Sec. 22). Hence, the function $z = \psi(w) = g(\sqrt[n]{w - w_0})$, which is the inverse of $w = f(z)$, $z \in U$, is n-valued and analytic in the annulus V, since it is a combination of a regular function and an analytic function (see Sec. 22).

Corollary 1 *Under the conditions of Theorem 1, point w_0 is an algebraic branch point of multiplicity n for the function $z = \psi(w)$, the inverse of $w = f(z)$, and in a neighborhood of point w_0 the following series expansion holds*:

$$\psi(w) = \sum_{k=0}^{\infty} c_k (\sqrt[n]{w - w_0})^k,$$

where $c_0 = z_0$ and $c_1 \neq 0$.

Indeed, $\psi(w) = g(\sqrt[n]{w - w_0})$ and $g(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k$, where $c_0 = g(0) = z_0$ and $c_1 \neq 0$, since if we use (32.3) and the formula for the derivative of the inverse function (see formula (13.2)), we have

$$c_1 = g'(0) = \frac{1}{\zeta'(z_0)} = \frac{1}{h_1(z_0)} \neq 0.$$

The proof of Theorem 1 leads to

Corollary 2 *Under the conditions of Theorem 1, there is a function $z = g(\zeta)$, $z_0 = g(0)$, that is regular at point $\zeta = 0$ and such that*

$$f(g(\zeta)) = f(z_0) + \zeta^n$$

in a neighborhood of point $\zeta = 0$. The derivative of f at point $\zeta = 0$ is

$$g'(0) = \sqrt[n]{\frac{n!}{f^{(n)}(z_0)}}.$$

Example 1. Suppose a point z_0 is a pole for $f(z)$. Consider the equation

$$f(z) = A. \tag{32.4}$$

Suppose U is a small neighborhood of point z_0. We will show that there is an α such that for every A that satisfies the inequality $|A| > \alpha$ Eq. (32.4) has exactly n different solutions belonging to U, where n is the order of the pole z_0 of $f(z)$.

(i) If $z_0 \neq \infty$, then $g(z) = 1/f(z)$ is regular at point z_0 and $g(z_0) = g'(z_0) = \ldots = g^{(n-1)}(z_0) = 0$, $g^{(n)}(z_0) \neq 0$ (see Sec. 18). From Sec. 13 and Theorem 1 it follows that the equation $g(z) = 1/A$, which is equivalent to Eq. (32.4), has exactly n different roots $z \in U$ if $1/|A| < \varepsilon$ for a positive ε.

(ii) If $z_0 = \infty$, we consider the one-to-one mapping $\zeta = 1/z$ of a neighborhood of point $z = \infty$ onto a neighborhood of point $\zeta = 0$ (Sec. 8). Then the number of solutions of Eq. (32.4) in a neighborhood of point $z = \infty$ coincides with the number of solutions of the equation $f(1/\zeta) = A$ in a neighborhood of point $\zeta = 0$. The function $h(\zeta) = f(1/\zeta)$ has an nth order zero at $\zeta = 0$ (see Sec. 18). Hence, as in the case (i), the equation $h(\zeta) = A$ has exactly n solutions. □

32.2 Univalent functions The definition of univalence in a domain was given in Sec. 8. Now we will introduce the concept of univalence at a point.

Definition. A function $f(z)$ is said to be *univalent at a point* z_0 if it is univalent in a neighborhood of point z_0.

It is obvious that a function univalent in a domain is univalent at each point of this domain. However, the converse is not generally true, i.e. a function that is univalent at each point of a domain may not be univalent in this domain (see Example 5 below).

Let us give the criteria of univalence at a point.

Theorem 2 *A function $f(z)$ that is regular at a point $z_0 \neq \infty$ is univalent at this point if and only if $f'(z_0) \neq 0$.*

Proof. Necessity. If $f'(z_0) = 0$ and $f(z) \not\equiv$ const, then, by Theorem 1, in any neighborhood of point z_0 there are two different points z_1 and z_2 such that $f(z_1)$ and $f(z_2)$, i.e. $f(z)$ is not univalent at point z_0. Obviously, the function $f(z) \equiv$ const is not univalent at z_0 either.

Sufficiency. If $f'(z_0) \neq 0$, then, by the theorem of Sec. 13, the function $f(z)$ is univalent at point z_0.

Corollary 3 *The function*

$$f(z) = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \ldots, \quad |z| > R,$$

regular at point $z = \infty$ is univalent at this point if and only if $c_{-1} = -\operatorname*{Res}_{z=\infty} f(z) \neq 0$.

Proof. Consider the function

$$g(\zeta) = f(1/\zeta) = c_0 + c_{-1}\zeta + c_{-2}\zeta^2 + \ldots, \quad |\zeta| < 1/R,$$

regular at point $\zeta = 0$. The function $\zeta = 1/z$ maps in a one-to-one manner the neighborhood $|z| > R$ of point $z = \infty$ onto the neighborhood $|\zeta| < 1/R$ of point $\zeta = 0$ (see Sec. 8). Hence, $f(z)$ is univalent at point $z = \infty$ if and only if $g(\zeta)$ is univalent at point $\zeta = 0$, i.e., by Theorem 2, we must ensure that $g'(0) = c_{-1} \neq 0$.

Corollary 4 *A function $f(z)$ with a pole at a point z_0 (finite or at infinity) is univalent at this point if and only if this pole is simple (of the first order).*

To prove this proposition we need only apply Theorem 2 (or Corollary 3 if $z_0 = \infty$) to the function $1/f(z)$. Corollary 4 also follows from Example 1.

Example 2. (a) The function $f(z) = z^2$ is univalent at every point of the extended complex plane except points 0 and ∞.

(b) The function $f(z) = 1/z^2$ is univalent at every point of the extended complex plane except at points 0 and ∞. □

Example 3. If a point z_0 is an essential singularity for a function $f(z)$, this function is not univalent at z_0. Indeed, in every neighborhood of point z_0 the equation $f(z) = A$ has, according to Picard's second theorem (Sec. 19), an infinite number of solutions for every value of A except, perhaps, one, i.e. $f(z)$ is not univalent at point z_0. □

Example 4. Suppose a function $f(z)$ is regular in a domain D everywhere except at two points, z_1 and z_2, that are poles for $f(z)$. Let us show that this function is not univalent in D. Indeed, if $|A|$ is large, then the equation $f(z) = A$ has at least two solutions $\tilde{z}_1$ and $\tilde{z}_2$ with point $\tilde{z}_j$ lying close to point z_j ($j = 1, 2$) (Example 1), i.e. $f(z)$ is not univalent in D. □

Example 5. (a) The function $f(z) = e^z$ is univalent at every point $z \neq \infty$, but is not univalent in the entire complex plane. Indeed, at all points $z_k = a + 2k\pi i$ ($k = 0, \pm 1, \pm 2, \ldots$) the function assumes the same value e^a.

(b) The function $f(z) = z^2$ is univalent at every point of the annulus $1 < |z| < 3$, but is not univalent in this annulus since $f(z)$ is an even function: $f(z) = f(-z)$. □

Let us now summarize. Suppose a function $f(z)$ is regular and univalent in a domain D with the points $z_1, z_2, \ldots, z_n$ deleted. Neither an essential singularity of $f(z)$ (Example 3) nor two poles (Example 4) can be among these points z_k ($k = 1, 2, \ldots, n$). Hence, $f(z)$ can have only one pole, which must be a first order one (Corollary 4).

Thus, the necessary conditions for the univalence of $f(z)$ in a domain D are the following:

(1) $f(z)$ must be regular in D everywhere except, perhaps, one point, a simple pole for $f(z)$.

(2) at every finite point $z \in D$ at which $f(z)$ is regular the derivative $f'(z)$ must be nonzero;

(3) if point $z = \infty$ belongs to D and at this point the function $f(z)$ is regular, then we must have $c_{-1} = -\operatorname*{Res}_{z=\infty} f(z) \neq 0$.

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Conditions (1)-(3) are not, generally speaking, sufficient for a function to be univalent in a domain (see Example 5). We will consider the sufficient conditions in Sec. 33.

32.3 The principle of domain preservation

Theorem 3 (the principle of domain preservation) *Suppose a function $f(z)$ is regular in a domain D and $f(z) \not\equiv$ const. Then under the mapping $w = f(z)$ the domain D is transformed into a domain.*

Proof. Let G be the image of D obtained as a result of the mapping $w = f(z)$. Let us show that G is an open set. Suppose point w_0 belongs to G, i.e. $w_0 = f(z_0)$, where $z_0 \in D$. By Theorem 1 and the theorem of Sec. 13, for any point w in a small neighborhood of point w_0 there is at least one point z in a neighborhood of point z_0 such that $w = f(z)$, i.e. $w \in G$. Thus, there is a neighborhood of w_0 belonging entirely to G.

The connectedness of G_0 follows from the continuity of the mapping $w = f(z)$, since under such a mapping the image of any continuous curve lying in D is a continuous curve consisting entirely of points belonging to set G. Hence, G is an open connected set, or a domain.

Corollary 5 *Let a function $f(z)$ be regular in a domain D of the extended complex plane except, perhaps, at the poles of $f(z)$, and $f(z) \not\equiv$ const. Then the function $w = f(z)$ maps D onto a domain in the extended complex z plane.*

Proof. Consider the case where $f(z)$ has one pole at a finite point $z_0 \in D$. For other cases the proof is similar.

Suppose D_0 is the domain D with point z_0 deleted. By Theorem 3, $w = f(z)$ maps D_0 onto the domain G_0. Example 1 implies that there is an annulus $R < |w| < \infty$ that belongs to G_0. Hence, the set $G = G_0 \cup \{w = \infty\}$ is a domain.

32.4 The maximum modulus principle

Theorem 4 *Suppose a function $f(z)$ is regular in a bounded domain D, is continuous up to the boundary of D, and is not a constant. Then the maximum of the modulus of this function,*

$$\max_{z \in \bar{D}} |f(z)|,$$

is attained only on the boundary of D.

Proof. Consider the point $z_0 \in D$. We wish to prove that there is a point $z_1 \in D$ such that $|f(z_1)| > |f(z_0)|$. By Theorem 3, the image of D obtained as a result of the mapping $w = f(z)$ is a domain G for which point $w_0 = f(z_0)$ is an interior point. This means we can select a point $w_1 \in G$ lying on the straight line connecting points 0 and w_0 and such that $|w_1| > |w_0|$ (Fig. 79). The point w_1 is the image of a point $z_1 \in D$, i.e. $w_1 = f(z_1)$. Hence, $|f(z_1)| > |f(z_0)|$. The proof of the theorem is complete.

Corollary 6 *If a function $f(z) \not\equiv$ const is regular in a domain D,* *then* $|f(z)|$ *cannot have a local maximum at an interior point of* D.

Indeed, Theorem 4 implies that in any neighborhood of point $z_0 \in D$ there is a point z_1 such that $|f(z_1)| > |f(z_0)|$.

Corollary 7 *If a function* $f(z) \not\equiv$ const *that is regular in a domain* D *has no zeros in* D, *then* $|f(z)|$ *cannot attain its minimum at an interior point of* D.

Indeed, in this case the function $1/f(z)$ is regular in D and, by

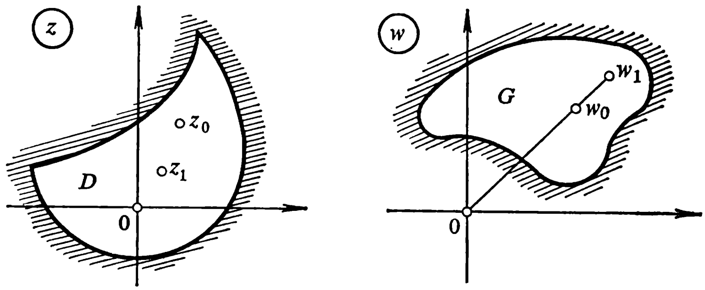


Fig. 79

Theorem 4, any neighborhood of a point $z_0 \in D$ contains a point $z_1 \in D$ such that $|1/f(z_1)| > |1/f(z_0)|$, i.e. $|f(z_1)| < |f(z_0)|$.

Example 6. Let a function $f(z) \not\equiv$ const be regular in a bounded domain D, is continuous up to the boundary Γ of D, and $|f(z)|_{z\in\Gamma} \equiv c =$ const. Let us show that $f(z)$ has at least one zero in D.

Indeed, if $f(z) \neq 0$ for all $z \in D$, then, by virtue of Corollary 7, we find that $|f(z)| > c$ for $z \in D$, which contradicts Theorem 4: $|f(z)| < c$ for $z \in D$. □

Schwarz's lemma *Suppose a function* $f(z)$ *is regular in the circle* $|z| < 1$, $f(0) = 0$, *and* $|f(z)| < 1$ *for* $|z| < 1$. *Then*

$$|f(z)| \leqslant |z|$$

in the entire circle $|z| < 1$. *If at least at one point* $z \neq 0$ *in the circle* $|z| < 1$ *we have* $|f(z)| = |z|$, *then*

$$f(z) = e^{i\alpha}z,$$

where α *is a real number.*

Proof. Consider the function $g(z) = f(z)/z$. It is regular in the circle $|z| < 1$, since $f(0) = 0$ (Sec. 18). On the boundary $|z| = \rho$, $0 < \rho < 1$, of this circle we have $|g(z)| = |f(z)|/|z| < 1/\rho$. Consequently, Theorem 4 states that $|g(z)| < 1/\rho$ in the entire circle $|z| \leqslant \rho$. Since ρ can be taken as close to unity as desired, we can write $|g(z)| \leqslant 1$, i.e. $|f(z)| \leqslant |z|$ for $|z| < 1$.

Moreover, if at a point z_0 ($|z_0| < 1$) the function $|g(z)|$ attains its maximum, i.e. $|g(z_0)| = 1$, then $g(z) \equiv$ const (Corollary 6), i.e. $g(z) = e^{i\alpha}$ and $f(z) = e^{i\alpha}z$.

The following maximum and minimum principle for harmonic functions is valid:

Theorem 5 *Suppose a function $u(x, y)$ that is harmonic in a bounded domain D is continuous up to the boundary of D and $u(x, y) \not\equiv$* const. *Then both the maximum and the minimum of this function are attained only on the boundary of D.*

Note that it is sufficient to prove the theorem for the case of a maximum, since the maximum of a harmonic function $u(x, y)$ coincides with the minimum of $-u(x, y)$, which is also a harmonic function.

Proof. Suppose the opposite statement is true, i.e. the maximum of $u(x, y)$ is attained at an interior point $z_0 = x_0 + iy_0$ of D. Consider a simply connected domain D lying inside D and containing point z_0. In D_1 there is a regular function $f(z)$ such that $\operatorname{Re} f(z) = u(x, y)$ (see Sec. 7). Then the function $g(z) = e^{f(z)}$ is regular in D_1 and its modulus, $|g(z)| = e^{u(x, y)}$, attains its maximum at point z_0. Hence, $g(z) \equiv$ const (Theorem 4), from which it follows that $f(z) \equiv$ const and $u(x, y) \equiv$ const for $z \in D_1$. Since D_1 is selected arbitrarily, we have $u(x, y) \equiv$ const for $z \in D$, which contradicts the hypothesis. The proof of the theorem is complete.

33 General Properties of Conformal Mappings

33.1 The definition of a conformal mapping In Sec. 8 we gave a definition of a conformal mapping for domains without the point at infinity. There we noted that such mappings are performed by univalent regular functions. For domains that include the point at infinity we introduce the following.

Definition 1. The mapping $w = f(z)$ of the domain D in the extended complex z plane onto a domain G in the extended complex w plane is said to be *conformal* if

(a) it is one-to-one, i.e. $f(z)$ is univalent in D;

(b) the function $f(z)$ is regular in D everywhere except, perhaps, a single point, where the function has a simple pole.

Let us now consider the local properties of a conformal mapping $w = f(z)$ in a neighborhood of a finite point z_0 at which $f(z)$ is regular. Since $f'(z_0) \neq 0$ is the criterion for the univalence of $f(z)$ at point z_0 (see Sec. 32), the geometrical meaning of the derivative (Sec. 8) yields the following two properties of conformal mappings:

(1) *Constancy of stretching.* The linear stretching at point z_0 is the same for all the curves passing through z_0 and is equal to $|f'(z_0)|$.

(2) *Preservation of the angle between curves.* All curves that pass through z_0 are rotated through the same angle $\arg f'(z_0)$.

Here are some properties of conformal mappings:

(3) *A mapping that is the inverse of a conformal mapping is also conformal.*

(4) *A combination of two conformal mappings is a conformal mapping.*

These properties follow from Definition 1 and the properties of univalent and inverse functions (Secs. 8, 13, and 32).

33.2 Conformality at the point at infinity We start with the concept of an angle between curves at the point at infinity.

Definition 2. The *angle between two curves*, γ_1 *and* γ_2, *that pass through the point* $z = \infty$ is defined as the angle between the images

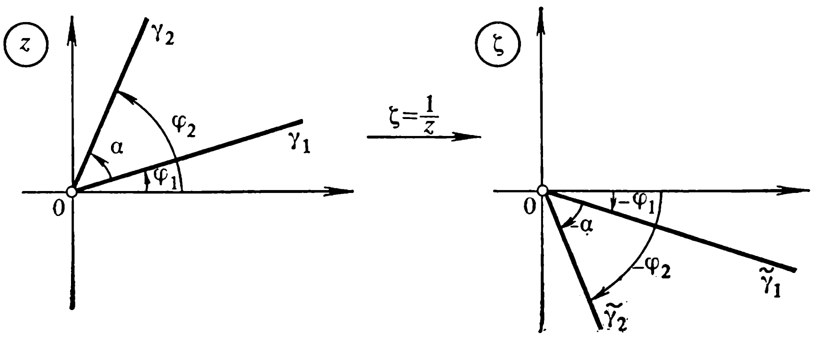


Fig. 80

of these curves obtained as a result of the mapping $\zeta = 1/z$ at point $\zeta = 0$.

Property 2 and this definition imply that the mapping $\zeta = 1/z$ preserves the angles between curves at each point in the extended complex plane.

Example 1. Suppose two rays γ_1 and γ_2 emerge from the same end point z_0. Then the angle between γ_1 and γ_2 at the point $z = \infty$ is equal to the angle between these rays at point z_0 taken with the opposite sign.

Proof. For the sake of simplicity we will restrict our discussion to the case where $z_0 = 0$. Suppose γ_j is the ray with $\arg z = \varphi_j (j = 1, 2)$. Then the angle between γ_1 and γ_2 (in the direction from γ_1 to γ_2) at point $z = 0$ is $\alpha = \varphi_2 - \varphi_1$ (Fig. 80). The image of γ_j under the mapping $\zeta = 1/z$ is the ray $\tilde{\gamma}_j$: $\arg \zeta = -\varphi_j$ $(j = 1, 2)$ (see Sec. 8); whence the angle between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ at point $\zeta = 0$ is $(-\varphi_2) - (-\varphi_1) = -\alpha$ (Fig. 80). Hence, by Definition 2, the angle between γ_1 and γ_2 at point $z = \infty$ is $-\alpha$. □

Definition 2 and Property 2 yields the following property of con ormal mappings:

(5) *A conformal mapping of a domain in the extended complex plane preserves the angles between curves at each point of the domain.*

Proof. By virtue of Definition 1 and Property 2, we need only prove the validity of the following propositions:

(a) If the function

$$f(z)=c_0+\frac{c_{-1}}{z}+\frac{c_{-2}}{z^2}+\ldots, \quad |z|>R,$$

is regular at point $z=\infty$ and $c_{-1}\neq 0$, then the mapping $w=f(z)$ preserves the angle between any two curves passing through point $z=\infty$.

(b) If $f(z)$ has a first order pole at a point z_0 (finite or at infinity), the mapping $w=f(z)$ preserves the angle between any two curves passing through point z_0.

We will prove proposition (a); proposition (b) can be proved similarly.

We write the function $w=f(z)$ as a composite of two functions: $\zeta=1/z$ and $w=g(\zeta)=f(1/\zeta)=c_0+c_{-1}\zeta+c_{-2}\zeta^2+\ldots$ By Definition 2, the mapping $\zeta=1/z$ preserves the angles between curves passing through $z=\infty$. The mapping $w=g(\zeta)$ preserves the angles between curves passing through point $\zeta=0$ since $g'(0)=$ $=c_{-1}\neq 0$ (Property 2). Hence, the mapping $w=f(z)$ preserves the angles between curves passing through point $z=\infty$.

Remark 1. It can be shown that the mapping $\zeta=1/z$ rotates Riemann's sphere about the diameter with the end points at $z=\pm 1$ (i.e. the images of these points on Riemann's sphere) through an angle of 180°. Hence, this mapping preserves the angle between any two curves at each point on Riemann's sphere. This makes Definition 2 look quite natural. It can be shown that all conformal mappings preserve the angles between curves on Riemann's sphere.

33.3 Correspondence between boundaries Let D and G be two simply connected, bounded domains whose boundaries Γ and $\tilde{\Gamma}$ are simple, closed, and piecewise smooth curves. Then we have

Theorem 1 (the theorem on the correspondence between boundaries) *If the function $w=f(z)$ maps a domain D conformally onto another domain G, then*

(1) *$f(z)$ can be continuously continued onto the closure of D, i.e. $f(z)$ can be redefined on Γ in such a way that it is continuous on $\bar{D}$;*

(2) *this function maps Γ onto $\tilde{\Gamma}$ in a one-to-one manner and with preservation of the sense.*

The proof of this theorem can be found in Hurwitz and Courant [1].

Here we will prove the converse of Theorem 1. Suppose D and G are simply connected, bounded domains with simple, closed, and piecewise smooth boundaries Γ and $\tilde{\Gamma}$, respectively. Then we have

Theorem 2 (a criterion of univalence of a function in a domain) *Suppose a function $w=f(z)$ that is regular in D and continuous up to Γ*

maps Γ *onto* $\tilde{\Gamma}$ *in a one-to-one manner and with preservation of sense. Then the function is univalent in D and maps D conformally onto G.*

Proof. We must prove that

(a) each point $w_0 \in G$ has corresponding to it only one point $z_0 \in D$ such that $f(z_0) = w_0$, i.e. the function $f(z) - w_0$ has exactly one zero in D;

(b) for each point w_1 that does not belong to G the function $f(z)$ does not admit the value w_1 for $z \in D$.

Let us prove proposition (a). By hypothesis, the function $f(z) - w_0$

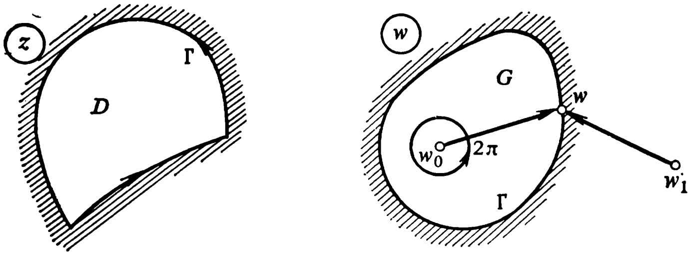


Fig. 81

does not vanish on Γ, since for $z \in \Gamma$ point $w = f(z)$ belongs to $\tilde{\Gamma}$, while $w_0 \in G$. According to the argument principle (Sec. 30), the number of zeros of $f(z) - w_0$ in D is

$$N = (1/2\pi)\, \Delta_\Gamma \arg [f(z) - w_0] = (1/2\pi)\, \Delta_{\tilde{\Gamma}} \arg (w - w_0).$$

Since point w_0 lies in the interior of the closed curve $\tilde{\Gamma}$ (Fig. 81), we can write $\Delta_{\tilde{\Gamma}} \arg (w - w_0) = 2\pi$, and $N = 1$.

Similarly, if point w_1 lies in the exterior of Γ, then $\Delta_{\tilde{\Gamma}} \arg (w - w_1) = 0$ (Fig. 81), and the equation $f(z) = w_1$ has not a single solution in D.

Remark 2. Theorems 1 and 2 are valid for domains in the extended complex plane with piecewise smooth boundaries: under a conformal mapping the boundary of a domain is transformed into the boundary of the image of the domain in a one-to-one manner and with preservation of sense (see Hurwitz and Courant [1]).

33.4 Riemann's mapping theorem The following theorem lies at the base of conformal mapping theory:

Theorem 3 (Riemann's mapping theorem) *Suppose D is a simply connected domain in the extended complex plane with at least two boundary points. Then*

(1) *there is a function* $w = f(z)$ *that maps D conformally onto the unit circle* $|w| < 1$;

(2) *this function is unique if*

$$f(z_0) = w_0, \quad \arg f'(z_0) = \alpha. \tag{33.1}$$

Here z_0 and w_0 are given points ($z_0 \in D$ and $|w_0| < 1$), and α is a given real number.

The following domains are exceptions:

(a) the entire extended complex plane;

(b) the entire extended complex plane with one point deleted.

These two domains cannot be mapped conformally onto the unit

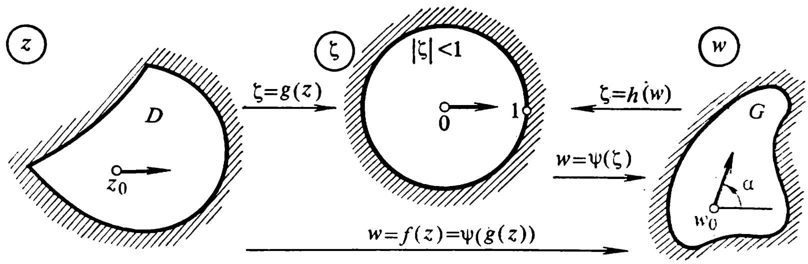


Fig. 82

circle $|w| < 1$. Indeed, suppose $w = f(z)$ maps the entire extended complex plane conformally onto the unit circle $|w| < 1$. Then this function is regular and bounded in the entire extended complex plane and, hence, is a constant, by Liouville's theorem (Sec. 19). Similarly, if $w = f(z)$ maps the entire extended complex plane with point z_0 deleted conformally onto the circle $|w| < 1$, then it is regular and bounded at $z \neq z_0$. This means that z_0 is a removable singular point for $f(z)$ (see Sec. 18), i.e. $f(z)$ is regular and bounded in the entire extended complex plane, or, by Liouville's theorem, a constant.

Note that if the boundary of a simply connected domain D contains two points, then it is a curve passing through these points. Theorem 3 states that such a domain can be mapped conformally onto a unit circle. The proof of Theorem 3 can be found in Hurwitz and Courant [1].

Theorem 3 has the following

Corollary *Suppose the boundaries of two simply connected domains D and G consist of at least two points. Then there is only one function $w = f(z)$ that maps D conformally onto G in a way such that*

$$f(z_0) = w_0, \quad \arg f'(z_0) = \alpha, \tag{33.2}$$

where $z_0 \in D$, $w_0 \in G$, and α is a real number.

Proof. Existence. By Theorem, 3, there is a function $\zeta = g(z)$ that maps D conformally onto the unit circle $|\zeta| < 1$ and is such that $g(z_0) = 0$ and $\arg g'(z_0) = 0$ (Fig. 82). Similarly, there is a function $\zeta = h(w)$ that maps G conformally onto the unit circle $|\zeta| < 1$ and is such that $h(w_0) = 0$ and $\arg h'(w_0) = -\alpha$. Then the function $w = \psi(\zeta)$, the inverse of $\zeta = h(w)$, maps the unit circle $|\zeta| < 1$ conformally onto G in a way such that $\psi(0) = w_0$ and $\arg \psi'(0) = \alpha$ (Fig. 82). Hence, the function $w = f(z) = \psi(g(z))$ maps D conformally onto G and satisfies conditions (33.2).

Uniqueness. Suppose there are two functions $w = f_j(z)$ $(j = 1, 2)$ that map D conformally onto G and are such that

$$f_j(z_0) = w_0, \quad \arg f_j'(z_0) = \alpha, \quad j = 1, 2.$$

Let us prove that $f_1(z) \equiv f_2(z)$ for $z \in D$.

By Theorem 3, there is only one function $\zeta = h(w)$ that maps domain G conformally onto the unit circle $|\zeta| < 1$ and is such that $h(w_0) = 0$ and $\arg h'(w_0) = 0$. The functions $\zeta = g_j(z) = h(f_j(z))$ $(j = 1, 2)$ map D conformally onto the unit circle $|\zeta| < 1$ and satisfy the following conditions:

$$g_j(z_0) = 0, \quad \arg g_j'(z_0) = \alpha, \quad j = 1, 2.$$

Hence, by Theorem 3, $g_1(z) \equiv g_2(z)$, i.e. $h(f_1(z)) \equiv h(f_2(z))$, whence $f_1(z) \equiv f_2(z)$.

Remark 3. Instead of the unit circle we can take another "standard" domain, e.g. the upper half-plane. In what follows we will usually consider mappings onto the unit circle or the upper half-plane.

Thus, if the boundaries of two simply connected domains D and G have at least two points each, there is a conformal mapping of D onto G, but the mapping is not unique. To make the mapping unique we must specify the conditions (33.2), i.e. *normalize* the conformal mapping. These *normalization conditions* contain three arbitrary, real parameters: u_0, v_0 $(w_0 = u_0 + iv_0)$, and α. Instead of (33.2) we can fix other conditions containing three independent real parameters. For instance:

(i) There is a unique function $w = f(z)$ that maps D conformally onto G and satisfies the conditions

$$f(z_0) = w_0, \quad f(z_1) = w_1,$$

where z_0 and w_0 are interior points of D and G, respectively, and z_1 and w_1 are boundary points of D and G.

(ii) There is a unique function $w = f(z)$ that maps D conformally onto G and satisfies the conditions

$$f(z_k) = w_k, \quad k = 1, 2, 3,$$

where z_1, z_2, and z_3 are three different boundary points of D, and w_1, w_2, and w_3 are three different boundary points of G, numbered in the order corresponding to the positive sense along the boundary curves of D and G.

When multiply connected domains are involved, the question of the existence of a conformal mapping is much more complex. Even for the simple doubly connected domains D: $\rho < |z| < R$

and G: $\rho_1 < |w| < R_1$ a conformal mapping of D onto G does not always exist (see Sec. 36). The theory of conformal mappings of n-connected domains is given in Hurwitz and Courant [1].

34 The Linear-Fractional Function

The function

$$w = \frac{az+b}{cz+d}, \quad ad - bc \neq 0, \tag{34.1}$$

where a, b, c, and d are complex numbers, is known as the *linear-fractional function*. The mapping, or transformation, performed by (34.1) is called the *linear-fractional* (*bilinear*, *Möbius*) *mapping*. The condition $ad - bc \neq 0$ implies that w cannot be a constant. We assume that if $c \neq 0$, then $w(\infty) = a/c$ and $w(-d/c) = \infty$, while if $c = 0$, then $w(\infty) = \infty$. Thus, the linear-fractional function is defined in the entire extended complex plane. If $c = 0$, the function (34.1) is the *linear function*, and the respective mapping is said to be linear.

Let us consider the main properties of the linear-fractional mapping.

34.1 Conformality

Theorem 1 *The linear-fractional function maps the extended complex plane conformally onto the extended complex plane.*

Proof. Obviously, the function (34.1) is regular in the entire extended complex plane except at the point $z = -d/c$, which is a first order pole.

Solving Eq. (34.1) for z, we arrive at the function

$$z = \frac{dw-b}{-cw+a}, \quad ad - bc \neq 0, \tag{34.2}$$

which is the inverse of (34.1). The function (34.2) is single-valued in the entire extended complex plane and is linear-fractional. Hence, the linear-fractional function is univalent in the extended complex plane.

Remark 1. The converse is also true, i.e. if a function $w = f(z)$ maps the extended complex plane conformally onto the extended complex plane, it is linear-fractional.

Indeed, by Definition 1 of Sec. 33, $f(z)$ is regular in the extended complex plane everywhere except at its simple pole. If point z_0 is finite and $\operatorname*{Res}_{z=z_0} f(z) = A$, then $g(z) = f(z) - A/(z - z_0)$ is regular in the entire extended complex plane. Liouville's theorem (Sec. 19) then states that $g(z)$ is a constant, i.e. $f(z)$ is a linear-fractional function. If $z_0 = \infty$, then $f(z)$ is an entire function and $f(z) = 0(z)$

$(z \to \infty)$ (Sec. 19). Then Liouville's theorem states that $f(z) = az + b$.

34.2 The group property

Theorem 2 *The linear-fractional mappings constitute a group, i.e.*

(1) *a linear-fractional transformation of a linear-fractional transformation is a linear-fractional transformation;*

(2) *the inverse of a linear-fractional transformation is also a linear-fractional transformation.*

Proof. Property 2 has been proved in Sec. 34.1. Let us prove Property 1. Suppose

$$\zeta = \frac{a_1 z + b_1}{c_1 z + d_1}, \quad a_1 d_1 - b_1 c_1 \neq 0, \tag{34.3}$$

$$w = \frac{a_2 \zeta + b_2}{c_2 \zeta + d_2}, \quad a_2 d_2 - b_2 c_2 \neq 0. \tag{34.4}$$

Substituting (34.3) into (34.4), we obtain

$$w = \frac{az + b}{cz + d}, \tag{34.5}$$

where $ad - bc = (a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) \neq 0$, i.e. the mapping (34.5) is linear-fractional.

Remark 2. The group of linear-fractional mappings is noncommutative. For instance, if $w(z) = 1/z$ and $\zeta(z) = z + 1$, then

$$w(\zeta(z)) = \frac{1}{z+1}, \quad \zeta(w(z)) = \frac{1}{z} + 1, \quad w(\zeta(z)) \neq \zeta(w(z)).$$

34.3 The circular property

Theorem 3 *The image of a straight line or a circle under the linear-fractional mapping is a straight line or a circle.*

Proof. We start with the linear mapping $w = az + b$ $(a \neq 0)$. This mapping is reduced to stretching (or contraction), rotation, and translation (see Sec. 8). Hence, the linear mapping transforms circles into circles and straight lines into straight lines.

When the linear-fractional function $w = \frac{az+b}{cz+d}$ is not linear $(c \neq 0)$, we can write

$$w = A + \frac{B}{z + z_0}, \tag{34.6}$$

where $A = a/c$, $B = (bc - ad)/c^2$, and $z_0 = d/c$. Then the mapping (34.6) is reduced to the following operations (performed in the specified order):

$$\zeta = z + z_0, \quad \eta = \frac{1}{\zeta}, \quad w = A + B\eta. \tag{34.7}$$

The first and third mappings in (34.7) possess the circular property since they are linear. We need only show that the second mapping possesses it, too, i.e.

$$w = \frac{1}{z} \tag{34.8}$$

has the circular property.

The equation of any circle or a straight line in the complex z plane has the form

$$\alpha (x^2 + y^2) + \beta x + \gamma y + \delta = 0 \tag{34.9}$$

(if $\alpha = 0$, then (34.9) is the equation of a straight line). Since

$$x^2 + y^2 = |z|^2 = z\bar{z}, \quad x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z}),$$

we can write Eq. (34.9) in the form

$$\alpha z\bar{z} + Dz + \bar{D}\bar{z} + \delta = 0, \tag{34.10}$$

where $D = \frac{1}{2}(\beta - i\gamma)$. Substitution of $1/w$ for z in (34.10) yields

$$\delta w\bar{w} + \bar{D}w + D\bar{w} + \alpha = 0. \tag{34.11}$$

Hence, the image of the circle (34.10) (or a straight line if $\alpha = 0$) obtained as a result of mapping (34.8) is the circle (34.11) (or a straight line if $\delta = 0$).

Note that the linear-fractional function $w = \frac{az+b}{cz+d}$ maps circles and straight lines that pass through the point $z = -d/c$ into straight lines, and other circles and straight lines into circles.

In what follows we will assume that a straight line is a circle with an infinitely large radius. This gives us another way in which to formulate the circular property, namely, under a linear-fractional mapping, circles are mapped into circles.

34.4 Preservation of symmetry Elementary geometry gives the following definition for the concept of symmetry (inversion) with respect to a circle. Suppose Γ is a circle of radius R centered at point O.

Definition. Points M and M^* are said to be *symmetric with respect to circle* Γ if they lie on a single ray that starts at point O and if $OM \times OM^* = R^2$ (Fig. 83).

For instance, every point on Γ is symmetric to itself with respect to Γ.

Thus, points z and z^* of the complex z plane are symmetric with respect to the circle Γ: $|z - a| = R$ if they lie on a single ray that starts at point a and if $|z - a|\,|z^* - a| = R^2$. Point $z = \infty$ is assumed to be symmetric to point a, the center of Γ.

This definition implies that the points z and z^* symmetric with respect to the circle $|z| = R$ are related through the formula

$$z^* = \frac{R^2}{\bar{z}}. \tag{34.12}$$

For instance, the points z and z^* symmetric with respect to the unit circle $|z| = 1$ (Fig. 84) are related through the formula

$$z^* = \frac{1}{\bar{z}}. \tag{34.13}$$

Since points z and $\bar{z}$ are symmetric with respect to the real axis, from (34.13) it follows that point $1/z$ is obtained from point z by

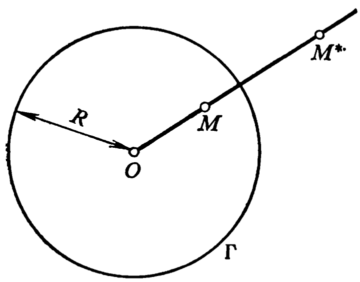


Fig. 83

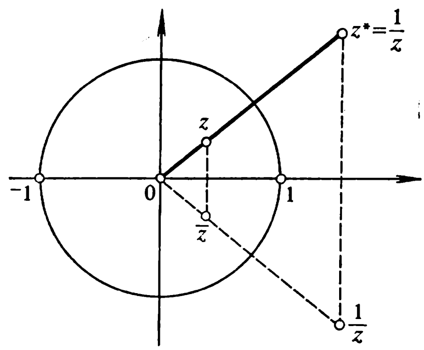


Fig. 84

two symmetry transformations: one with respect to the real axis and the other with respect to the unit circle $|z| = 1$ (in any order) (Fig. 84).

From (34.12) it follows that points z and z^* symmetric with respect to the circle $|z - a| = R$ are related through the formula

$$z^* = a + \frac{R^2}{\bar{z} - \bar{a}}. \tag{34.14}$$

All linear-fractional mappings possess the following property of symmetry preservation:

Theorem 4 *A pair of points symmetric with respect to a circle is mapped by a linear-fractional transformation into a pair of points symmetric with respect to the image of the circle.*

Here a circle may be a straight line.

First let us prove the following

Lemma *Points M and M^* are symmetric with respect to a circle Γ if and only if any circle γ that passes through these points intersects Γ at a right angle.*

Proof of lemma. Necessity. Suppose points M and M^* are symmetric with respect to a circle Γ of radius R centered at point O (Fig. 85). Consider circle γ that passes through points M and M^*. Through point O we draw a straight line that touches circle γ at point P.

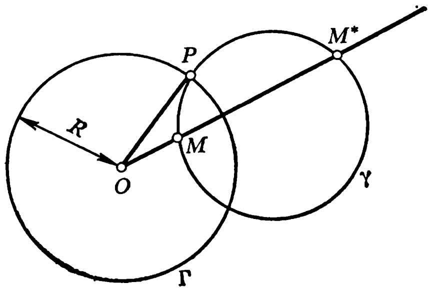


Fig. 85

According to a theorem of elementary geometry (the square of a tangent is equal to the product of a secant by its external part), we have $OP^2 = OM \times OM^*$. This product is equal to R^2, since points M and M^* are symmetric with respect to Γ. Then $OP = R$, i.e. point P lies on Γ. Thus, a tangent to γ is a radius of Γ and, hence γ and Γ intersect at a right angle at point P.

Sufficiency. Suppose any circle γ that passes through points M and M^* intersects circle Γ at a right angle (Fig. 85). Then the straight line (the particular case of a circle) that passes through M and M^* also intersects Γ at a right angle, i.e. it passes through the center O of Γ. Moreover, points M and M^* lie on a single ray that starts at point O, since otherwise the circle of radius $MM^*/2$, which pass through M and M^*, would not intersect Γ at a right angle.

It now remains to be proved that $OM \times OM^* = R^2$. Suppose the circle γ, which passes through points M and M^*, intersects Γ at point P (Fig. 85). Then OP is a tangent to γ and, hence, $OP^2 = OM \times OM^*$, by the theorem on the square of a tangent (see the proof of necessity).

Proof of Theorem 4. Suppose points z and z^* are symmetric with respect to the circle Γ. Let the linear-fractional function $w = f(z)$ map Γ into $\widetilde{\Gamma}$ and points z and z^* into points w and w^*, respectively. By the circular property of the linear-fractional mapping, $\widetilde{\Gamma}$ is a circle. We must prove that points w and w^* are symmetric with respect to $\widetilde{\Gamma}$. In view of the lemma, it is sufficient to prove that any circle $\widetilde{\gamma}$ that passes through points w and w^* intersects $\widetilde{\Gamma}$ at a right angle.

The circle γ passing through points z and z^* is the preimage of $\widetilde{\gamma}$

in the linear-fractional mapping $w = f(z)$. It intersects Γ at a straight angle. Hence, $\tilde{\gamma}$ intersects $\tilde{\Gamma}$ at a straight angle, too, since the linear-fractional mapping is conformal in the entire extended complex plane and, therefore, preserves the angles between curves at every point.

34.5 The linear-fractional transfoımation that maps three points into three points.

Theorem 5 *There is only one linear- ractional transformation that maps three given points z_1, z_2, and z_3 into three points w_1, w_2, and w_3, respectively. This transformation is given by the formula*

$$\frac{w-w_1}{w-w_2} \times \frac{w_3-w_2}{w_3-w_1} = \frac{z-z_1}{z-z_2} \times \frac{z_3-z_2}{z_3-z_1}. \tag{34.15}$$

Proof. Theorem 2 implies that the function $w = f(z)$ given by (34.15) is linear-fractional. It is also clear that $w_k = f(z_k)$ $(k = 1, 2, 3)$.

Let us prove that if the linear-fractional function $w = f_1(z)$ satisfies the same conditions as $w = f(z)$, namely, $w_k = f_1(z_k)$ $(k = 1, 2, 3)$, then $f_1(z) \equiv f(z)$. Let $z = \psi(w)$ be the inverse of $w = f(z)$. Then $\psi(f_1(z))$ is linear-fractional:

$$\psi(f_1(z)) = \frac{az+b}{cz+d}$$

and $\psi(f_1(z_k)) = z_k$, i.e.

$$\frac{az_k+b}{cz_k+d} = z_k, \quad k = 1, 2, 3.$$

This implies that

$$cz_k^2 + (d-a) z_k - b = 0,$$

i.e. the quadratic equation $cz^2 + (d-a) z - b = 0$ has three different roots. Hence, $c = 0$, $d = a$, $b = 0$, and $\psi(f_1(z)) \equiv z$, whence $f_1(z) \equiv f(z)$.

Corollary 1 *The function $w = f(z)$ given by* (34.15) *maps the circular domain whose boundary passes through the points z_k $(k = 1, 2, 3)$ conformally onto the circular domain whose boundary passes through the points w_k $(k = 1, 2, 3)$.*

Here and in what follows, a circular domain means the interior of a circle or its exterior or a half-plane.

Remark 3. Theorem 5 implies that a linear-fractional transformation $w = w(z)$ can have no more than two fixed points z_1 and z_2, i.e. such points that $w(z_k) = z_k$ $(k = 1, 2)$ if $w(z) \not\equiv z$. A linear-fractional transformation with the points z_1 and z_2 fixed is given by

$$\frac{w-z_1}{w-z_2} = A\,\frac{z-z_1}{z-z_2},$$

where A is a complex number.

Example 1. The linear-fractional transformation that maps point

z_1 into point $w=0$ and point z_2 into point $w=\infty$ has the form

$$w = A\frac{z-z_1}{z-z_2}, \tag{34.16}$$

where A is a complex number. □

34.6 Examples of linear-fractional transformations

Example 2. The linear-fractional transformation that maps the half-plane $\operatorname{Im} z > 0$ onto the unit circle $|w| < 1$ has the form

$$w = \frac{z-z_0}{z-\bar{z}_0}e^{i\alpha}, \tag{34.17}$$

where $\operatorname{Im} z_0$ is positive and α is a real number.

Proof. Suppose the linear-fractional function $w = w(z)$ maps the half-plane $\operatorname{Im} z > 0$ onto the unit circle $|w| < 1$ in a way such that $w(z_0) = 0$ ($\operatorname{Im} z_0 > 0$). Then, in view of the property of symmetry preservation, $w(\bar{z}_0) = \infty$ and, by (34.16),

$$w = A\frac{z-z_0}{z-\bar{z}_0}. \tag{34.18}$$

Let us show that $|A| = 1$. Since points on the real axis are mapped into points on the boundary of the unit circle, i.e. $|w| = 1$ for real $z = x$, from (34.18) we have

$$1 = \left| A\frac{x-z_0}{x-\bar{z}_0}\right| = |A|\frac{|x-z_0|}{|x-\bar{z}_0|} = |A|$$

($|x-z_0| = |x-\bar{z}_0|$). Hence $A = e^{i\alpha}$ and (34.18) is simply (34.17). □

Remark 4. Under the mapping (34.17), the angle of rotation of curves at point z_0 is $\alpha - \pi/2$ (Fig. 86), since (34.17) yields $\arg w'(z_0) = \alpha - \pi/2$ (see Example 5 in Sec. 8).

Remark 5. Every conformal mapping of the half-plane $\operatorname{Im} z > 0$ onto the unit circle $|w| < 1$ has the form (34.17).

Indeed, by Riemann's mapping theorem (Sec. 33), there is only one conformal mapping $w = w(z)$ that transforms the half-plane $\operatorname{Im} z > 0$ onto the circle $|w| < 1$ and satisfies the conditions $w(z_0) = 0$ and $\arg w'(z_0) = \alpha - \pi/2$. Hence, this mapping must be (34.17).

The same is true of formulas (34.19) and (34.21) below.

Example 3. The linear-fractional function that maps the circle $|z| < 1$ onto the circle $|w| < 1$ has the form

$$w = \frac{z-z_0}{1-z\bar{z}_0}e^{i\alpha}, \tag{34.19}$$

where $|z_0| < 1$ and α is a real number.

Proof. Suppose the linear-fractional function $w = w(z)$ maps $|z| < 1$ onto $|w| < 1$ in a way such that $w(z_0) = 0$ ($|z_0| < 1$).

Then, by the property of symmetry preservation (Sec. 34.4), $w(1/\bar{z}_0) = \infty$ and from (34.16) we obtain

$$w = A\frac{z-z_0}{1-z\bar{z}_0} \tag{34.20}$$

Let us show that $|A| = 1$. Since all the points on the boundary of the unit circle are mapped into points on the boundary of the

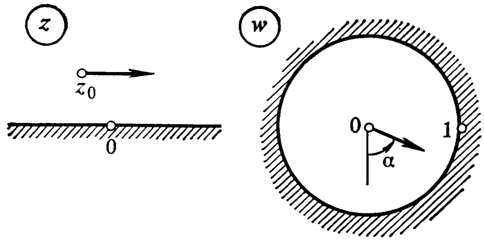


Fig. 86

other unit circle, i.e. $|w| = 1$ for $z = e^{i\varphi}$, from (34.20) we obtain

$$1 = \left| A\frac{e^{i\varphi}-z_0}{1-e^{i\varphi}\bar{z}_0} \right| = |A| \frac{|e^{i\varphi}-z_0|}{|e^{i\varphi}|\,|e^{-i\varphi}-\bar{z}_0|} = |A|,$$

$$(|e^{i\varphi}-z_0| = |\overline{e^{i\varphi}-z_0}| = |e^{-i\varphi}-\bar{z}_0|).$$

Hence, $A = e^{i\alpha}$ and from (34.20) we obtain (34.19). □

Remark 6. Under the mapping (34.19), the angle of rotation of curves at point z_0 is α (Fig. 87), since (34.19) yields $\arg w'(z_0) = \alpha$ (Example 5 in Sec. 8).

Example 4. The linear-fractional mapping of the half-plane $\operatorname{Im} z > 0$ into the half-plane $\operatorname{Im} w > 0$ has the form

$$w = \frac{az+b}{cz+d}, \tag{34.21}$$

where a, b, c, and d are real numbers, and $ad - bc > 0$.

Proof. Suppose the linear-fractional function $w = w(z)$ maps the half-plane $\operatorname{Im} z > 0$ onto the half-plane $\operatorname{Im} w > 0$. Take three different points z_1, z_2, z_3 on the boundary of $\operatorname{Im} z > 0$, i.e. the z_k are different real numbers. The images of these points are boundary points of $\operatorname{Im} w > 0$, i.e. the $w_k = w(z_k)$ are three real numbers. Then the function $w = w(z)$ is defined by (34.15), from which we obtain (34.21), where a, b, c, and d are real numbers.

Let us show that $ad - bc > 0$. In view of the principle of correspondence of boundaries (Sec. 33), the conformal mapping $w = w(z)$ transforms the real axis $\operatorname{Im} z = 0$ into the real axis $\operatorname{Im} w = 0$ with preservation of sense. Hence, for real $z = x$ we have arg $w'(x) > 0$, i.e.

$$w'(x) = \frac{ad - bc}{(cx+d)^2} > 0.$$

whence $ad - bc > 0$. □

Example 5. The conformal mapping $w = w(z)$ of the circle $|z| < 1$

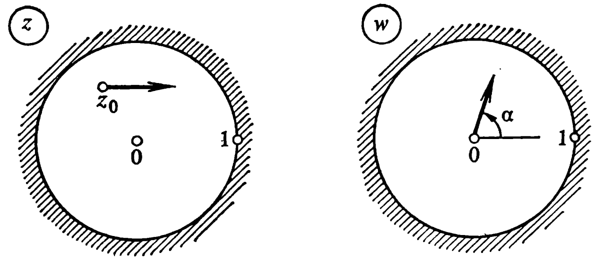


Fig. 87

onto the circle $|w| < 1$ that satisfies the conditions $w(z_0) = w_0$ and $\arg w'(z_0) = \alpha$ is given by the formula

$$\frac{w - w_0}{1 - w\bar{w}_0} = \frac{z - z_0}{1 - z\bar{z}_0} e^{i\alpha}. \qquad (34.22)$$

Proof. The function

$$\zeta = g(z) = \frac{z - z_0}{1 - z\bar{z}_0} e^{i\alpha}$$

maps the circle $|z| < 1$ onto the circle $|\zeta| < 1$ in a way such that $g(z_0) = 0$ and $\arg g'(z_0) = \alpha$ (see Example 3). The function

$$\zeta = h(w) = \frac{w - w_0}{1 - w\bar{w}_0}$$

maps the circle $|w| < 1$ onto the same circle $|\zeta| < 1$ in a way such that $h(w_0) = 0$ and $\arg h'(w_0) = 0$ (Example 3). Hence, the function $w = w(z)$ given by (34.22) maps the circle $|z| < 1$ onto the circle $|w| < 1$ in a way such that $w(z_0) = w_0$ and $\arg w'(z_0) = \alpha$. □

Example 6. The conformal mapping $w = w(z)$ of the half-plane $\operatorname{Im} z > 0$ onto the half-plane $\operatorname{Im} w > 0$ that satisfies the conditions $w(z_0) = w_0$ and $\arg w'(z_0) = \alpha$ is defined thus:

$$\frac{w - w_0}{w - \bar{w}_0} = \frac{z - z_0}{z - \bar{z}_0} e^{i\alpha}.$$

The proof of the proposition is similar to the proof of formula (34.22). □

35 Conformal Mapping Performed by Elementary Functions

35.1 The function $w = z^2$ Let us study the properties of the function $w = z^2$ (some were discussed in Sec. 8).

(1) *Univalence.* The reader will recall that the function $w = z^2$ is univalent in a domain D if and only if there are no two different points z_1 and z_2 in D related through the formula

$$z_1 = -z_2. \tag{35.1}$$

Condition (35.1) implies that points z_1 and z_2 are symmetric with respect to point $z = 0$. Thus, the function $w = z^2$ is univalent in

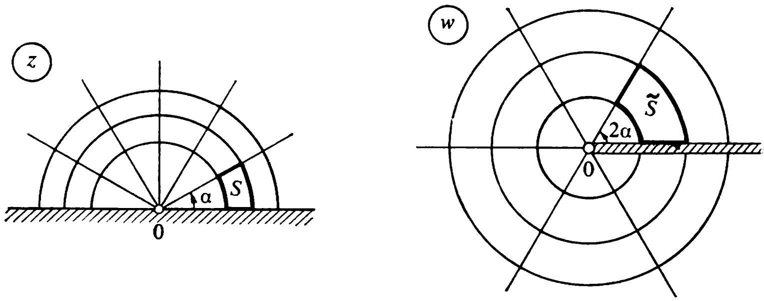


Fig. 88

a domain D if and only if this domain has not a single pair of points that are symmetric with respect to point $z = 0$. For instance, the function $w = z^2$ is univalent in the half-plane whose boundary passes through the point $z = 0$.

Example 1. (a) The function $w = z^2$ maps the upper half-plane conformally onto the complex w plane with a cut along the ray $[0, +\infty)$, domain G (see Fig. 35, Sec. 8).

(b) The function $w = z^2$ maps the lower half-plane conformally onto the same domain G (see Fig. 36, Sec. 8). □

Let us study the mapping of the coordinate chart performed by the function $w = z^2$ for the polar and Cartesian systems of coordinates.

(2) *The images of rays* $\arg z = \alpha$ *and arcs of circles* $|z| = \rho$. The lines $\arg z = \text{const}$ and $|z| = \text{const}$ constitute a coordinate chart in the complex z plane (polar coordinates). In Sec. 8 it was shown that the function $w = z^2$ maps (a) the ray $\arg z = \alpha$ into the ray $\arg w = 2\alpha$, and (b) the arc $|z| = \rho, \alpha \leqslant \arg z \leqslant \beta$, with $\beta - \alpha < \pi$, into the arc $|w| = \rho^2, 2\alpha \leqslant \arg w \leqslant 2\beta$ in a one-to-one manner.

Example 2. From Properties 1 and 2 it follows that the function $w = z^2$ maps the annular sector S: $\rho_1 < |z| < \rho_2$, $0 < \arg z < \alpha \leqslant \pi$, with $0 \leqslant \rho_1 < \rho_2 \leqslant +\infty$, conformally onto the annular sector $\tilde{S}$: $\rho_1^2 < |w| < \rho_2^2$, $0 < \arg w < 2\alpha$ (Fig. 88). □

(3) *The images of straight lines* $\operatorname{Re} z = c$, $\operatorname{Im} z = c$. Let us show that the function $w = z^2$ maps (a) the straight line $\operatorname{Re} z = c$ into the parabola

$$v^2 = 2p\left(\frac{p}{2} - u\right), \tag{35.2}$$

and (b) the straight line $\operatorname{Im} z = c$ into the parabola

$$v^2 = 2p\left(u + \frac{p}{2}\right) \tag{35.3}$$

in a one-to-one manner. Here $p = 2c^2$ and $w = u + iv$.

Indeed,

$$w = u + iv = z^2 = (x + iy)^2 = x^2 - y^2 + 2xyi,$$

i.e. $u = x^2 - y^2$ and $v = 2xy$. If $\operatorname{Re} z = x = c$, $-\infty < y < +\infty$, then

$$u = c^2 - y^2, \quad v = 2cy,$$

from which formula (35.2) follows. Similarly, for $\operatorname{Im} z = y = c$ we have (35.3).

If $c = 0$, then $p = 0$, and the parabola (35.2) degenerates into the ray $(-\infty, 0]$ traversed twice, i.e. the straight line $\operatorname{Re} z = 0$ is mapped into the ray $(-\infty, 0]$ traversed twice (Fig. 37, Sec. 8). Similarly, the straight line $\operatorname{Im} z = 0$ is mapped into the ray $[0, +\infty)$ traversed twice (Fig. 88).

Note that any parabola of the form (35.2) intersects any parabola of the form (35.3) at a right angle, in view of the fact that angles are

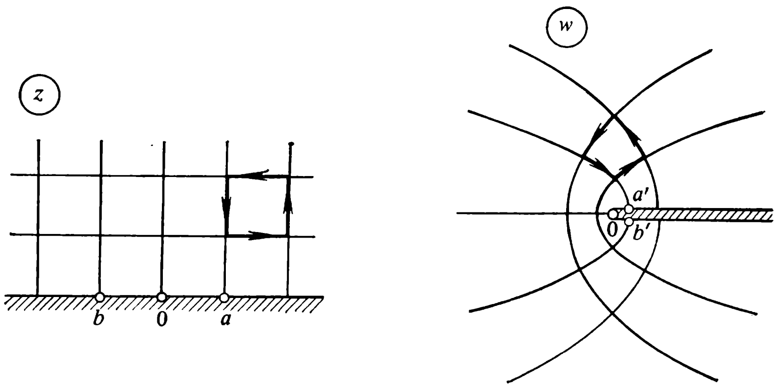


Fig. 89

preserved in conformal mappings. The foci of the parabolas (35.2) and (35.3) lie at the same point $w = 0$.

Example 3. Properties 1 and 3 imply that the function $w = z^2$ maps the rectangle depicted in Fig. 89 onto the curvilinear quadrangle bounded by arcs of the parabolas (35.2) and (35.3). □

35.2 The function $w = \sqrt{z}$ The properties of the function $w = \sqrt{z}$, which is the inverse of $w = z^2$, were discussed in Secs. 13 and 22. The reader will recall that $\sqrt{z}$ is analytic in the complex z

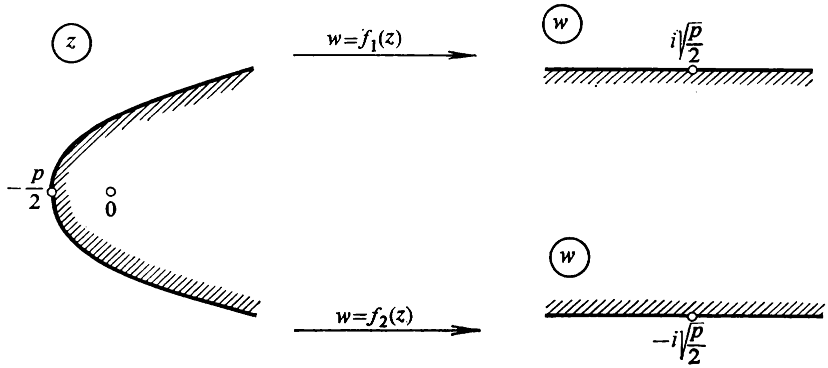


Fig. 90

plane with points $z = 0$ and $z = \infty$ deleted, while in the complex z plane with a cut connecting points 0 and ∞ it splits into two regular branches.

Example 4. Let D be the complex z plane with a cut along the ray $[0, +\infty)$ (Fig. 47, Sec. 13). In this domain the function $\sqrt{z}$ splits into two regular branches, $f_1(z)$ and $f_2(z) = -f_1(z)$, where $f_1(x + i0) = \sqrt{x} > 0$ for positive x's, i.e. the function $f_1(z)$ assumes positive values on the upper bank of the cut. The function $w = f_1(z)$ maps D conformally onto the upper half-plane $\operatorname{Im} w > 0$, while the function $w = f_2(z)$ maps D conformally onto the lower half-plane $\operatorname{Im} w < 0$ (Fig. 47, Sec. 13). □

Example 5. Let D be the complex z plane with a cut along the ray $(-\infty, 0]$ (Fig. 48, Sec. 13). In this domain the function $\sqrt{z}$ splits into two regular branches, $f_1(z)$ and $f_2(z) = -f_1(z)$, where $f_1(1) = 1$. The function $w = f_1(z)$ maps D conformally onto the half-plane $\operatorname{Re} z > 0$, while the function $w = f_2(z)$ maps D conformally onto the half-plane $\operatorname{Re} z < 0$ (Fig. 48, Sec. 13). □

Example 6. Let D be the exterior of the parabola $y^2 = 2p(x + p/2)$ ($p > 0$ and $z = x + iy$), i.e. the domain $y^2 > 2p(x + p/2)$ (Fig. 90). In this domain, $\sqrt{z}$ splits into two regular branches, $f_1(z)$ and $f_2(z) = -f_1(z)$, where $f_1(-p/2) = i\sqrt{p/2}$. Example 3 (Fig. 89) implies that $w = f_1(z)$ maps D conformally onto the half-plane $\operatorname{Im} w > \sqrt{p/2}$, while $w = f_2(z)$ maps D conformally onto the half-plane $\operatorname{Im} w < -\sqrt{p/2}$ (Fig. 90). □

Example 7. Let D be the half-plane $\operatorname{Im} z > 0$ with a cut along the segment $[0, ih]$ $(h > 0)$ (Fig. 91). Let us find the conformal mapping of D onto the upper half-plane $\operatorname{Im} w > 0$.

(a) The function $\zeta = z^2$ maps D conformally onto the complex ζ plane with a cut along the ray $[-h^2, +\infty)$ (Fig. 91), domain D_1;

(b) the function $\eta = \zeta + h^2$ (a translation) maps D_1 conformally onto the complex η plane with a cut along the ray $[0, +\infty)$ (Fig. 91), domain D_2;

(c) the function $w = \sqrt{\eta}$ (precisely, its regular branch in D_2 that assumes positive values on the upper bank of the cut along $[0, +\infty)$) maps D_2 conformally onto the half-plane $\operatorname{Im} w > 0$ (Example 4).

Hence, the combination of mappings (*a*)-(*c*), i.e. the function $w = \sqrt{z^2 + h^2}$, maps D conformally onto the half-plane $\operatorname{Im} w > 0$ (Fig. 91). □

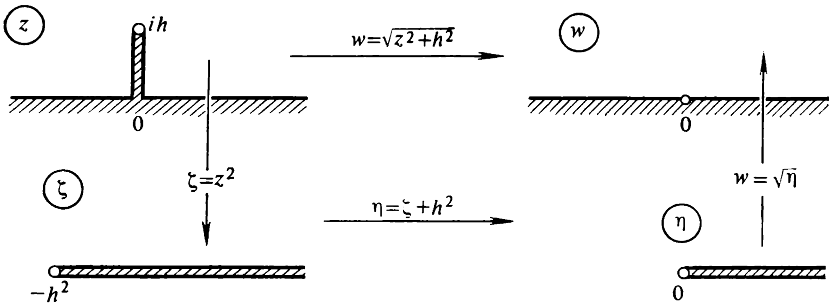


Fig. 91

35.3. The function $w = z^\alpha$ The properties of the power function z^α were discussed in Sec. 22. There we considered the following example:

Example 8. The function $w = z^\alpha$, $\alpha > 0$, maps the sector $0 < \arg z < \beta \leqslant 2\pi$, with $\beta \leqslant 2\pi/\alpha$, conformally onto the sector $0 < \arg w < \alpha\beta$ (Fig. 61, Sec. 22). □

Remark 1. In Example 8 and in what follows the symbol z^α stands for the following function:

$$z^\alpha = |z|^\alpha e^{i\alpha \arg z}, \tag{35.4}$$

defined in the sector $0 < \arg z < 2\pi$.

We note the following particular case of Example 8.

Example 9. Suppose S is the sector $0 < \arg z < \beta \leqslant 2\pi$. Then the function $w = z^{\pi/\beta}$ maps S conformally onto the upper half-plane $\operatorname{Im} w > 0$. □

Let us consider a domain D that is bounded by two arcs of circles that intersect at points a and b at an angle α (Fig. 92). This domain is called a *lune*. We will show that the lune D can be mapped conformally onto the upper half-plane via a linear-fractional and a power function.

Let us apply the linear-fractional transformation $\zeta = \frac{z-a}{z-b}$, with $\zeta(a) = 0$ and $\zeta(b) = \infty$. This transformation maps the arcs bounding D into arcs intersecting at point $\zeta = 0$ at an angle α (see Sec. 34). Hence, the lune D is transformed into the sector $\beta < \arg \zeta < \beta + \alpha$, where β is a real number.

The rotation $\eta = \zeta e^{-i\beta}$ maps this sector into a new sector, $0 < \arg \eta < \alpha$, which the function $w = \eta^{\pi/\alpha}$ maps onto the half-plane $\operatorname{Im} w > 0$ (Example 9). Thus, the function

$$w = \left(\frac{z-a}{z-b}\, e^{-i\beta}\right)^{\pi/\alpha}$$

maps the lune D conformally onto the upper half-plane. □

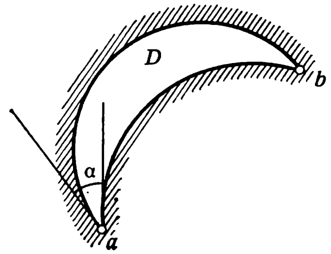


Fig. 92

Example 10. The following functions perform conformal mappings of the domains depicted in Fig. 93 onto the upper half-plane:

(a) $w = \left(\frac{z}{1-z}\right)^{4/3}$, (b) $w = \left(\frac{1-z}{1+z}\right)^{2/3}$, (c) $w = \left(\frac{z-1}{z+1}\right)^{2/3}$,

(d) $w = \sqrt{\frac{z}{1-z}}$, (e) $w = \sqrt{i\,\frac{z+1}{z-1}}$, (*f*) $w = \sqrt{\frac{z-2}{z-1}}$. □

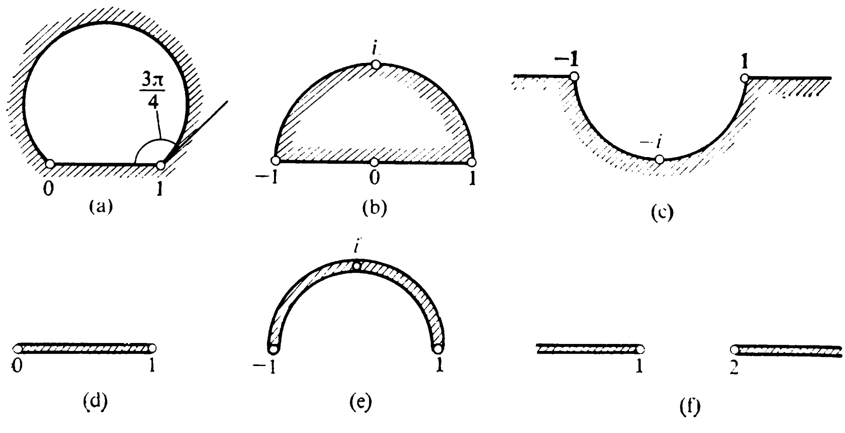


Fig. 93

35.4 The function $w = e^z$ Some of the properties of the function $w = e^z$ were discussed in Sec. 8. Let us recall these properties.

(1) *Univalence.* The function $w = e^z$ is univalent in a domain D if and only if this domain has not a single pair of points z_1 and z_2 that obey the relationship

$$z_1 - z_2 = 2k\pi i, \quad k = \pm 1, \pm 2, \ldots . \tag{35.5}$$

For instance, $w = e^z$ is univalent in the strip $0 < \operatorname{Im} z < 2\pi$ and maps this strip conformally onto the complex w plane with a cut along the ray $[0, +\infty)$ (Fig. 38, Sec. 8).

Let us study the mapping of the coordinate chart $\operatorname{Re} z = \text{const}$, $\operatorname{Im} z = \text{const}$ performed by $w = e^z$.

(2) *The images of straight lines* $\operatorname{Re} z = c$, $\operatorname{Im} z = c$. In Sec. 8 it was found that $w = e^z$ maps (a) the segment $\operatorname{Re} z = c$, $a \leqslant \operatorname{Im} z \leqslant b$, $b - a < 2\pi$, into the arc $|w| = e^c$, $a \leqslant \arg w \leqslant b$, and (b) the straight line $\operatorname{Im} z = c$ into the ray $\arg w = c$ in a one-to-one manner.

Example 11. Properties 1 and 2 imply that the function $w = e^z$ maps the rectangle $c_1 < \operatorname{Re} z < c_2$, $a < \operatorname{Im} z < b$, with $-\infty \leqslant c_1 < c_2 \leqslant +\infty$ and $b - a \leqslant 2\pi$, conformally onto the annular sector $e^{c_1} < |w| < e^{c_2}$, $a < \arg w < b$. Some particular cases of such mappings are shown in Fig. 94. □

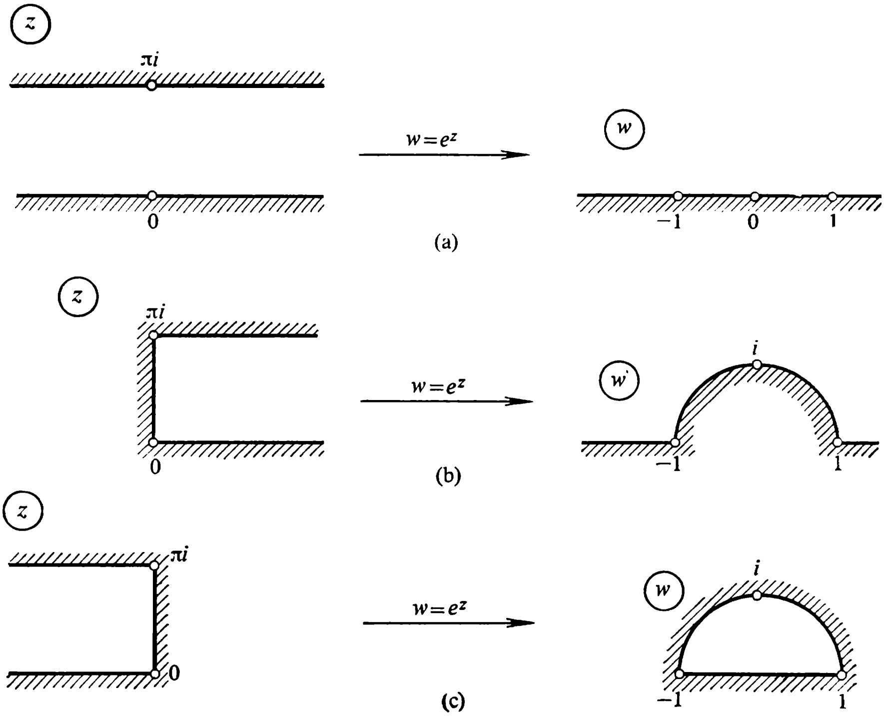


Fig. 94

35.5 **The function** $w = \ln z$ The properties of the function $w = \ln z$, which is the inverse of $w = e^z$, were discussed in Secs. 13 and 21. The reader will recall that $w = \ln z$ is analytic in the complex z plane with points $z = 0, \infty$ deleted, while in the complex z plane with a cut from point $z = 0$ to point $z = \infty$ it splits into an infinite number of regular branches.

Here are two examples of conformal mappings performed by $w = \ln z$ and discussed in Secs. 13 and 21.

Example 12. Suppose D is the complex z plane with a cut along the ray $[0, +\infty)$ (Fig. 49, Sec. 13). In this domain the function $\ln z$ splits into the regular branches

$$(\ln z)_k = \ln |z| + i (\arg z)_0 + 2k\pi i, \quad k = 0, \pm 1, \pm 2, \ldots,$$

where $0 < (\arg z)_0 < 2\pi$. The function $w = (\ln z)_k$ maps D conformally onto the strip $2k\pi < \operatorname{Im} w < 2(k+1)\pi$ (Fig. 49). □

Example 13. The function $w = \ln z$ maps the sector $0 < \arg z < \alpha \leqslant 2\pi$ conformally onto the strip $0 < \operatorname{Im} w < \alpha$ (Fig. 60, Sec. 21). Here $\ln z = \ln |z| + i \arg z$, $0 < \arg z < \alpha$. □

35.6 The Zhukovskii (Joukowski) function Consider the function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{35.6}$$

This function is known as the *Zhukovskii* (or *Joukowski*) *function*, named after N. E. Zhukovskii, a Russian applied mathematician and aerodynamicist who was the first to use it widely in aerodynamics. This function is regular at all points except $z = 0$ and $z = \infty$, with $w'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$. These two points are first order poles for the Zhukovskii function. Hence, function (35.6) is univalen at all points except ± 1, since $w'(z) \neq 0$ for $z \neq \pm 1$, and is not univalent at $z = \pm 1$, since $w'(\pm 1) = 0$ (see Sec. 32.2). Let us prove the following property.

(1) *Univalence.* The Zhukovskii function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ is univalent in a domain D if and only if this domain does not contain a pair of different points z_1 and z_2 that obey the following relationship:

$$z_1 z_2 = 1. \tag{35.7}$$

Indeed, suppose $\frac{1}{2}\left(z_1 + \frac{1}{z_1}\right) = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)$. Then $(z_1 - z_2) \left(1 - \frac{1}{z_1 z_2}\right) = 0$, from which it follows that either $z_1 = z_2$ or $z_1 z_2 = 1$.

Geometrically, Eq. (35.7) means that point $z_2 = 1/z_1$ is obtained from point z_1 by a double symmetry transformation: with respect to the circle $|z| = 1$ and with respect to the straight line $\operatorname{Im} z = 0$ (Fig. 84, Sec. 34). Thus, the Zhukovskii function is univalent in a domain if and only if the domain contains not a single pair of different points in which one of the points can be obtained from the other by the above-mentioned symmetry transformation: with respect to the unit circle and with respect to the real axis.

Example 14. The Zhukovskii function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ is univalent in the following domains:

(a) $|z| > 1$, the exterior of the unit circle;
(b) $|z| < 1$, the unit circle;
(c) $\operatorname{Im} z > 0$, the upper half-plane;
(d) $\operatorname{Im} z < 0$, the lower half-plane. □

Remark 2. Suppose $\tilde{D}$ is the domain whose points are $1/z$, $z \in D$. Then the Zhukovskii function is univalent in D if and only if D and $\tilde{D}$ have no common points. The images of D and $\tilde{D}$ obtained as a result of the mapping performed by the Zhukovskii function coincide, since $w(z) = w(1/z)$.

(2) *The images of circles and rays.* Let us find the images of circles $|z| = \rho$ and rays $\arg z = \alpha$ (the polar coordinate chart) obtained as a result of mapping performed by the Zhukovskii function. Putting $z = re^{i\varphi}$ and $w = u + iv$ in (35.6), we obtain $u + iv = \frac{1}{2}\left(re^{i\varphi} + \frac{1}{r}e^{-i\varphi}\right)$, whence

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\varphi, \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\varphi. \tag{35.8}$$

Take the circle

$$z = \rho e^{i\varphi}, \quad 0 \leqslant \varphi \leqslant 2\pi \tag{35.9}$$

(ρ is positive and fixed). From (35.8) it follows that the Zhukovskii function maps (35.9) into the ellipse

$$u = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\varphi, \quad v = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\varphi, \quad 0 \leqslant \varphi \leqslant 2\pi, \tag{35.10}$$

with the semiaxes $a_\rho = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)$ and $b_\rho = \frac{1}{2}\left|\rho - \frac{1}{\rho}\right|$ and the foci at $w = \pm 1$, since $a_\rho^2 - b_\rho^2 = 1$. By solving Eqs. (35.10) for φ and assuming that $\rho \neq 1$, we can rewrite the equation of the ellipse in canonical form:

$$\frac{u^2}{a_\rho^2} + \frac{v^2}{b_\rho^2} = 1. \tag{35.11}$$

Note that if we replace ρ with $1/\rho$ ($\rho \neq 1$), the ellipse (35.10) does not change, but its orientation changes to the opposite. Figure 95 shows circles $|z| = \rho$, $\rho > 1$, oriented clockwise and their images, ellipses (35.11); from (35.10) we can see that the ellipses are

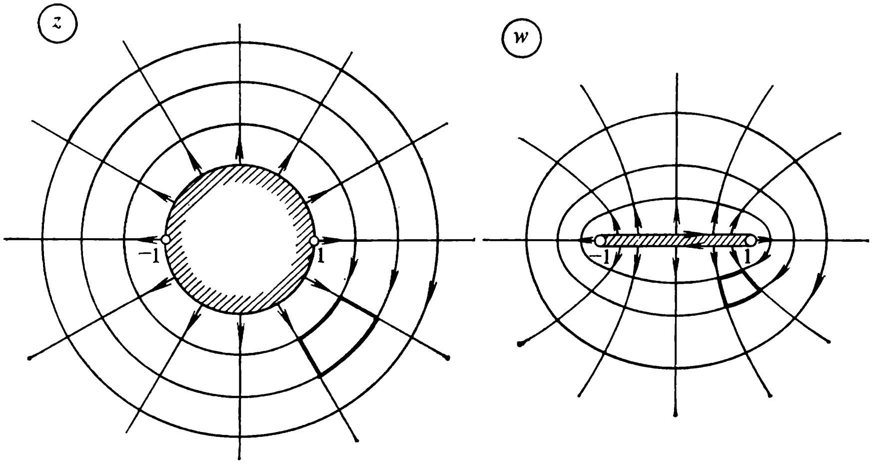


Fig. 95

also oriented clockwise. Figure 96 shows circles $|z| = \rho$, $0 < \rho < 1$, and their images, ellipses (35.11)—the orientation has changed: a circle $|z| = \rho$ oriented counterclockwise is mapped into an ellipse (35.11) oriented clockwise.

For $\rho = 1$ the ellipse degenerates into the segment $[-1, 1]$ traversed twice, i.e. the circle $|z| = 1$ is mapped into the segment $[-1, 1]$ traversed twice (see Figs. 95 and 96).

Consider the ray

$$z = re^{i\alpha}, \quad 0 < r < +\infty \tag{35.12}$$

(α is fixed). Under the mapping performed by the Zhukovskii function the ray is mapped into the curve (see (35.8))

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\alpha, \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\alpha, \quad 0 < r < +\infty. \tag{35.13}$$

Solving Eqs. (35.13) for r and assuming that $\alpha \neq k\pi/2$ (k is an integer), we obtain

$$\frac{u^2}{\cos^2\alpha} - \frac{v^2}{\sin^2\alpha} = 1. \tag{35.14}$$

The curve given by (35.14) is a hyperbola with the foci at points $w = \pm 1$ and the asymptotes $v = \pm u \tan\alpha$.

If $0 < \alpha < \pi/2$, then curve (35.13) is the right branch of hyperbola (35.14), i.e. the ray (35.12) for $0 < \alpha < \pi/2$ is mapped into the right branch of hyperbola (35.14) (the orientation is shown in

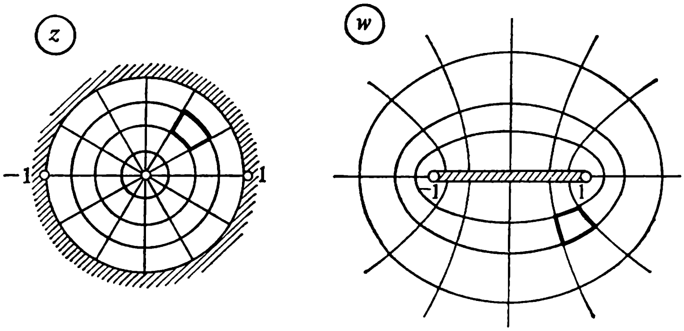


Fig. 96

Fig. 97). If we substitute $\pi - \alpha$ for α in (35.13), we have the left branch of the same hyperbola (35.14), whence the ray (35.12) for $\pi/2 < \alpha < \pi$ is mapped into the left branch of hyperbola (35.14), but the orientation changes to the opposite.

Let us consider the rays (35.12) at $\alpha = k\pi/2$ (k is an integer). From (35.13) it follows that the ray $\arg z = \pi/2$ is mapped into the

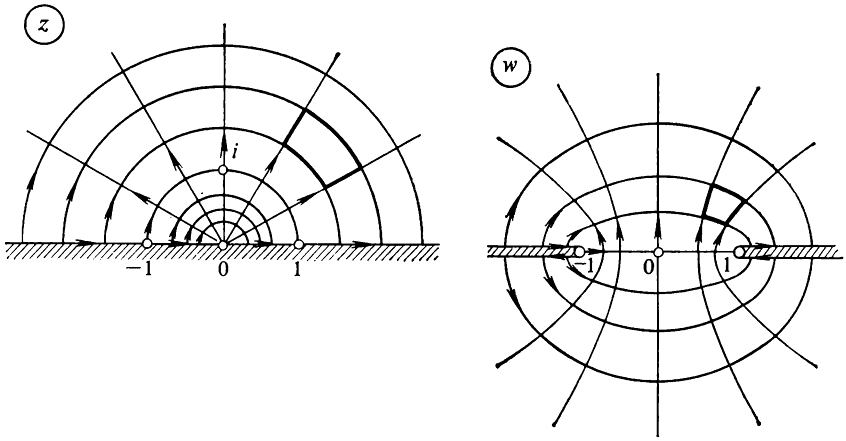


Fig. 97

imaginary axis $\operatorname{Re} w = 0$ (Fig. 97). The ray $\arg z = 3\pi/2$ is also mapped into the imaginary axis $\operatorname{Re} w = 0$. For $\alpha = 0$ the curve (35.13) degenerates into the ray $[1, +\infty)$ traversed twice (or the ray is folded) (Fig. 97), i.e. the ray $\arg z = 0$ is mapped into the ray $[1, +\infty)$ traversed twice, namely, the ray $[1, +\infty)$ is mapped into the ray $[1, +\infty)$ and the half-interval $(0, 1]$ into the ray $(+\infty, 1]$ (Fig. 97). Similarly, the ray $\arg z = \pi$ is mapped into the ray $(-\infty, -1]$ traversed twice (Fig. 97).

Thus, the Zhukovskii function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps circles $|z| = \rho$ into ellipses (35.11) and rays $\arg z = \alpha$ into branches of hyperbolas (35.14); the foci of all ellipses (35.11) and hyperbolas (35.14) are at points $w = \pm 1$, and any ellipse (35.11) intersects any hyperbola (35.14) at a right angle.

Example 15. Suppose D is the exterior of the unit circle (Fig. 95). Let us find the domain onto which D is mapped by the Zhukovskii function, which is univalent in D (see Example 14). Here are two ways in which this can be done.

(1) The images of circles $|z| = \rho$, $\rho > 1$, are ellipses (35.11), which fill the entire complex w plane with a cut along the segment $[-1, 1]$. Hence, the Zhukovskii function maps the exterior of the unit circle conformally onto the exterior of the segment $[-1, 1]$ (Fig. 95).

(2) The ray

$$z = re^{i\alpha}, \quad 1 < r < +\infty \tag{35.15}$$

is transformed into the curve

$$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\alpha, \quad v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\alpha,$$
$$1 < r < +\infty \tag{35.16}$$

which is a part (a half) of hyperbola (35.13). When α changes from 0 to 2π, the curves (35.16) fill the entire complex w plane with a cut along the segment $[-1, 1]$ (Fig. 95). Hence, the Zhukovskii function maps the exterior of the unit circle conformally onto the exterior of the segment $[-1, 1]$ (Fig. 95).

Note that the circle $|z| = 1$ oriented clockwise (the boundary of domain D) is mapped into the cut along the segment $[-1, 1]$; precisely, the semicircle $|z| = 1$, $\operatorname{Im} z \geqslant 0$ is mapped into the upper bank of the cut and the semicircle $|z| = 1$, $\operatorname{Im} z \leqslant 0$ is mapped into the lower bank of the cut (Fig. 98). In other words, the circle $|z| = 1$ is "flattened" into the cut along the segment $[-1, 1]$ with preservation of sense. □

Example 16. As in Example 15, we can show that the Zhukovskii function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps the unit circle $|z| < 1$ conformally onto the exterior of the segment $[-1, 1]$ (Fig. 96). This proposition also follows from Example 15 and Remark 2.

Note that under this mapping the circle $|z| = 1$ oriented counterclockwise (the boundary of the circle $|z| < 1$) is transformed into the cut along the segment $[-1, 1]$ oriented clockwise. Precisely, the semicircle $|z| = 1$, $\operatorname{Im} z \geqslant 0$ is mapped into the lower bank of the cut and the semicircle $|z| = 1$, $\operatorname{Im} z \leqslant 0$ into the upper bank (Fig. 99). □

Example 17. As in Example 15, we find that the Zhukovskii function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps the upper half-plane $\operatorname{Im} z > 0$

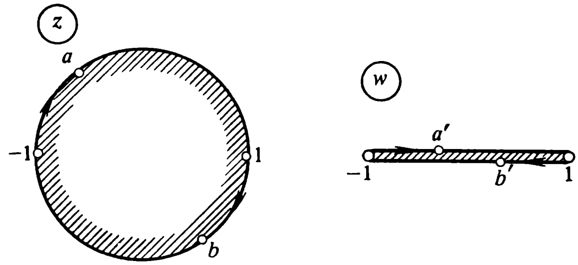


Fig. 98

conformally onto the complex w plane with cuts along the rays $(-\infty, 1]$ and $[1, +\infty)$ (Fig. 97). Under this mapping

(a) the ray $(-\infty, -1]$ is transformed into the upper bank of the cut along the ray $(-\infty, -1]$,

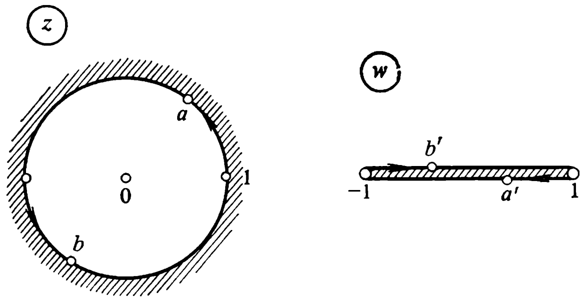


Fig. 99

(b) the half-interval $[-1, 0)$ into the lower bank of the cut $(-\infty, -1]$,

(c) the half-interval $(0, 1]$ into the lower bank of the cut $[1, +\infty)$,

(d) the ray $[1, +\infty)$ into the upper bank of the cut $[1, +\infty)$ (Fig. 100). □

Example 18. From Example 17 and Remark 2 it follows that the Zhukovskii function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps the lower half-plane

Im $z < 0$ conformally onto the complex w plane with cuts along the rays $(-\infty, -1]$ and $[1, +\infty)$. □

In Figs. 95-97 the annular sectors in the complex z plane and their images in the complex w plane obtained as a result of mappings per-

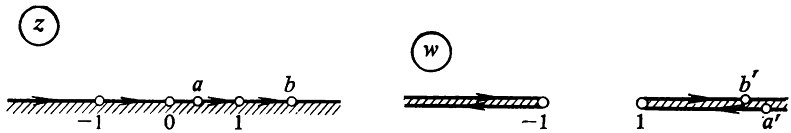


Fig. 100

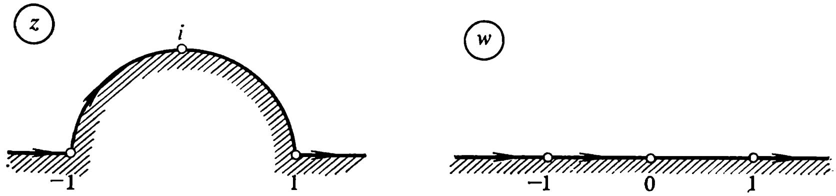
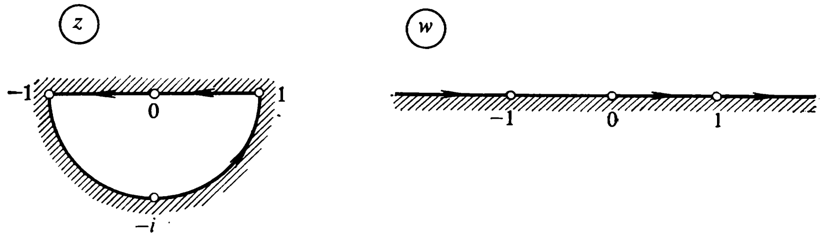
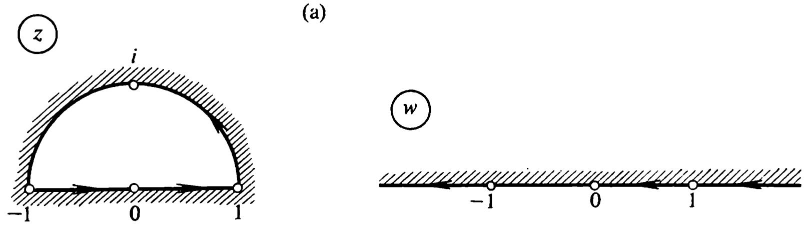


Fig. 101



(a)



(b)

Fig. 102

formed by the Zhukovskii function are shown by heavy lines. The following particular cases of such mappings are often used in the conformal mapping of various domains.

Example 19. The Zhukovskii function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps conformally

(a) the domain $\operatorname{Im} z > 0$, $|z| > 1$ (Fig. 101) onto the upper half-plane $\operatorname{Im} w > 0$ (from Fig. 95);

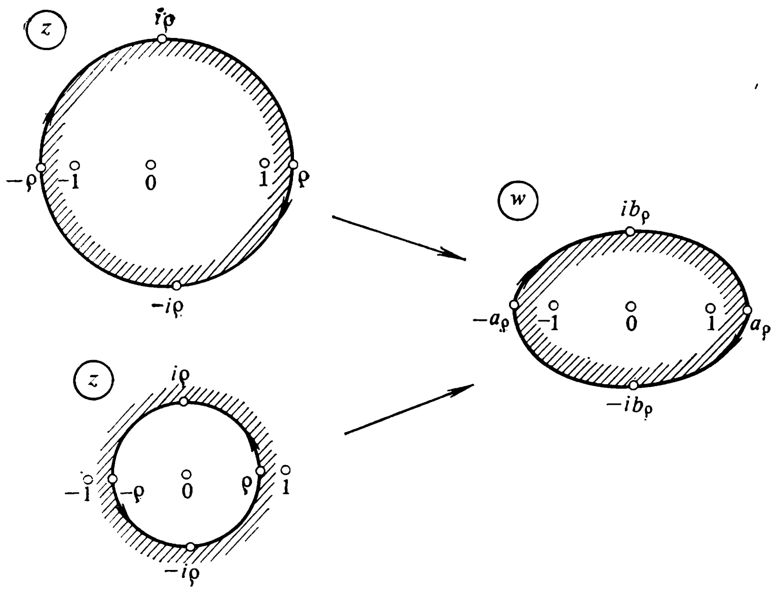


Fig. 103

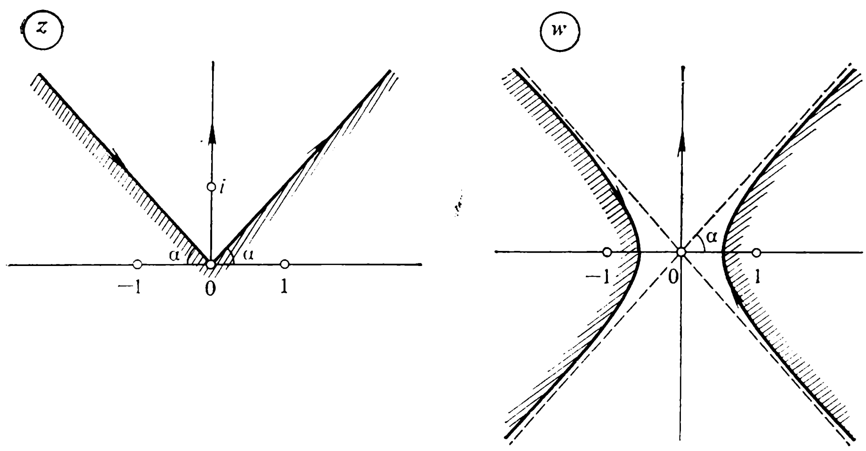


Fig. 104

(b) the semicircle $|z| < 1$, $\operatorname{Im} z < 0$ (Fig. 102) onto the upper half-plane $\operatorname{Im} w > 0$; the semicircle $|z| < 1$, $\operatorname{Im} z > 0$ (Fig. 102) onto a lower half-plane $\operatorname{Im} w < 0$ (from Fig. 96).

(c) the domain $|z| > \rho > 1$ (Fig. 103) onto the exterior of the ellipse (35.11) (from Fig. 95); the circle $|z| < \rho < 1$ (Fig. 103) onto the exterior of the ellipse (35.11) (from Fig. 96);

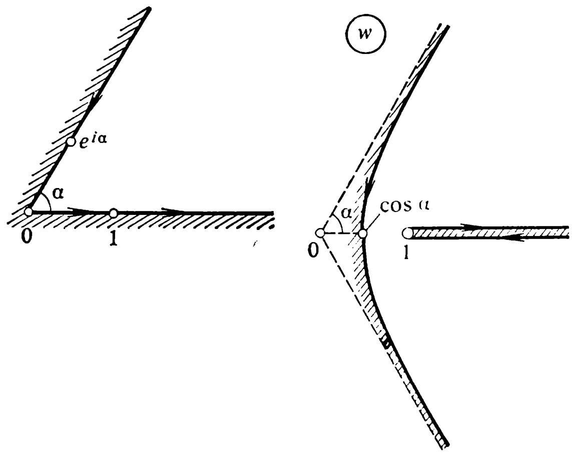


Fig. 105

(d) the sector $\alpha < \arg z < \pi - \alpha$, with $0 < \alpha < \pi/2$ (Fig. 104), onto the exterior of the hyperbola (35.14) (from Fig. 97);

(e) the sector $0 < \arg z < \alpha$, with $0 < \alpha < n/2$ (Fig. 105), onto the interior of the right branch of the hyperbola (35.14) with a cut along the ray $[1, +\infty)$ (from Fig. 97);

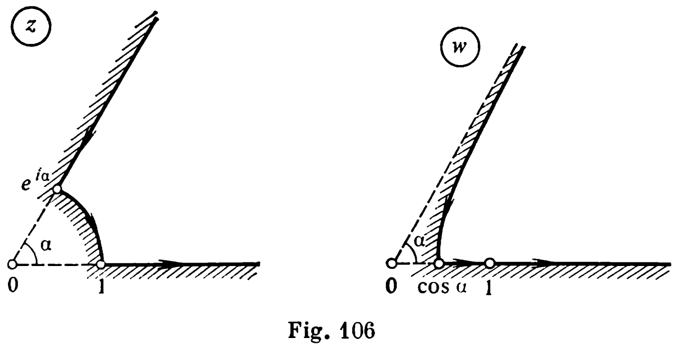


Fig. 106

(f) the sector $0 < \arg z < \alpha$, $|z| > 1$, with $0 < \alpha < \pi/2$ (Fig. 106), onto the domain $u^2/\cos^2\alpha - v^2/\sin^2\alpha > 1$, with $u > 0$ and $v > 0$ $(w = u + iv)$ (from Fig. 97). □

35.7 The inverse of the Zhukovskii function Solving the equation $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ for z, we arrive at $z = w + \sqrt{w^2 - 1}$, which means that the function

$$w = z + \sqrt{z^2 - 1} \tag{35.17}$$

is the inverse of the Zhukovskii function. Hence the mapping performed by (35.17) is the inverse of the mapping performed by the Zhukovskii function.

Some properties of the function (35.17) were discussed in Sec. 24. We recall that this function is analytic in the complex z plane with

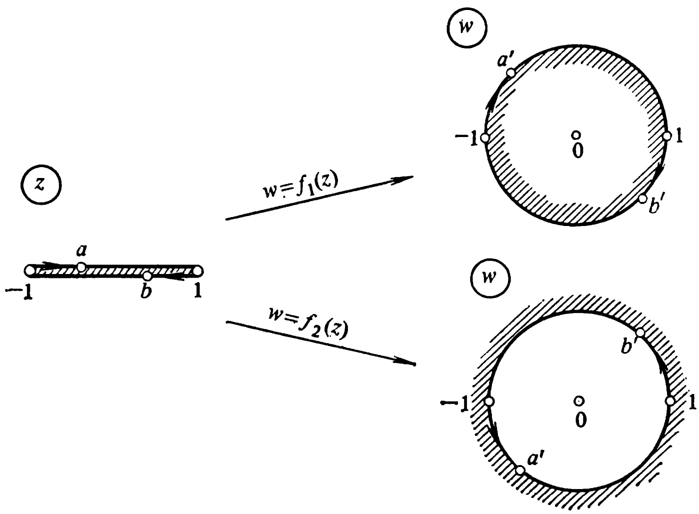


Fig. 107

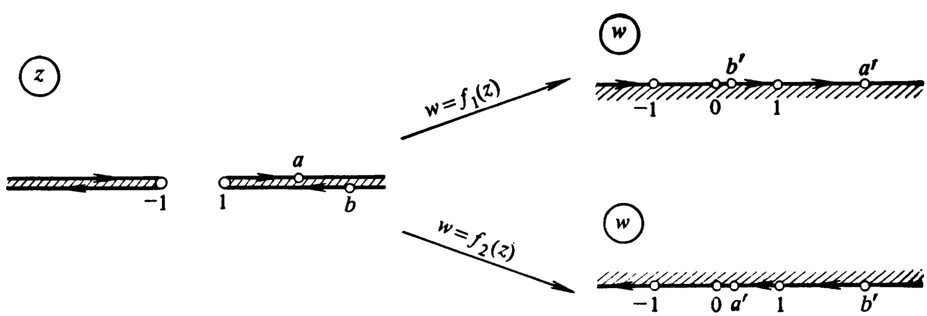


Fig. 108

points $z = \pm 1$ deleted, while in the complex z plane with a cut connecting points $z = \pm 1$ it splits into two regular branches.

Example 20. Suppose D is the complex z plane with a cut along the segment $[-1, 1]$ (Fig. 107). In this domain the function $z +$

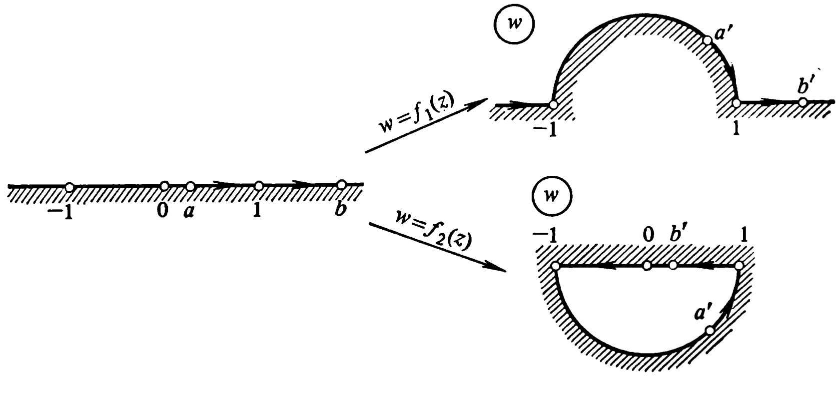


Fig. 109

$\sqrt{z^2 - 1}$ splits into two regular branches, $f_1(z)$ and $f_2(z)$, where $f_1(\infty) = \infty$ and $f_2(\infty) = 0$ (see Sec. 24). From Examples 15 and 16 (Figs. 98 and 99) it follows that the function $w = f_1(z)$ maps D conformally onto the exterior of the unit circle, while the function $w = f_2(z)$ maps D conformally onto the circle $|w| < 1$ (Fig. 107). □

Example 21. Let D be the complex z plane with cuts along the rays $(-\infty, -1]$ and $[1, +\infty)$ (Fig. 108). In this domain the function $z + \sqrt{z^2 - 1}$ splits into two regular branches, $f_1(z)$ and $f_2(z)$, with $f_1(0) = i$ and $f_2(0) = -i$ (see Sec. 24). From Examples 17 and 18 it follows that the function $w = f_1(z)$ maps D conformally onto the upper half-plane $\operatorname{Im} w > 0$, while the function $w = f_2(z)$ maps D onto the lower half-plane $\operatorname{Im} w < 0$ (Fig. 108). □

Example 22. In the half-plane $\operatorname{Im} z > 0$ the function $z + \sqrt{z^2 - 1}$ splits into two regular branches, $f_1(z)$ and $f_2(z)$, with $f_1(0) = i$ and $f_2(0) = -i$. The mappings performed by these functions are shown in Fig. 109 (cf. Figs. 101 and 102a). □

35.8 Trigonometric and hyperbolic functions Let us study the conformal mappings performed by trigonometric and hyperbolic functions.

Example 23. We wish to show that the function $w = \cosh z$ maps the semistrip $0 < \operatorname{Im} z < \pi$, $\operatorname{Re} z > 0$ conformally onto the upper half-plane $\operatorname{Im} w > 0$ (Fig. 110).

Indeed, the function $w = \cosh z = \frac{1}{2}(e^z + e^{-z})$ is a combination of two functions:

$$\zeta = e^z, \quad w = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right).$$

As a result of performing the mappings $\zeta = e^z$ (Fig. 94b) and $w = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)$ (Fig. 101) sequentially we obtain the mapping shown in Fig. 110. □

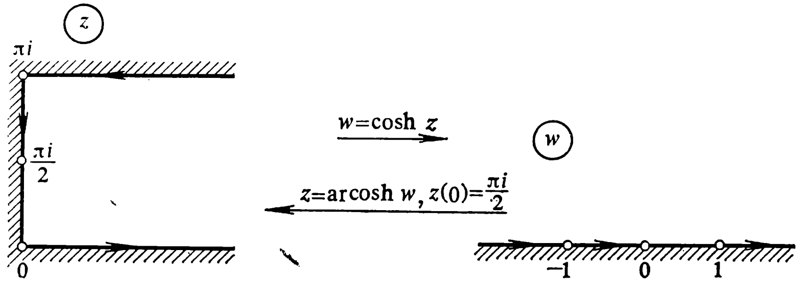


Fig. 110

Example 24. We wish to show that the function $w = \cos z$ maps the semistrip $-\pi < \operatorname{Re} z < 0$, $\operatorname{Im} z > 0$ conformally onto the upper half-plane $\operatorname{Im} w > 0$ (Fig. 111).

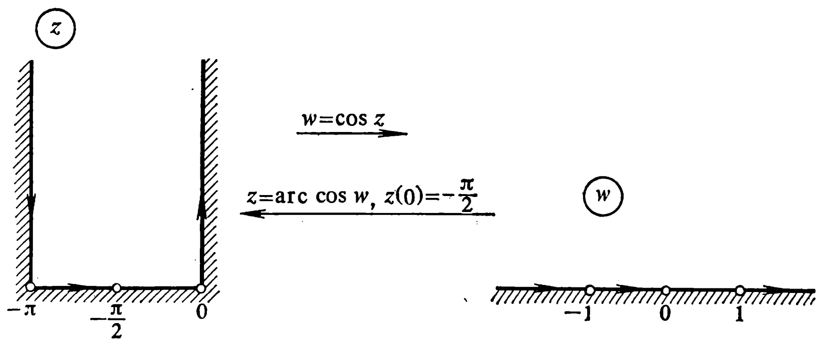


Fig. 111

Indeed, since $\cos z = \cosh(-iz)$, performing first the mapping $\zeta = -iz$ (rotation about point $z = 0$ by an angle of $-\pi/2$) and then the mapping $w = \cosh \zeta$ (Fig. 110), we arrive at the mapping shown in Fig. 111. □

Example 25. Let us show that the function $w = \sin z$ maps the semistrip $-\pi/2 < \operatorname{Re} z < \pi/2$, $\operatorname{Im} z > 0$ conformally onto the upper half-plane $\operatorname{Im} w > 0$ (Fig. 112).

Indeed, we can write $\sin z = \cos(z - \pi/2)$. This means that if we first perform the translation $\zeta = z - \pi/2$ and then the mapping

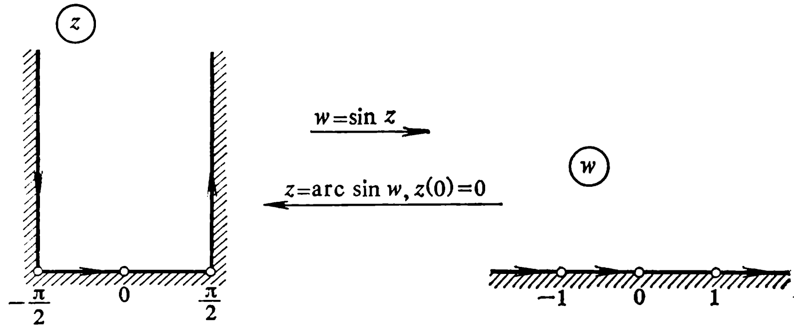


Fig. 112

$w = \cos \zeta$ (Fig. 111), we arrive at the mapping shown in Fig. 112. □

Example 26. Let us show that the function $w = \tan z$ maps the strip $-\pi/4 < \operatorname{Re} z < \pi/4$ conformally onto the unit circle $|w| < 1$

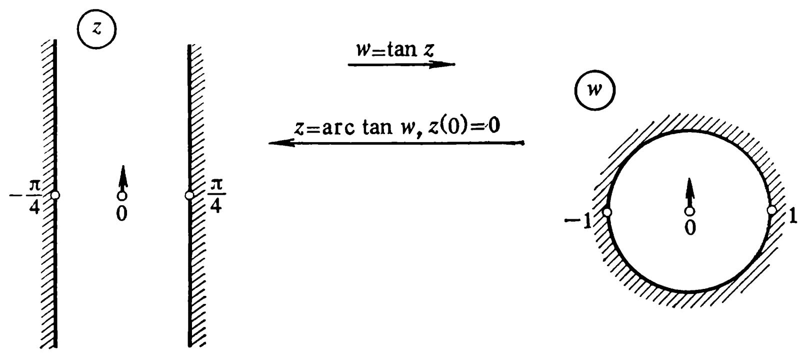


Fig. 113

(Fig. 113). Note that this mapping satisfies the conditions $w(0) = 0$ and $\arg w'(0) = 0$.

Indeed, since

$$\tan z = \frac{\sin z}{\cos z} = \frac{1}{i} \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = (-i) \frac{e^{2iz} - 1}{e^{2iz} + 1},$$

we can think of the mapping $w = \tan z$ as the combination of three mappings:

$$\zeta = 2iz, \quad \eta = e^{\zeta}, \quad w = (-i)\frac{\eta - 1}{\eta + 1}.$$

Performing these mappings in the above-mentioned order, we arrive at the mapping shown in Fig. 113. □

The examples we have considered show that mappings performed by trigonometric and hyperbolic functions are reduced to combinations of the mappings discussed in Secs. 34 and 35.1-35.7.

35.9 Various examples The conformal mappings discussed in Secs. 34 and 35.1-35.8 are classical, so to say. They can be used to

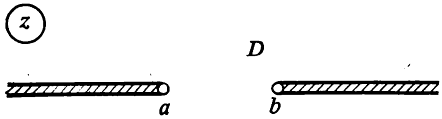


Fig. 114

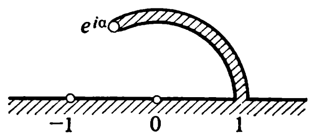


Fig. 115

find the conformal mappings of other simple domains. Example 27-33 below give the conformal mappings $w = w(z)$ of a given domain D in the complex z plane onto the upper half-plane $\operatorname{Im} w > 0$.

Example 27. Suppose D is the complex z plane with cuts along the rays $(-\infty, a]$ and $[b, +\infty)$, with $-\infty < a < b < +\infty$ (Fig. 114). There are two ways in which we can perform the mapping.

(a) Just as in Example 10 (Fig. 93f), we find that

$$w = \sqrt{\frac{z-b}{z-a}}, \quad \text{where} \quad w\left(\frac{a+b}{2}\right) = i.$$

(b) The linear function $\zeta = \left(z - \frac{a+b}{2}\right)\frac{2}{b-a}$ (a translation and a stretching) maps D onto the complex ζ plane with cuts along the rays $(-\infty, -1]$ and $[1, +\infty)$. Then, as in Example 21 (Fig. 108), we perform the mapping $w = \zeta + \sqrt{\zeta^2 - 1}$, with $w|_{\zeta=0} = i$. □

Example 28. Suppose D is the half-plane $\operatorname{Im} z > 0$ with a cut along the arc $|z| = 1$, $0 \leqslant \arg z \leqslant \alpha$, with $0 < \alpha < \pi$ (Fig. 115). There are two ways in which we can perform the mapping.

(a) The function $\zeta = (z-1)/(z+1)$ maps the half-plane $\operatorname{Im} z > 0$ onto the half-plane $\operatorname{Im} \zeta > 0$ (Sec. 33), while the cut along the arc is mapped into a cut along the segment $[0, ih]$, since $1 \to 0$ and $-1 \to \infty$, with $h = \tan(\alpha/2)$. After this, as in Example 7 (Fig. 91), we perform the mapping $w = \sqrt{\zeta^2 + h^2}$, where $w(x + i0) > 0$ for $x > 1$.

(b) The Zhukovskii function $\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps D onto the complex ζ plane with cuts along the rays $(-\infty, -1]$ and $[\cos\alpha, +\infty)$ (from Fig. 97). After this we follow Example 27. □

Example 29. Suppose D is the strip $0 < \operatorname{Im} z < \pi$ with a cut along the segment $[0, id]$, where $0 < d < \pi$ (Fig. 116). The function $\zeta = e^z$ maps D onto the domain shown in Fig. 115. After this we follow Example 28. □

Example 30. Suppose D is the strip $-\pi < \operatorname{Im} z < \pi$ with a cut along the ray $[a, +\infty)$, where a is a real number (Fig. 117). The

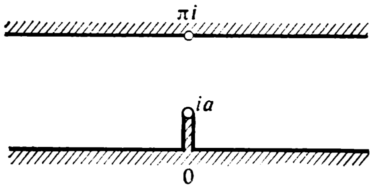


Fig. 116

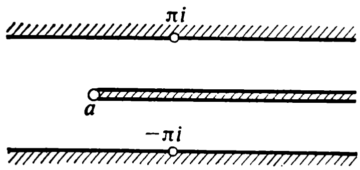


Fig. 117

function $\zeta = e^z$ maps D onto the complex ζ plane with cuts along the rays $(-\infty, 0]$ and $[e^a, +\infty)$. After this we follow Example 27. □

Example 31. Suppose D is the semistrip $0 < \operatorname{Im} z < \pi$, $\operatorname{Re} z > 0$ with a cut along the segment $[\pi i/2, \alpha + \pi i/2]$, with $\alpha > 0$ (Fig. 118).

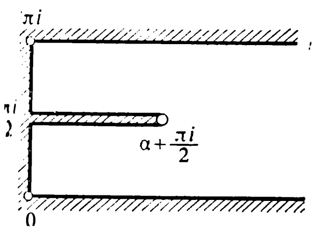


Fig. 118

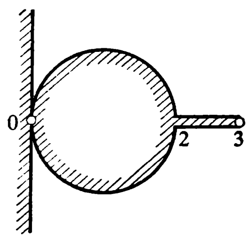


Fig. 119

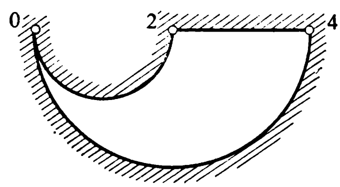


Fig. 120

The function $\zeta = \cosh z$ maps D onto the half-plane $\operatorname{Im} \zeta > 0$ with a cut along the segment $[0, i \sinh \alpha]$ (from Example 23). After this we follow Example 7. □

Example 32. Suppose D is the domain $\operatorname{Re} z > 0$, $|z - 1| > 1$ with a cut along the segment $[2, 3]$ (Fig. 119). The function $\zeta = 1/z$ maps D onto the strip $0 < \operatorname{Re} \zeta < 1/2$ with a cut along the segment $[1/3, 1/2]$, domain D_1. Then a linear function can be used to map D_1 onto the domain shown in Fig. 116. After this we can follow Example 29. □

Example 33. Suppose D is the domain $|z - 1| > 1$, $|z - 2| < 2$, $\operatorname{Im} z < 0$ (Fig. 120). The function $\zeta = 1/z$ maps D onto the semistrip

D_1: $1/4 < \operatorname{Re} \zeta < 1/2$, $\operatorname{Im} \zeta > 0$. Then a linear function can be used to map D_1 onto the domain shown in Fig. 111. After this we can follow Example 24. □

In Examples 34-37 below the functions $w = w(z)$ map a given domain D in the complex z plane onto the unit circle $|w| < 1$.

Example 34. Suppose D is the complex z plane with a cut along the segment $[a, b]$, where $-\infty < a < b < +\infty$ (Fig. 121). The linear function $\zeta = \left(z - \frac{a+b}{2}\right)\frac{2}{b-a}$ (a translation and a stretch-

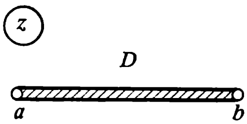


Fig. 121

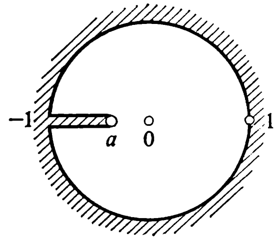


Fig. 122

ing) maps D onto the exterior of the segment $[-1, 1]$. Then, just as in Example 20 (Fig. 107), we employ the function $w = \zeta + \sqrt{\zeta^2 - 1}$, with $w(\infty) = 0$. □

Example 35. Suppose D is the circle $|z| < 1$ with a cut along the segment $[-1, a]$, where $-1 < a < 0$ (Fig. 122). The Zhukovskii

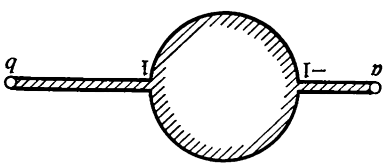


Fig. 123

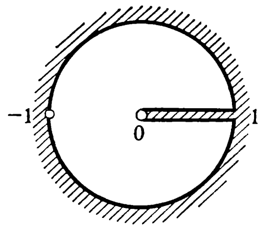


Fig. 124

function $\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps D onto the complex ζ plane with a cut along the segment $\left[\frac{1}{2}\left(a + \frac{1}{a}\right), 1\right]$ (from Fig. 96). After this we can follow Example 34. □

Example 36. Suppose D is the domain $|z| > 1$ with cuts along the segments $[a, -1]$ and $[1, b]$, where $-\infty < a < -1$ and $1 < b < +\infty$ (Fig. 123). The Zhukovskii function $\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps D onto the exterior of the segment $[a', b']$, where $a' =$

$\frac{1}{2}\left(a+\frac{1}{a}\right)$ and $b'=\frac{1}{2}\left(b+\frac{1}{b}\right)$ (from Fig. 95). After this we can follow Example 34. □

Example 37. Suppose D is the circle $|z|<1$ with a cut along the segment [0, 1] (Fig. 124). The Zhukovskii function $\zeta=\frac{1}{2}\left(z+\frac{1}{z}\right)$ maps the domain D onto the complex ζ plane with a cut along the ray $[-1, +\infty)$ (from Fig. 96), domain D_1. The function $\eta=\sqrt{\zeta+1}$, with $\eta|_{\zeta=-5}=2i$, maps D_1 onto the half-plane $\operatorname{Im}\eta>0$. Finally, the function $w=(\eta-i)/(\eta+i)$ maps the half-plane $\operatorname{Im}\eta>0$ onto the circle $|w|<1$ (see Sec. 34). □

Various examples of conformal mappings performed by elementary functions can be found in Koppenfels and Stallmann [1].

Example 38. Suppose D is the domain $\operatorname{Im} z<0$, $|z+il|>R$, where $l>R>0$ (Fig. 125). This domain can be called a *noncon-*

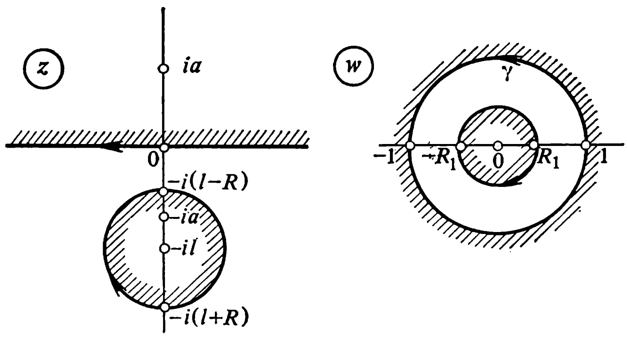


Fig. 125

centric annulus (a straight line is a circle with an infinite radius). Let us find the function that maps D conformally onto a concentric annulus. We start by determining two points that are symmetric with respect to the straight line $\operatorname{Im} z=0$ and with respect to the circle $|z+il|=R$ simultaneously. These points must lie on a common perpendicular to the straight line and the circle (Sec. 34), i.e. on the imaginary axis. From the fact that these two points are symmetric to the straight line $\operatorname{Im} z=0$ we conclude that they are $+ia$ and $-ia$, with $a>0$, while from the fact that they are symmetric with respect to the circle $|z+il|=R$ we conclude that $(l+a)(l-a)=R^2$, whence $a=\sqrt{l^2-R^2}$. Let us show that the sought-for function is

$$w=\frac{z+ia}{z-ia}. \tag{35.18}$$

Indeed, under such a mapping the straight line $\operatorname{Im} z=0$ becomes the circle γ in Fig. 125. By the property of symmetry preservation (see Sec. 34), points $z = \pm ia$ are mapped into the points $w = 0$ and $w = \infty$ symmetric with respect to γ. Hence, $w = 0$ is the center of circle γ. Since point $w(0) = -1$ belongs to γ, we find that γ is the circle $|w| = 1$ in Fig. 125. Reasoning along the same lines, we conclude that the circle $|z + il| = R$ is mapped via (35.18) into the circle $|w| = R_1$, where $R_1 = \frac{R-l-a}{R+l+a}$. By the property of boundary correspondence (see Sec. 33), the function (35.18) maps D conformally onto the concentric annulus $R_1 < |w| < 1$ (Fig. 125). □

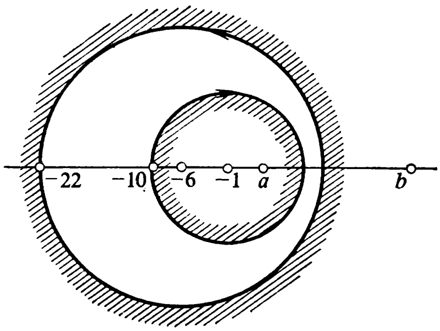


Fig. 126

Example 39. Suppose D is the nonconcentric circle $|z + 1| > 9$, $|z + 6| < 16$ (Fig. 126). Let us find the function that maps D conformally onto a concentric annulus. To this end we determine two points, a and b, that are symmetric with respect to the circles $|z+1| = 9$ and $|z+6| = 16$ simultaneously. These points lie on a common perpendicular to the circles, i.e. on the real axis (Fig. 126), which means that a and b are real. The fact that they are symmetric with respect to the given circles yields (see Sec. 34)

$$(a + 1)(b + 1) = 81, \quad (a + 6)(b + 6) = 256.$$

Solving this system, we find that $a = 2$ and $b = 26$. As in Example 38, we can prove that the function $w = \frac{z-2}{z-26}$ maps D conformally onto the concentric annulus $1/3 < |w| < 1/2$. □

Example 40. Suppose D is the domain $|z - ih| > \sqrt{1 + h^2}$, with h real. The boundary of D is the circle γ centered at point ih and passing through the points $z = \pm 1$, $z = ia$, and $z = -i/a$, with $a = h + \sqrt{1 + h^2}$ (Fig. 127). We wish to show that the Zhukovskii function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ is univalent in D.

We start the mapping $\zeta = 1/z$, with the points ζ placed in the same complex z plane. Under such a mapping the points $z = \pm 1$ remain in place, while the point $z = ia$ is mapped into point $z = -i/a$, which means that γ is mapped into itself (Sec. 34). Point $z = \infty$ is mapped into the point $\zeta = 0$, which lies inside γ. Hence, the exterior D of γ is mapped into the interior $\tilde{D}$ of γ. Since D and $\tilde{D}$ have no common points, Remark 2 enables us to conclude that the Zhukovskii function is univalent in D (and in $\tilde{D}$).

Let us find the image of D obtained as a result of the mapping performed by the Zhukovskii function. We notice that we can write $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ as $\frac{w-1}{w+1} = \left(\frac{z-1}{z+1}\right)^2$. For this reason we can think of the Zhukovskii function as a combination of two functions:

$$w = \frac{1+\zeta}{1-\zeta}, \quad \zeta = \left(\frac{z-1}{z+1}\right)^2.$$

Under the mapping $\zeta = \left(\frac{z-1}{z+1}\right)^2$ the circle γ is mapped into the cut along a ray $\tilde{\gamma}$ connecting points $\zeta = 0$ and $\zeta = \infty$. Under the

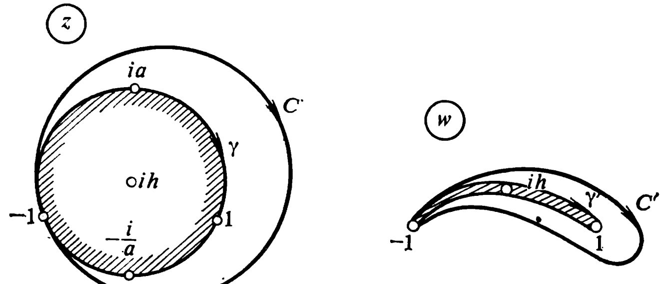


Fig. 127

mapping $w = \frac{1+\zeta}{1-\zeta}$ the ray $\tilde{\gamma}$ is mapped into the arc γ' of a circle with its ends at points $w = \pm 1$. Since the function $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$ maps point $z = ia$ into point $w = \frac{1}{2}\left(ia - \frac{i}{a}\right) = ih$, the arc γ' passes through the point $w = ih$. Hence, the Zhukovskii function maps the exterior of γ conformally onto the exterior of the arc γ', which has its ends at points $w = \pm 1$ and passes through point $w = ih$ (Fig. 127).

Note that the Zhukovskii function maps a circle C that is close to γ and touches it at point $z = -1$ into a curve C' (Fig. 127) that resembles the cross section of an airfoil. N. E. Zhukovskii used curves like C' to calculate the aerodynamic lift on an airfoil (see Lavrent'ev and Shabat [1]). □

36 The Riemann-Schwarz Symmetry Principle

In this section we will study a method of analytic continuation using symmetry considerations. The method is known as the Riemann-Schwarz symmetry principle and enables simplifying considerably the solution of problems that involve the conformal mapping of regions symmetric with respect to a straight line.

36.1 Symmetry with respect to the real axis We start with

Lemma 1 *Suppose a curve γ separates a bounded domain D into two domains, D_1 and D_2* (Fig. 128), *and suppose we have a function*

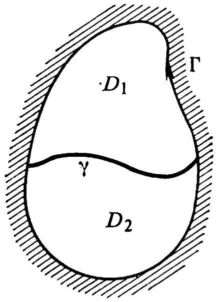


Fig. 128

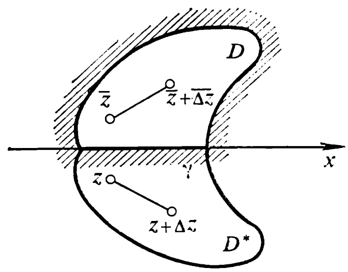


Fig. 129

$f(z)$ that is regular in both D_1 and D_2 and is continuous in D. Then the function is regular in D everywhere.

Proof. Without loss of generality we may assume that $f(z)$ is continuous in D up to the boundary Γ of D. Consider the function

$$F(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}\,d\zeta. \tag{36.1}$$

It is regular in D (see Sec. 16). If we prove that $F(z)\equiv f(z)$ for all $z\in D$, the lemma will have been proved.

Adding and subtracting integrals along γ, we can write (36.1) for $z\in D_1$ as

$$F(z)=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta+\frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta, \tag{36.2}$$

where Γ_j is the boundary of D_j $(j=1, 2)$. Here the integral along Γ_1 is equal to $f(z)$ and the integral along Γ_2 is zero (see Sec. 10), i.e. $F(z)=f(z)$ for $z\in D_1$. Similarly, $F(z)=f(z)$ for $z\in D_2$. Hence, $F(z)\equiv f(z)$ for all $z\in D$, since, by the hypothesis of the lemma, the function $f(z)$ is continuous in D.

Corollary 1 *Suppose two domains, D_1 and D_2, have no common points and border on a curve γ* (Fig. 128). *Suppose we have two functions, $f_1(z)$ and $f_2(z)$, that are regular in D_1 and D_2, respectively, and are ontinuous up to γ. If the values of these functions coincide on γ, then*

$$F(z)=\begin{cases} f_1(z), & z\in D_1\cup\gamma,\\ f_2(z), & z\in D_2\end{cases} \tag{36.3}$$

is the analytic cintinuation of $f_1(z)$ *from domain* D_1 *into the domain* $D_1 \cup \gamma \cup D_2$.

In this case we will say that $f_2(z)$ is the analytic continuation of $f_1(z)$ from D_1 into D_2 across curve γ.

In the theorem we will now prove, D is a domain whose boundary contains a segment γ of the real axis, and D^* is the domain symmetric to D with respect to the real axis (Fig. 129).

Theorem 1 *Let a function* $f(z)$ *be regular in a domain* D *whose boundary contains a segment* γ *of the real axis, and suppose domains* D *and* D^* *have no common points. If* $f(z)$ *is continuous up to* γ *and assumes real values on* γ, *then it can be continued analytically into the domain* $D \cup \gamma \cup D^*$. *The continuation is given by the formula*

$$F(z)=\begin{cases} f(z), & z\in D\cup\gamma,\\ \overline{f(\bar{z})}, & z\in D^*.\end{cases} \tag{36.4}$$

Proof. Let us prove that the function

$$f_1(z)=\overline{f(\bar{z})}, \quad z\in D^*, \tag{36.5}$$

at each point $z \in D^*$ has a derivative $f_1'(z)$. Consider the quotient

$$\frac{f_1(z+\Delta z)-f_1(z)}{\Delta z}=\frac{\overline{f(\overline{z+\Delta z})}-\overline{f(\bar{z})}}{\Delta z}=\left[\overline{\frac{f(\bar{z}+\overline{\Delta z})-f(\bar{z})}{\overline{\Delta z}}}\right]. \tag{36.6}$$

Since $z \in D^*$, we can write $\bar{z} \in D$, and for a small Δz the point $\bar{z} + \overline{\Delta z}$ also belongs to D (Fig. 129). Hence, as $\Delta z \to 0$ the quotient (36.6) tends to finite limit equal to $\overline{f'(\bar{z})}$, i.e. $f_1'(z) = \overline{f'(\bar{z})}$. Thus, $f_1(z)$ is differentiable and, therefore, regular in D^*.

We will now show that the function $F(z)$, defined by (36.4) is continuous in the domain $D \cup \gamma \cup D^*$. Indeed, from the fact that $f(z)$ is continuous up to γ it follows that $\lim\limits_{z\to x} f(z) = f(x)$, $x \in \gamma$, whence we find that $\lim\limits_{z\to x} f_1(z) = \overline{\lim\limits_{z\to x} f(\bar{z})} = \overline{f(x)}$, i.e. $f_1(z)$ is continuous up to γ. Since $f(x) = \overline{f(x)}$ by the hypothesis of the theorem, we can write $f_1(z)|_{z\in\gamma} = f(z)|_{z\in\gamma}$. Hence, $F(z)$ is regular in $D \cup \gamma \cup D^*$, by virtue of Lemma 1, and is the analytic continuation of $f(z)$.

Note that under the hypothesis of Theorem 1, $f_1(z)$ defined by (36.5) is the analytic continuation of $f(z)$ (from D into D^* across γ).

Example 1. If an entire function $f(z)$ assumes real values on the real axis, then for all values of z we have $f(\bar{z}) = \overline{f(z)}$. For instance, $e^{\bar{z}} = \overline{e^{z}}$, $\sin \bar{z} = \overline{\sin z}$, $\cos \bar{z} = \overline{\cos z}$, $\sinh \bar{z} = \overline{\sinh z}$, and $\cosh \bar{z} = \overline{\cosh z}$. □

Example 2. Suppose $f(z)$ is regular in the half-plane $\operatorname{Im} z > 0$ everywhere except at a simple pole at point z_0 ($\operatorname{Im} z_0 > 0$), is continuous up to the real axis, and assumes real values on the real axis. Let us show that if this function remains bounded as $z \to \infty$, $\operatorname{Im} z \geqslant 0$, then

$$f(z) = \frac{A}{z - z_0} + \frac{\bar{A}}{z - \bar{z}_0} + C, \tag{36.7}$$

where $A = \operatorname*{Res}_{z \to z_0} f(z)$, and C is a real number.

Indeed, in this case the function given by (36.4) is regular in the entire complex plane everywhere except at simple poles at z_0 and $\bar{z}_0$, with $\operatorname*{Res}_{z \to \bar{z}_0} F(z) = \bar{A}$ (by virtue of (36.4)). Whence, the function $g(z) = F(z) - \frac{A}{z - z_0} - \frac{\bar{A}}{z - \bar{z}_0}$ is entire and bounded. Therefore, by Liouville's theorem (Sec. 19), $g(z) \equiv \text{const}$, from which (36.7) follows. □

Example 3. Suppose $f(z)$ is regular in the half-plane $\operatorname{Im} z > 0$, is continuous up to the segments γ_1: $(-\infty, a)$ and γ_2: $(b, +\infty)$,

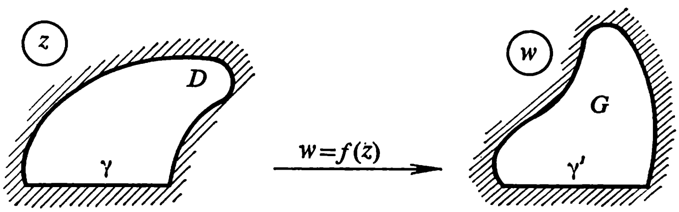


Fig. 130

with $-\infty < a \leqslant b < +\infty$, and assumes real values on these segments. Then the function given by (36.4) is regular in the entire complex z plane with a cut along the segment $[a, b]$. □

Example 4. Let D_0 be the complex z plane with a cut along the segment $[-1, 1]$ and with point $z = \infty$ deleted. We wish to prove that in D_0 we can isolate a regular branch of $\sqrt{z^2 - 1}$.

Indeed, by the monodromy theorem (Sec. 24), in the half-plane $\operatorname{Im} z > 0$ we can isolate a regular branch $f(z)$ of $\sqrt{z^2 - 1}$. Suppose, for the sake of definiteness, $f(x) > 0$ for $x > 1$. Then $f(x) < 0$ for $x < -1$ (Sec. 24), and Example 3 yields that (36.4) is a regular branch of $\sqrt{z^2 - 1}$ in D_0. □

36.2 Applications of the symmetry principle Suppose the conditions of Theorem 1 are met and also (Fig. 130)

(a) the function $w = f(z)$ maps domain D conformally onto a domain G in the upper half-plane $\operatorname{Im} w > 0$;

(b) the image of segment γ is a segment γ' of the real axis $\operatorname{Im} w = 0$ (γ' is a section of the boundary of G).

Then Theorem 1 yields

Corollary 2 *The function* $F(z)$ *defined by* (36.4) *maps domain* $D_0 = D \cup \gamma \cup D^*$ *conformally onto the domain* $G_0 = G \cup \gamma' \cup G^*$, *where* G^* *is the domain that is symmetric to* G *with respect to the real axis* $\operatorname{Im} w = 0$ (Fig. 131).

Example 5. Mapping the exterior of a cross onto a half-plane. Suppose D_0 is the complex z plane with cuts along the ray $[-4, +\infty)$

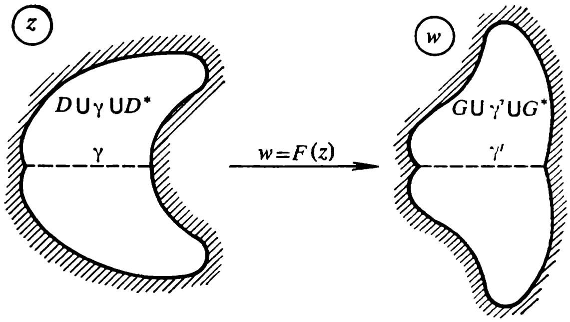


Fig. 131

and the segment $[-3i, 3i]$ (Fig. 132). Let us find the function that maps D_0 onto the upper half-plane $\operatorname{Im} w > 0$.

At first it seems that $\eta = z^2$ is the natural mapping (cf. Example 7 in Sec. 35). But this mapping is not conformal because the function z^2

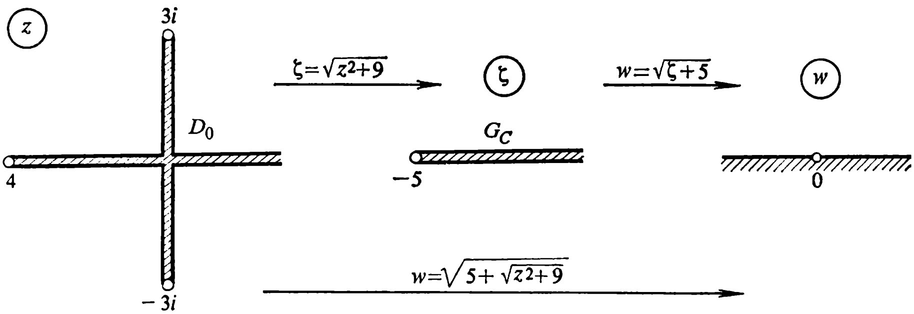


Fig. 132

is not univalent in D_0 (e.g. $(4i)^2 = (-4i)^2 = -16$). We will then take only a "half" of D_0. Suppose D is the upper half-plane $\operatorname{Im} z > 0$ with a cut along the segment $[0, 3i]$ (Fig. 133). In such a domain the function $\eta = z^2$ is univalent.

From Example 7 in Sec. 35 it follows that the function $\zeta = f(z) = \sqrt{z^2 + 9}$ maps D conformally onto the domain G: $\operatorname{Im} \zeta > 0$

(Fig. 133). Here $f(z)$ is a regular branch of $\sqrt{z^2+9}$ such that $f(x+i0)>0$ for $x>0$. Under such a mapping the segment γ: $(-\infty, -4)$ is mapped into the interval γ': $(-\infty, -5)$ (Fig. 133). In view of Corollary 2, the function $\zeta = F(z) = \sqrt{z^2+9}$ maps the domain $D_0 = D \cup \gamma \cup D^*$ conformally onto the domain $G_0 = G \cup \gamma' \cup G^*$, which is the complex ζ plane with a cut along the ray $[-5, +\infty)$ (Fig. 132). Here $F(z)$ is the analytic continuation

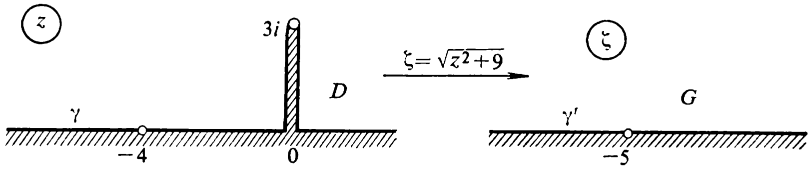


Fig. 133

of $f(z)$ into D_0, i.e. $F(z)$ is the regular branch of $\sqrt{z^2+9}$ in D_0 such that $F(x+i0)>0$ for $x>0$.

Using the function $w=\sqrt{\zeta+5}$ to map G_0 onto the half-plane $\operatorname{Im} w>0$, we arrive at the function $w=\sqrt{5+\sqrt{z^2+9}}$, which maps D_0 conformally onto the half-plane $\operatorname{Im} w>0$ (Fig. 132). □

Example 6. Mapping the interior of a parabola on a half-plane. Suppose D_0 is the domain $y^2<2p(x+p/2)$, with $z=x+iy$, $p>0$ (Fig. 134). Let us find the function that maps D_0 conformally onto the half-plane $\operatorname{Im} w>0$.

The parabola $y^2=2p(x+p/2)$ becomes a straight line under the mapping $\zeta=\sqrt{z}$ (see Example 6 in Sec. 35). But D_0 contains the branch point $z=0$ of $\sqrt{z}$. Then we take a "half" of D_0, i.e. we assume that D is the domain $y^2<2p(x+p/2)$, $y>0$ (Fig. 135). Let us find the function that maps D conformally onto the upper half-plane.

(1) The function $\zeta=\sqrt{z}$ maps D (Fig. 135) conformally onto the semistrip Π: $0<\operatorname{Im}\zeta<\sqrt{p/2}$, $\operatorname{Re}\zeta>0$ (see Example 3 in Sec. 35).

(2) The function $\eta=\cosh(\pi\sqrt{2}\zeta/\sqrt{p})$ maps Π (Fig. 135) conformally onto the half-plane $\operatorname{Im}\eta>0$ (see Example 23 in Sec. 35). Thus, the function $\eta=\cosh(\pi\sqrt{2z}/\sqrt{p})$ maps D conformally onto the half-plane $\operatorname{Im}\eta>0$ in a way such that the interval γ: $(-p/2, +\infty)$ is mapped into the interval γ': $(-1, +\infty)$ (Fig. 135). By Corollary 2, the function $\eta=\cosh(\pi\sqrt{2z}/\sqrt{p})$ maps D_0 conformally onto the complex plane with a cut along the ray $(-\infty, -1]$ (Fig. 134).

(3) The function $w = \sqrt{-\eta - 1}$ maps the complex η plane with a cut along the ray $(-\infty, -1)$ onto the half-plane $\operatorname{Im} w > 0$ (Fig. 134).

The final result is as follows: the function $w = i\sqrt{2} \cosh (\pi\sqrt{z}/\sqrt{2p})$ maps D_0 conformally onto the half-plane $\operatorname{Im} w > 0$ (Fig. 134). □

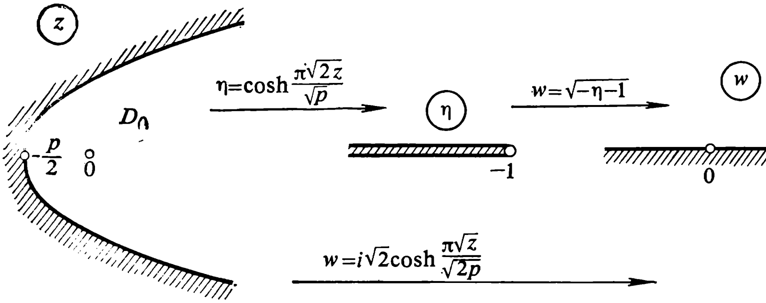


Fig. 134

Example 7. Mapping the interior of the right branch of a hyperbola onto a half-plane. Let us find the function that maps the domain D_0: $x^2/\cos^2 \alpha - y^2/\sin^2 \alpha > 1$, $x > 0$ (Fig. 136), where $z = x + iy$, $0 < \alpha < \pi/2$, conformally onto the half-plane $\operatorname{Im} w > 0$.

The function $\tau = z + \sqrt{z^2 - 1}$, which is the inverse of the Zhukovskii function (see Sec. 35), "straightens out" the hyperbola.

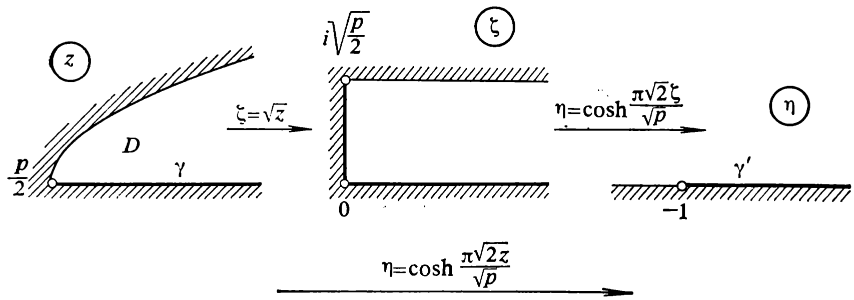


Fig. 135

But D_0 contains the branch point $z = 1$ of this function. For this reason we take only a half of D_0; suppose D is the domain $x^2/\cos^2 \alpha - y^2/\sin^2 \alpha > 1$, $x > 0$, $y > 0$ (Fig. 137). Let us find the function that maps D conformally onto the upper half-plane.

Carrying out the following mappings one after the other:

$$\tau = z + \sqrt{z^2 - 1} = e^{\operatorname{arcosh} z}, \quad \eta = \tau^{\pi/\alpha}, \quad \zeta = \frac{1}{2}\left(\eta + \frac{1}{\eta}\right)$$

(see Examples 19f, 9 and 19a in Sec. 35), we find that the function $\zeta = \cosh\left(\frac{\pi}{\alpha}\operatorname{arcosh} z\right)$ maps D conformally onto the half-plane $\operatorname{Im}\zeta > 0$, with the interval γ: $(\cos\alpha, +\infty)$ mapped into the interval γ': $(-1, +\infty)$ (Fig. 137). By Corollary 2, the function $\zeta =$

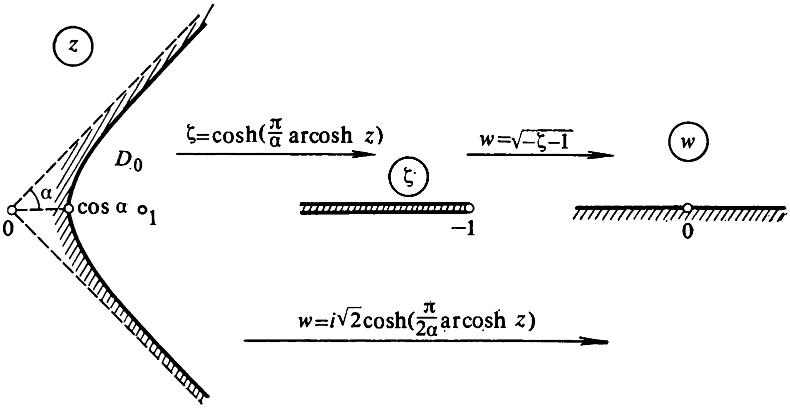


Fig. 136

$\cosh\left(\frac{\pi}{\alpha}\operatorname{arcosh} z\right)$ maps D_0 onto a domain G_0, the complex ζ plane with a cut along the ray $(-\infty, -1]$ (Fig. 136).

The function $w = \sqrt{-\zeta - 1}$ maps G_0 onto the half-plane $\operatorname{Im} w > 0$ (Fig. 136). Thus, the function $w = i\sqrt{2}\cosh\left(\frac{\pi}{2\alpha}\operatorname{arcosh} z\right)$ maps D_0 conformally onto the half-plane $\operatorname{Im} w > 0$ (Fig. 136). □

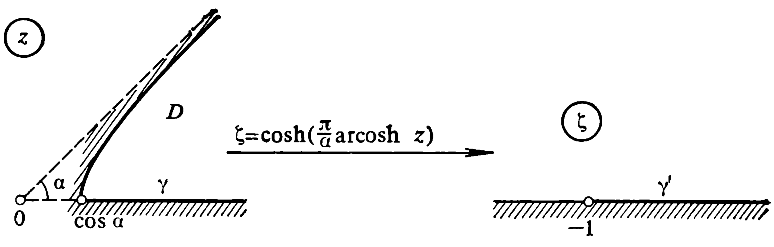


Fig. 137

Remark. Theorem 1 and Corollary 2 can easily be employed in the case where γ and γ' are arcs of circles (in particular, segments of straight lines). For this we must use a linear-fractional transformation that maps γ and γ' into segments of the real axis and then use the property of symmetry preservation in linear-fractional mappings. For more details see Lavrent'ev and Shabat [1].

Example 8. Let D be the complex z plane with cuts along the segments $[0, e^{2k\pi i/n}]$, $k = 0, 1, \ldots, n-1$ (Fig. 138). We wish to find the function that maps D conformally onto the exterior of the unit circle.

Consider the sector D_0: $0 < \arg z < 2\pi/n$ (Fig. 139). We want to find the function $w = f_0(z)$ that maps D_0 conformally onto the

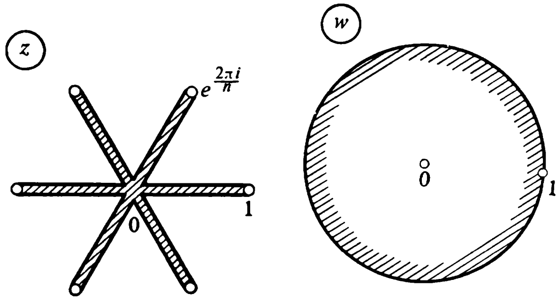


Fig. 138

sector G_0: $0 < \arg w < 2\pi/n$, $|w| > 1$ (Fig. 139), and satisfies the following conditions

$$f_0(1) = 1, \quad f_0(\infty) = \infty, \quad f_0(e^{2\pi i/n}) = e^{2\pi i/n}. \tag{36.8}$$

Carrying out the following mappings one after another:

$$\zeta = z^{n/2}, \quad \eta = \zeta + \sqrt{\zeta^2 - 1}, \quad w = \eta^{2/n}$$

(see Examples 9 and 19a in Sec. 35), we find that the sought-for function is $w = f_0(z) = (z^{n/2} + \sqrt{z^n - 1})^{2/n}$ (Fig. 139). Here $f_0(z)$

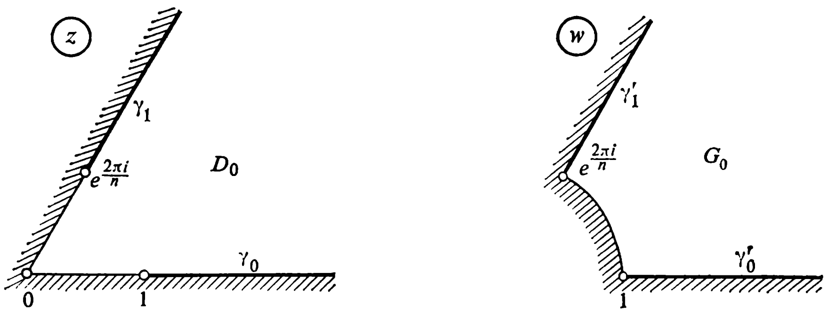


Fig. 139

is the regular branch of $(z^{n/2} + \sqrt{z^n - 1})^{2/n}$ that satisfies the conditions (36.8) in D_0. The function maps the ray γ_0: $(1, +\infty)$ into the ray γ_0': $(1, +\infty)$ and the ray γ_1: $(e^{2\pi i/n}, +\infty\, e^{2\pi i/n})$ into the ray γ_1': $(e^{2\pi i/n}, +\infty\, e^{2\pi i/n})$.

We wish to show that there is an analytic continuation $F(z)$ of the function $f_0(z)$ into the domain D and that the function $w = F(z)$ maps D conformally onto the domain $|w| > 1$ (Fig. 138).

Let us take the sector D_1: $2\pi/n < \arg z < 4\pi/n$ adjacent to D_0 and the sector G_1: $2\pi/n < \arg w < 4\pi/n$, $|w| > 1$. The points of D_1 are obtained from the points of D_0 by multiplying the latter by $e^{2\pi i/n}$; the points of G_1 and G_0 are related in a similar manner. Hence, the function

$$w = f_1(z) = e^{2\pi i/n} f_0(e^{-2\pi i/n} z), \quad z \in D_1,$$

maps D_1 conformally onto G_1, with $f_1(z) \equiv f_0(z)$, $z \in \gamma_1$, i.e. $f_1(z)$ is the analytic continuation of $f_0(z)$ from D_0 into D_1 across γ_1.

Similarly, if D_k is the sector (angle) $2k\pi/n < \arg z < 2(k+1)\pi/n$ and G_k is the sector $2k\pi/n < \arg w < 2(k+1)\pi/n$, $|w| > 1$, the function

$$w = f_k(z) = e^{2k\pi i/n} f_0(e^{-2k\pi i/n} z), \quad z \in D_k,$$

maps D_k conformally on G_k ($k = 2, 3, \ldots, n$), with $f_k(z) \equiv f_{k-1}(z)$, $z \in \gamma_k$, where γ_k is the ray $(e^{2k\pi i/n}, +\infty\, e^{2k\pi i/n})$.

Obviously, D_n coincides with D_0 and

$$f_n(z) = e^{2n\pi i/n} f_0(e^{-2n\pi i/n} z) = f_0(z), \quad z \in D_0.$$

Hence, the function $w = F(z) = f_k(z)$, $z \in D_k \cup \gamma_k$, $k = 0, 1, \ldots, n-1$, is regular in D and maps D conformally onto $|w| > 1$.

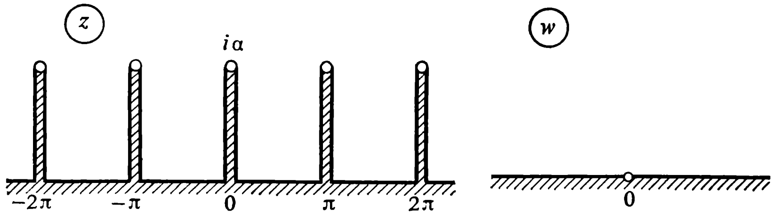


Fig. 140

Thus, the function $w = (z^{n/2} + \sqrt{z^n - 1})^{2/n}$ maps D conformally onto the domain $|w| > 1$. □

Example 9. Let D be the half-plane $\operatorname{Im} z > 0$ with cuts along the segments $[k\pi, k\pi + i\alpha]$, $k = 0, \pm 1, \pm 2, \ldots$, $0 < \alpha < +\infty$ (Fig. 140). We wish to find the function that maps D conformally onto the half-plane $\operatorname{Im} w > 0$.

We start with the semistrip D_0: $-\pi < \operatorname{Re} z < 0$, $\operatorname{Im} z > 0$ (Fig. 141). Let us find the function $w = f_0(z)$ that maps D_0 conformally onto the semistrip G_0: $-\pi < \operatorname{Re} w < 0$, $\operatorname{Im} w > 0$ (Fig. 141) and satisfies the conditions

$$f_0(i\alpha) = 0, \quad f_0(-\pi + i\alpha) = -\pi, \quad f_0(\infty) = \infty. \tag{36.9}$$

Carrying out the following mappings one after another:

$$\zeta = \cos z, \quad \eta = \zeta/\cosh \alpha, \quad w = \operatorname{arc}\cos \eta$$

(see Example 25 in Sec. 35), we arrive at the sought-for mapping:

$$w = f_0(z) = \operatorname{arc}\cos \frac{\cos z}{\cosh \alpha},$$

where $f_0(z)$ is the regular branch of $\operatorname{arc}\cos \frac{\cos z}{\cosh \alpha}$ in D_0 that satisfies the conditions (36.9). Under this mapping the ray γ_0: $(i\alpha, i\alpha +$

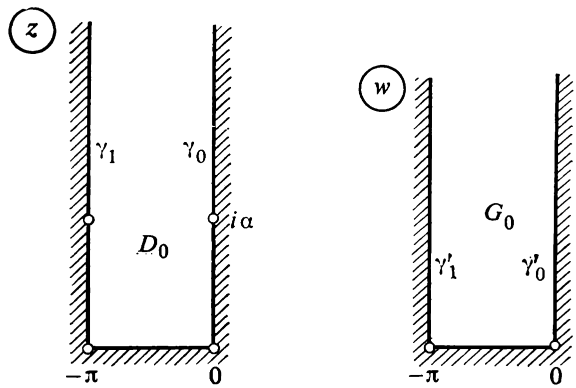


Fig. 141

$i\infty)$ is mapped into the ray γ'_0: $(0, 0 + i\infty)$ and the ray γ_{-1}: $(-\pi + i\alpha, -\pi + i\infty)$ into the ray γ'_{-1} : $(-\pi, -\pi + i\infty)$.

We can show that there is an analytic continuation $F(z)$ of $f_0(z)$ into D and that the function $w = F(z)$ maps D conformally onto the half-plane $\operatorname{Im} w > 0$.

Let us take the semistrip D_1: $0 < \operatorname{Re} z < \pi$, $\operatorname{Im} z > 0$ adjacent to D_0. The points of D_1 can be obtained from the points of D_0 by adding π to the latter; the domains G_1 and G_0 are related in a similar manner. Hence, the function

$$w = f_1(z) = f_0(z - \pi) + \pi, \quad z \in D_1$$

maps D_1 conformally onto G_1, with $f_1(z) \equiv f_0(z)$ on the ray γ_0, i.e. $f_1(z)$ is the analytic continuation of $f_0(z)$ from D_0 into D_0 across γ_0.

Similarly, if D_k is the semistrip $(k-1)\pi < \operatorname{Re} w < k\pi$, $\operatorname{Im} w > 0$ and G_k is the semistrip $(k-1)\pi < \operatorname{Re} w < k\pi$, $\operatorname{Im} w > 0$, then the function

$$w = f_k(z) = f_0(z - k\pi) + k\pi, \quad z \in D_k,$$

maps D_k conformally onto G_k, $k = 0, \pm 1, \pm 2, \ldots$, with $f_k(z) \equiv f_{k-1}(z)$ on the ray γ_{k-1}: $((k-1)\pi + i\alpha, \quad (k-1)\pi + i\infty)$. Hence, the function

$$w = F(z) = f_k(z), \quad z \in D_k \cup \gamma_k, \quad k = 0, \pm 1, \pm 2, \ldots,$$

is regular in D and maps D conformally onto the half-plane $\operatorname{Im} w > 0$ (Fig. 140). Thus, the function

$$w = \arccos \frac{\cos z}{\cosh \alpha}$$

maps D conformally on the upper half-plane $\operatorname{Im} w > 0$. □

Example 10. Suppose $w = f(z)$ is the function that maps the annulus K: $\rho < |z| < R$ conformally onto the annulus K': $\rho' < |w| < R'$. Let us prove that these annuluses are similar, i.e. $\rho/\rho' = R/R'$.

Proof. Two cases are possible: (1) the circle $|z| = \rho$ is mapped into the circle $|w| = \rho'$, and (2) the circle $|z| = \rho$ is mapped into the circle $|w| = R'$.

We start with the first case. By the Riemann-Schwarz symmetry principle, there is an analytic continuation $F_1(z)$ of $f(z)$ into the annulus K_1: $\rho_1 < |z| < R$, with $\rho_1 = \rho^2/R$. The function $w = F_1(z)$ maps K_1 conformally onto the annulus K_1'': $\rho_1' < |w| < R'$, where $\rho_1' = (\rho')^2/R$, in such a way that the circle $|z| = \rho_1$ is mapped into the circle $|w| = \rho_1''$. Similarly, there is an analytic continuation $F_2(z)$ of $F_1(z)$ (and $f(z)$) into the annulus K_2: $r_2 < |z| < R$, where $r_2 = \rho^4/R^3$, and so on. In this way we establish that there is an analytic continuation $F(z)$ of $f(z)$ into the annulus $0 < |z| < R$, with $\lim_{z \to 0} F(z) = 0$. Then point $z = 0$ is a removable singular point for $F(z)$ (see Sec. 18), i.e. the function $w = F(z)$ maps the circle $|z| < R$ conformally onto the circle $|w| < R'$, with $F(0) = 0$. Hence, $F(z)$ is a linear-fractional function and $F(\infty) = \infty$ (see Sec. 34), i.e. $f(z) = Az$, whence $\rho/\rho' = R/R'$. □

37 The Schwarz-Cristoffel Transformation Formula

We will study the conformal mapping $w = f(z)$ of the upper half-plane $\operatorname{Im} z > 0$ onto a polygon Π in the complex w plane. We will use the following notations (Fig. 142): the A_k are the vertices of Π, $k = 1, 2, \ldots, n$, the $\pi\alpha_k$ are the angles of Π at the A_k, with $\sum_{k=1}^{n} \alpha_k = n - 2$, and the a_k are the preimages of the A_k in relation to the mapping $w = f(z)$, i.e. $f(a_k) = A_k$.

37.1 The Schwarz-Cristoffel theorem

Theorem 1 *Suppose a function* $w = f(z)$ *maps the half-plane* $\mathrm{Im}\, z > 0$ *conformally onto a bounded polygon* Π, $0 < \alpha_k \leqslant 2$, $a_k \neq \infty$ $(k = 1, 2, \ldots, n)$. *Then the Schwarz-Cristoffel transformation formula is valid*:

$$f(z) = c \int_{z_0}^{z} (\zeta - a_1)^{\alpha_1 - 1} (\zeta - a_2)^{\alpha_2 - 1} \ldots (\zeta - a_n)^{\alpha_n - 1} \, d\zeta + c_1, \quad (37.1)$$

where c *and* c_1 *are real, and the integral is taken along a curve lying in the half-plane* $\mathrm{Im}\, z > 0$.

Proof. According to Riemann's mapping theorem (Sec. 33), there is a function $w = f(z)$ that maps the half-plane $\mathrm{Im}\, z > 0$ conform-

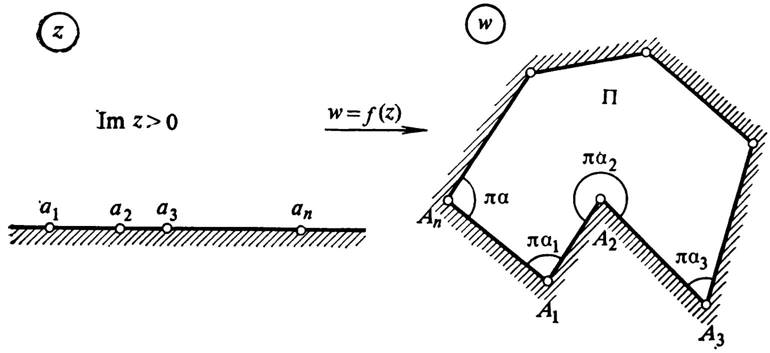


Fig. 142

ally onto a bounded polygon Π in a way such that $a_k \neq \infty$ $(k = 1, 2, \ldots, n)$. Let us investigate the properties of this function.

(1) Suppose $F(z)$ is an analytical function with the initial element $f(z)$, $\mathrm{Im}\, z > 0$. We will show that $F(z)$ is analytic in the entire extended complex plane with the point a_k $(k = 1, 2, \ldots, n)$ deleted.

We will use the Riemann-Schwarz symmetry principle (see Sec. 36). The function $w = f(z)$ maps the interval γ_k: (a_k, a_{k+1}) into the interval Γ_k: (A_k, A_{k+1}), with $a_{n+1} = a_1$ and $A_{n+1} = A_1$, and one of the intervals γ_k, $k = 1, 2, \ldots, n$, containing the point $z = \infty$ in its interior. According to the Riemann-Schwarz symmetry principle, there is an analytic continuation $f_k^*(z)$ of the function $f(z)$ into the half-plane $\mathrm{Im}\, z < 0$ across γ_k. The function $w = f_k^*(z)$ maps the half-plane $\mathrm{Im}\, z < 0$ conformally onto the polygon Π_k^*, which is symmetric to Π with respect to the straight line l_k of which the interval Γ_k is a part (Fig. 143).

Moreover, in view of the Schwarz-Cristoffel symmetry principle, there is an analytic continuation $f_{kj}(z)$ of $f_k^*(z)$ into the half-plane

$\operatorname{Im} z > 0$ across γ_j. The function $w = f_{kj}(z)$ maps the half-plane $\operatorname{Im} z > 0$ conformally onto the polygon Π_{kj}, which is symmetric to the polygon Π_k^* with respect to l_j^* (Fig. 143). Repeating this process, we can find the analytic continuation of $f_{kj}(z)$ into the half-plane $\operatorname{Im} z < 0$ across γ_s, and so on. All these continuations define

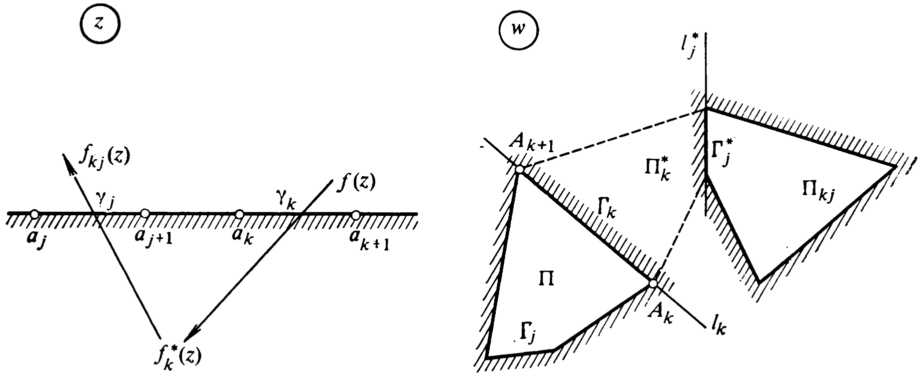


Fig. 143

a function $F(z)$ that is analytic in the entire extended complex plane with the points a_k $(k = 1, 2, \ldots, n)$ deleted.

(2) Let us prove that $F''(z)/F'(z)$ is single-valued and regular in the entire extended complex z plane with the points a_k $(k = 1, 2, \ldots, n)$ deleted. We will employ the same notations. Let us show that

$$f_k^*(z) = e^{i\beta_k}\overline{f(\bar{z})} + B_k, \quad \operatorname{Im} z < 0, \tag{37.2}$$

where β_k is real. Indeed, by the Riemann-Schwarz symmetry principle, the points $w = f(\bar{z})$ and $w^* = f_k^*(z)$ $(\operatorname{Im} z < 0)$ are symmetric with respect to the straight line l_k. The linear function $\zeta = (w - A_k)e^{-i\varphi_k}$, where $\varphi_k = \arg(A_{k+1} - A_k)$, maps l_k into the real axis $\operatorname{Im}\zeta = 0$, and points symmetric with respect to l_k into points symmetric with respect to $\operatorname{Im}\zeta = 0$ (see Sec. 36). Hence

$$(w^* - A_k)e^{-i\varphi_k} = \overline{[(w - A_k)e^{-i\varphi_k}]},$$

from which (37.2) follows.

Reasoning along the same lines, we obtain

$$f_{kj}(z) = e^{i\beta_j}\overline{f_k^*(\bar{z})} + B_j, \quad \operatorname{Im} z > 0. \tag{37.3}$$

Combining (37.2) with (37.3), we obtain

$$f_{kj}(z) = e^{i\beta_{kj}} f(z) + B_{kj}, \quad \operatorname{Im} z > 0.$$

Following exactly the same line of reasoning, we find that

$$\tilde{f}(z) = e^{i\beta} f(z) + B, \quad \operatorname{Im} z > 0,$$

for any element $\tilde{f}(z)$, $\operatorname{Im} z > 0$, of function $F(z)$, whence $\tilde{f}''(z)/\tilde{f}'(z) = f''(z)/f'(z)$. Similarly, for any element $\tilde{f}^*(z)$ $\operatorname{Im} z < 0$ of function $F(z)$ we obtain $(\tilde{f}^*)''/(\tilde{f}^*)' = (f_k^*)''/(f_k^*)'$. Thus, the function $g(z) = F''(z)/F'(z)$ is single-valued.

Let us prove that this function is regular in the entire extended complex z plane with the points a_k $(k = 1, 2, \ldots, n)$ deleted. Indeed, the function $f(z)$ is regular in the half-plane $\operatorname{Im} z > 0$ and $f'(z) \neq 0$, since the mapping $w = f(z)$, $\operatorname{Im} z > 0$, is conformal. For this reason the function $g(z) = f''(z)/f'(z)$ is regular in the half-plane $\operatorname{Im} z > 0$. The fact that $g(z)$ is regular in the half-plane $\operatorname{Im} z < 0$ can be proved similarly.

Further, in view of the Riemann-Schwarz symmetry principle, the function $f(z)$ is regular and univalent at each point of the real axis $\operatorname{Im} z = 0$ except at the points a_k $(k = 1, 2, \ldots, n)$. Whence $f'(z) \neq 0$ for $\operatorname{Im} z = 0$, $z \neq \infty$, $z \neq a_k$ $(k = 1, 2, \ldots, n)$, and $f(z)$ can be expanded about point $z = \infty$ in the series

$$f(z) = c_0 + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \ldots, \quad |z| > R,$$

with $c_{-1} \neq 0$. Hence, the function $f''(z)/f'(z)$ and, therefore, $g(z) = F''(z)/F'(z)$ are regular in the entire extended complex z plane with the points a_k $(k = 1, 2, \ldots, n)$ deleted and can be expanded in a power series about $z = \infty$:

$$\frac{F''(z)}{F'(z)} = -\frac{2}{z} + \frac{b_{-2}}{z^2} + \ldots, \quad |z| > R.$$

(3) Let us prove that in a neighborhood of point a_k the function $f(z)$ has the form

$$f(z) = A_k + (z - a_k)^{\alpha_k} h_k(z), \tag{37.4}$$

where $h_k(z)$ is regular at a_k, $h_k(a_k) \neq 0$.

Consider the semicircle K: $|z - a_k| < \varepsilon$, $\operatorname{Im} z > 0$, with ε small and positive (Fig. 144). The function $w = f(z)$ maps K conformally onto a domain Δ, the part of the polygon Π that lies in the neighborhood of A_k.

The function $\zeta = (w - A_k)^{1/\alpha_k}$ maps Δ conformally onto a domain G, the part of the half-plane that lies in the neighborhood of point $\zeta = 0$. Hence, the function $\zeta = g_k(z) = [f(z) - A_n]^{1/\alpha_k}$ maps the semicircle K conformally onto G in a way such that $g_k(a_k) = 0$, and the image of the interval γ: $(a_k - \varepsilon, a_k + \varepsilon)$ is the interval γ' containing point $\zeta = 0$ (Fig. 144). In view of the Riemann-Schwarz symmetry principle, the function $g_k(z)$ is regular at point $z = a_k$ and $g_k'(a_k) \neq 0$, i.e. $g_k(z) = (z - a_k)\,\tilde{g}_k(z)$, where $\tilde{g}_k(z)$

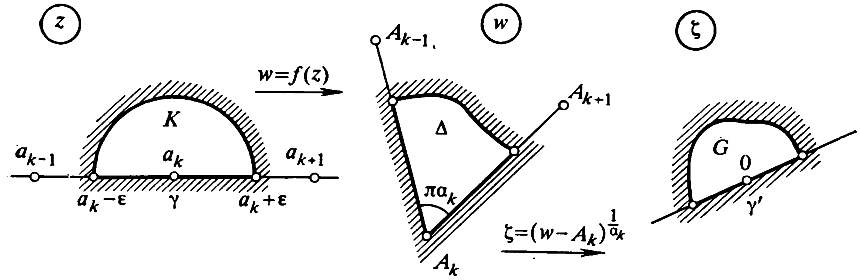


Fig. 144

is regular at point $z = a_k$ and $\tilde{g}_k(a_k) \neq 0$. The formula

$$[f(z) - A_k]^{1/\alpha_k} = (z - a_k)\,\tilde{g}_k(z)$$

yields (37.4).

(4) Let us prove that

$$\frac{f''(z)}{f'(z)} = \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - a_k}. \tag{37.5}$$

From (37.4) we obtain

$$\frac{f''(z)}{f'(z)} = \frac{\alpha_k - 1}{z - a_k} + \frac{\psi_k'(z)}{\psi_k(z)},$$

where $\psi_k(z) = \alpha_k h_k(z) + (z - a_k)\,h_k'(z)$ is regular at point $z = (a_k$, and $\psi_k(a_k) = \alpha_k h_k(a_k) \neq 0$. Hence, the function $g(z) = F''(z)/F'(z)$, t equal to $f''(z)/f'(z)$ in the neighborhood of a_k (Property 2), has a first order pole at a_k and $\operatorname*{Res}_{z=a_k} g(z) = \alpha_k - 1$. If we combine this with Property 2, we find that the function

$$H(z) = \frac{F''(z)}{F'(z)} - \sum_{k=1}^{n} \frac{\alpha_k - 1}{z - a_k}$$

is regular in the entire extended complex plane and tends to zero as $z \to \infty$. Hence, by Liouville's theorem (Sec. 19), $H(z) \equiv 0$, which for $\operatorname{Im} z > 0$ yields (37.5).

(5) Let us prove the validity of the Schwarz-Cristoffel transformation formula (37.1). Integrating (37.5) along a curve that lies in the

half-plane $\operatorname{Im} z > 0$ with one of its ends at the fixed point z_0 and the other at point z, we obtain

$$\ln f'(z) = \sum_{k=1}^{n} (\alpha_k - 1) \ln (z - a_k) + \tilde{c},$$

whence

$$f'(z) = c (z - a_1)^{\alpha_1 - 1} (z - a_2)^{\alpha_2 - 1} \ldots (z - a_n)^{\alpha_n - 1}.$$

Integration of the last formula results in the Schwarz-Cristoffel transformation formula (37.1). The proof of Theorem 1 is complete.

37.2 Calculating the parameters in the Schwarz-Cristoffel transformation formula Formula (37.1) makes it possible to find the form of the function $w = f(z)$ that maps the half-plane $\operatorname{Im} z > 0$ conformally onto a bounded polygon Π. Thus, if the polygon is given, i.e. if we know the vertices A_k and the angles $\pi\alpha_k$ $(k = 1, 2, \ldots, n)$, the problem of finding the function $f(z)$ is reduced to finding the points a_k $(k = 1, 2, \ldots, n)$ and the constants c and c_1. Any three points out of the collection of the a_k $(k = 1, 2, \ldots, n)$ can be selected arbitrarily (Sec. 33), but then the remaining a_k and c and c_1 are determined uniquely. Let us study one method by which we can determine these points and constants.

We write formula (37.1) thus:

$$f(z) = c \int_{z_0}^{z} h(\zeta)\, d\zeta + c_1, \tag{37.6}$$

where $h(z)$ is a regular branch of the function $(z - a_1)^{\alpha_1 - 1} (z - a_2)^{\alpha_2 - 1} \ldots (z - a_n)^{\alpha_n - 1}$ in the half-plane $\operatorname{Im} z > 0$. Since the various branches of this function differ by constant factors (see Sec. 24), in the above formula only c (precisely, $\arg c$) depends on which branch $h(z)$ is selected.

We will assume, for the sake of definiteness that points a_1, a_2, and a_3 are given, with $a_1 < a_2$, and that $z_0 = a_1$ (the constant c_1 depends on the choice of z_0). Putting $z = a_1$ in (37.6), we find that $c_1 = f(a_1) = A_1$. Let us find $\arg c$. Note that $\arg h(x) = \theta = \text{const}$ for $a_1 < x < a_2$. From (37.6) we have

$$A_2 - A_1 = f(a_2) - f(a_1) = ce^{i\theta} \int_{a_1}^{a_2} |h(t)|\, dt,$$

whence $\arg c = \arg (A_2 - A_1) - \theta$. Thus, we can write (37.6) as

$$f(z) = Ae^{i\alpha} \int_{a_1}^{z} h(\zeta)\, d\zeta + A_1, \tag{37.7}$$

where A is positive, and $\alpha = \arg (A_2 - A_1) - \theta$.

In formula (37.7) we still have to find $n - 2$ unknown parameters, namely, the positive number A and the real numbers $a_4, a_5, \ldots, a_n$. From (37.7) we obtain

$$A_{k+1}-A_k=Ae^{i\alpha}\int_{a_k}^{a_{k+1}} h(t)\,dt.$$

Bearing in mind that $\arg h(x) = \text{const}$ in the interval (a_k, a_{k+1}), we obtain

$$|A_{k+1}-A_k|=A\int_{a_k}^{a_{k+1}} |h(t)|\,dt, \quad k=1, 2, \ldots, n. \tag{37.8}$$

Here $a_{n+1} = a_1$, $A_{n+1} = A_1$, and one of the intervals (a_k, a_{k+1}) $k = \ , 2, \ldots, n$, contains point $z = \infty$.

The parameters $A, a_4, a_5, \ldots, a_n$ can be found by solving the system (37.8). Theorem 1 implies that this system has a unique solution. In reality, however, it is not often that one can find these parameters by solving (37.8). Other methods for finding the parameters in the Schwarz-Cristoffel transformation formula are available for simple polygons (see Examples 2 and 4 below).

37.3 Mapping a half-plane onto a triangle and a rectangle Theorem 1 deals with the case where all the a_k are finite. This leads to the following

Corollary 1 *Suppose a function $w = f(z)$ maps the half-plane $\operatorname{Im} z > 0$ conformally onto a bounded polygon Π in a way such that the $a_k \neq \infty$ $(k = 1, 2, \ldots, n - 1)$ but $a_n = \infty$. Then the following formula is valid:*

$$f(z)=c\int_{z_0}^{z}(\zeta-a_1)^{\alpha_1-1}(\zeta-a_2)^{\alpha_2-1}\ldots(\zeta-a_{n-1})^{\alpha_{n-1}-1}\,d\zeta+c_1. \tag{37.9}$$

We can obtain this formula from (37.1) via a linear-fractional mapping of the half-plane $\operatorname{Im} z > 0$ onto the half-plane $\operatorname{Im} \zeta > 0$ that maps point $z = a_n$ into point $\zeta = \infty$ (see Lavrent'ev and Shabat [1]). Note that there is one factor less in (37.9) compared to (37.1). Therefore, it is usually more convenient to use formula (37.9) instead of (37.1). We will assume that $z_0 = a_1$, so that $c_1 = A_1$.

Example 1. Mapping a half-plane onto a triangle. Let us find the function $w = f(z)$ that maps the half-plane $\operatorname{Im} z > 0$ conformally onto the bounded triangle Π with vertices at A_1, A_2, and A_3, where $A_1 = 0$, $A_2 = 1$, and $\operatorname{Im} A_3 > 0$ (Fig. 145). Here $0 < \alpha_k < 1$ $(k = 1, 2, 3)$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

We put $a_1 = 0$, $a_2 = 1$, $a_3 = \infty$ (Fig. 145). From (37.9) we have

$$f(z) = c \int_0^z \zeta^{\alpha_1 - 1} (\zeta - 1)^{\alpha_2 - 1} \, d\zeta = A \int_0^z \zeta^{\alpha_1 - 1} (1 - \zeta)^{\alpha_2 - 1} \, d\zeta,$$

where $A = ce^{\pi(\alpha_2 - 1) i}$, and the integrand assumes positive values in the interval (0, 1). From

$$A_2 - A_1 = 1 = A \int_0^1 t^{\alpha_1 - 1} (1 - t)^{\alpha_2 - 1} \, dt = A \times B(\alpha_1, \alpha_2)$$

we find that $A = 1/B(\alpha_1, \alpha_2)$, where $B(\alpha_1, \alpha_2)$ is the beta function (see Kudryavtsev [1]). Thus, the function

$$w = \frac{1}{B(\alpha_1, \alpha_2)} \int_0^z \zeta^{\alpha_1 - 1} (1 - \zeta)^{\alpha_2 - 1} \, d\zeta \tag{37.10}$$

maps the half-plane $\operatorname{Im} z > 0$ conformally onto the triangle Π. □

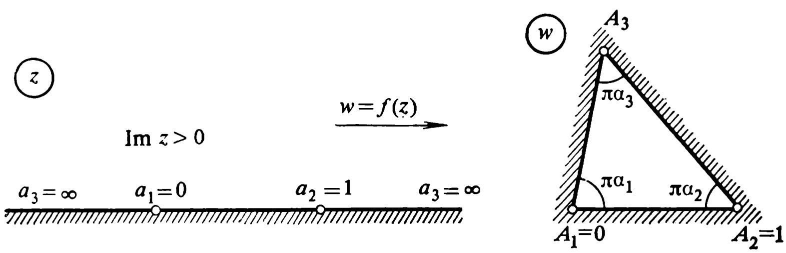


Fig. 145

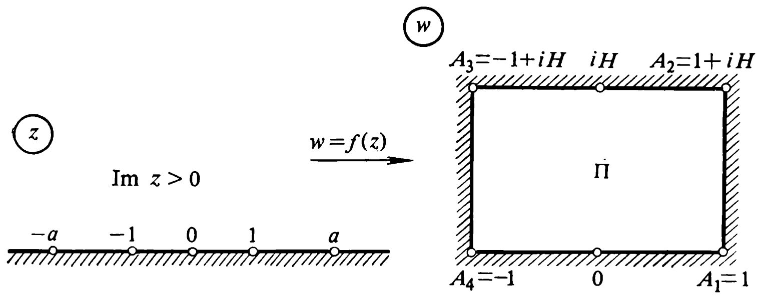


Fig. 146

Example 2. Mapping a half-plane onto a rectangle. Let us find the function $w = f(z)$ that maps the half-plane $\operatorname{Im} z > 0$ conformally onto the bounded rectangle Π with vertices at points A_k $(k = 1, 2, 3, 4)$, where $A_1 = 1$, $A_2 = 1 + iH$, $A_3 = -1 + iH$, $A_4 = -1$, $H > 0$, and $\alpha_k = 1/2$ $(k = 1, 2, 3, 4)$ (Fig. 146).

Let us take the right half of the rectangle Π, the rectangle Π_+ with vertices at points 0, 1, $1+iH$, and iH (Fig. 147). Suppose $w = f(z)$ maps the first quadrant $\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$ onto Π_+ in a way such that $f(0) = 0$, $f(1) = 1$, and $f(\infty) = iH$. It maps the interval γ: $(0, +i\infty)$ into the interval γ': $(0, iH)$, and the point $z = a$, $1 < a < +\infty$, is the preimage of point $w = 1 + iH$ (Fig. 147).

Using the Riemann-Schwarz symmetry principle (Sec. 36), we continue $f(z)$ analytically into the half-plane $\operatorname{Im} z > 0$ and denote

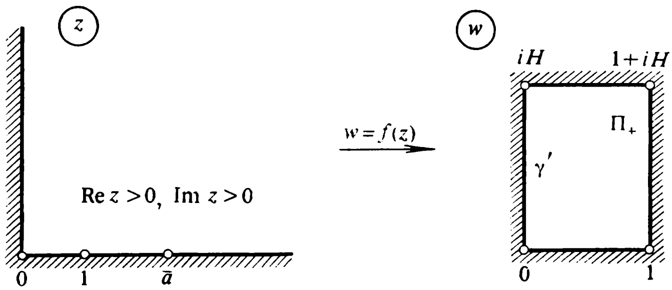


Fig. 147

the analytic continuation by $f(z)$, too. The function $w = f(z)$ maps the half-plane $\operatorname{Im} z > 0$ conformally onto Π (Fig. 146) in a way such that $f(0) = 0$, $f(\pm 1) = \pm 1$, $f(\pm a) = \pm 1 + iH$, and $f(\infty) = iH$.

Formula (37.1) yields

$$w = f(z) = A \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}, \tag{37.11}$$

where $k = 1/a$, $0 < k < 1$, and $\sqrt{(1-t^2)(1-k^2t^2)} > 0$ at $0 < t < 1$. Here, in view of (37.8), the parameters k, A, and H are related through the following equations:

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{1}{A}, \quad \int_1^{1/k} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} = \frac{H}{A}.$$

The integral (37.11) at $A = 1$ is known as *Legendre's normal elliptic integral of the first kind*. The inverse of (37.11), i.e. $z = \psi(w)$, is known as *Jacobi's elliptic function*. It maps the rectangle Π conformally onto the half-plane $\operatorname{Im} z > 0$.

Here are the main properties of $\psi(z)$:

(1) It is regular in the complex z plane everywhere except at the points

$$z = 2n + iH(2k + 1)$$

(here k and n are integers), which are simple poles for this function.

(2) It is two periods, $T_1 = 4$ and $T_2 = 2Hi$, i.e.

$$\psi(z + 4n + 2Hki) \equiv \psi(z)$$

(here k and n are integers).

We can prove these properties in the same way as we proved Theorem 1. Note that elliptic functions enable finding the conformal mapping of the interior of an ellipse onto a half-plane (see Lavrent'ev and Shabat [1]). Elliptic functions are considered in greater detail in Hurwitz and Courant [1] and Privalov [1].

37.4 Mapping a half-plane onto an open polygon Let us consider an open polygon Π that does not contain point $w = \infty$ in its interior. Suppose that one or more vertices of this polygon lie at point $w = \infty$. If $\pi\alpha_j$ is the angle at the vertex $A_j = \infty$ of Π, then $-2 \leqslant \alpha_j \leqslant 0$ (Sec. 33). This sign convention retains the relationship $\sum\limits_{k=1}^{n} \alpha_k = n - 2$.

We can prove that both Theorem 1 and Corollary 1 are valid, i.e. we have

Theorem 2 *The conformal mapping $w = f(z)$ of the half-plane* $\operatorname{Im} z > 0$ *onto Π is performed by*

(a) *function* (37.1) *if* $a_k \neq \infty$ $(k = 1, 2, \ldots, n)$,

(b) *function* (37.9) *if* $a_k \neq \infty$ $(k = 1, 2, \ldots, n - 1)$ *but* $a_n = \infty$.

The *proof* of the theorem is similar to that of Theorem 1 (for more details see Lavrent'ev and Shabat [1]).

Example 3. Suppose Π is a triangle with vertices at $A_1 = 0$, $A_2 = 1$, and $A_3 = \infty$ and $0 < \alpha_1 \leqslant 2$, $0 < \alpha_2 \leqslant 2$, and $-2 \leqslant \alpha_3 \leqslant 0$, with $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Theorem 2 and Example 1 imply that the function (37.10) maps the half-plane $\operatorname{Im} z > 0$ conformally onto Π.

Figure 148 shows examples of such triangles:

(a) $\alpha_1 = \frac{3}{4}, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = -\frac{1}{4}$;

(b) $\alpha_1 = \frac{5}{4}, \quad \alpha_2 = \frac{3}{4}, \quad \alpha_3 = -1$;

(c) $\alpha_1 = 2, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = -\frac{3}{2}$;

(d) $\alpha_1 = \frac{3}{2}, \quad \alpha_2 = \frac{3}{2}, \quad \alpha_3 = -2$.

Formula (37.10) then yields

$$\text{(a)}\quad w=\frac{2}{\pi}\,[\arcsin\sqrt[4]{z}-\sqrt[4]{z}\sqrt{1-\sqrt{z}}];$$

$$\text{(b)}\quad w=\frac{1}{\pi}\left[\arctan\frac{\zeta\sqrt{2}}{1-\zeta^2}+\operatorname{artanh}\frac{\zeta\sqrt{2}}{1+\zeta^2}-\frac{\zeta 2\sqrt{2}}{1+\zeta^4}\right],$$

$$\zeta=\sqrt[4]{\frac{z}{1-z}};$$

$$\text{(c)}\quad w=1-\left(1+\frac{z}{2}\right)\sqrt{1-z};$$

$$\text{(d)}\quad w=\frac{2}{\pi}\,[\arcsin\sqrt{z}+(2z-1)\sqrt{z\,(1-z)}].$$

Here we have taken the branches of multiple-valued functions that are positive for $z=x$, $0<x<1$. □

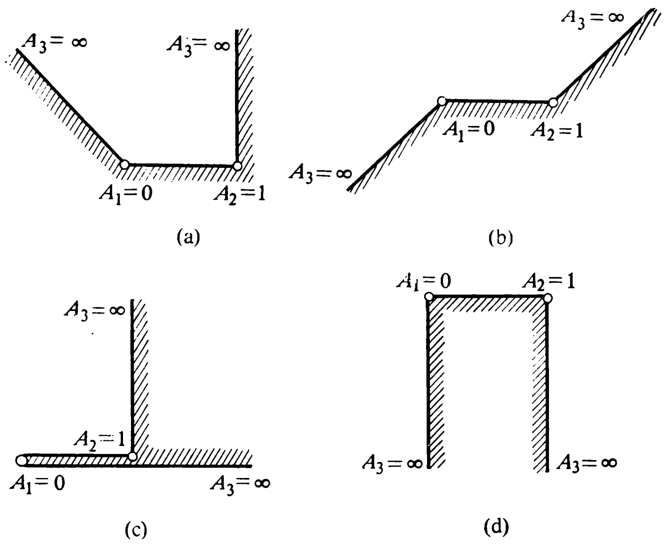


Fig. 148

Example 4. Let Π (a rectangle) be the strip $0<\operatorname{Im} w<\pi$ with a cut along the ray $(-\infty+ih\pi,\ ih\pi]$, $0<h<1$ (Fig. 149). Here $A_1=ih\pi, A_2=\infty, A_3=\infty, A_4=\infty, \alpha_1=2, \alpha_2=\alpha_3=\alpha_4=0$. We put $a_1=0$, $a_2=1$, and $a_3=\infty$. Then $a_4=-b$, with $0<b<+\infty$. In view of Theorem 2 and formula (37.9), the conformal mapping of the half-plane $\operatorname{Im} z > 0$ onto Π is performed by the function

$$w = f(z) = c \int_0^z \frac{\zeta\, d\zeta}{(\zeta - 1)(\zeta + b)} + ih\pi. \qquad (37.12)$$

Let us find arg c (cf. Sec. 37.2). We take the point z_1 in the interval $(0, 1)$: $z_1 = x_1$, $0 < x_1 < 1$. Its image is point $w_1 = f(z_1)$, which

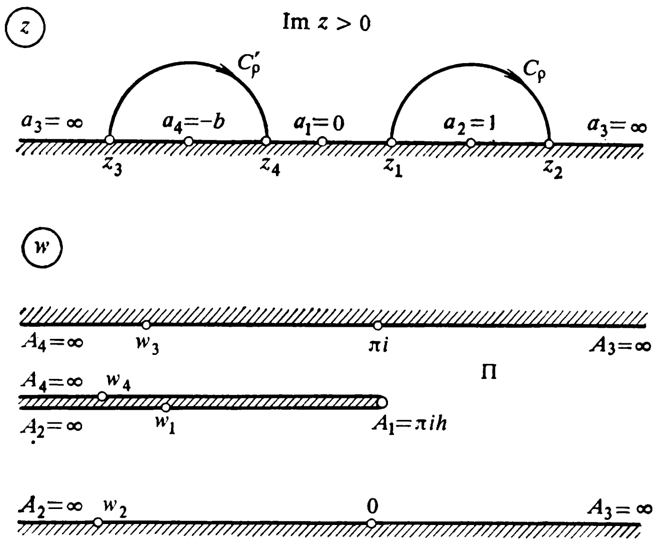


Fig. 149

lies on the side A_1A_2 of Π (Fig. 149), i.e. $\operatorname{Re} w_1 < 0$, $\operatorname{Im} w_1 = h\pi$. Then (37.12) yields

$$w_1 - A_1 = c \int_0^{x_1} \frac{t\, dt}{(t-1)(t+b)}.$$

Here $w_1 - A_1 = \operatorname{Re} w_1 < 0$, and the integrand is negative ($0 \leqslant t \leqslant x_1 < 1$). Hence c is positive.

Thus, two unknown parameters are left in (37.12), namely, $c < 0$ and $b > 0$. Let us find them.

Suppose $z_1 = 1 - \rho$ and $z_2 = 1 + \rho$, where $\rho > 0$ is small. Then point $w_1 = f(z_1)$ lies on side A_1A_2 and point $w_2 = f(z_2)$ on side

A_2A_3 (Fig. 149), whence $\operatorname{Im}(w_2 - w_1) = -\pi h$. From (37.12) we have

$$w_2 - w_1 = f(z_2) - f(z_1) = c \int\limits_{C_\rho} \frac{\zeta \, d\zeta}{(\zeta-1)(\zeta+b)}, \tag{37.13}$$

where C_ρ is the semicircle $|\zeta| = \rho$, $\operatorname{Im} \zeta \geqslant 0$ oriented clockwise (Fig. 149).

We consider the integral (37.13). Point $\zeta = 1$ is a first order pole for the integrand and

$$\operatorname*{Res}_{\zeta=1} \frac{\zeta}{(\zeta-1)(\zeta+b)} = \frac{1}{1+b}.$$

Hence,

$$\frac{\zeta}{(\zeta-1)(\zeta+b)} = \frac{1}{1+b} \frac{1}{\zeta-1} + g(\zeta),$$

where $g(\zeta)$ is regular at point $\zeta = 1$ and, therefore, is bounded in a neighborhood of this point: $|g(\zeta)| \leqslant M$. This means we can write the integral (37.13) as a sum of two integrals. The first is

$$\frac{c}{1+b} \int\limits_{C_\rho} \frac{d\zeta}{\zeta-1} = -\frac{c\pi i}{1+b}. \tag{37.14}$$

For the second integral we have the estimate

$$\int\limits_{C_\rho} g(\zeta) \, d\zeta = O(\rho) \quad (\rho \to 0) \tag{37.15}$$

since $\left| \int\limits_{C_\rho} g(\zeta) \, d\zeta \right| \leqslant M\pi\rho$.

From (37.13)-(37.15) we obtain

$$\operatorname{Im}(w_2 - w_1) = -\pi h = -\frac{\pi c}{1+b} + O(\rho) \quad (\rho \to 0),$$

whence, as $\rho \to 0$, we find that

$$\frac{c}{1+b} = h. \tag{37.16}$$

Note that the method of obtaining (37.16) from (37.12) we have just discussed can be applied to any polygon, with w in the neighborhood of the vertex A_k if $\alpha_k = 0$. Let us apply it to Π in the neighborhood of point A_4. We have

$$w_4 - w_3 = f(z_4) - f(z_3) = c \int\limits_{C'_\rho} \frac{\zeta \, d\zeta}{(\zeta-1)(\zeta+b)},$$

where $z_3 = -b - \rho$, $z_4 = -b + \rho$, $\mathrm{Im}\,(w_4 - w_3) = \pi\,(h-1)$, and C'_ρ is the semicircle $|\zeta + b| = \rho$, $\mathrm{Im}\,\zeta \geqslant 0$ (Fig. 149). This implies that

$$\mathrm{Im}\,(w_4 - w_3) = \pi\,(h-1) = -\frac{\pi bc}{1+b} + O(\rho) \quad (\rho \to 0)$$

and, as $\rho \to 0$, we find that

$$\frac{bc}{1+b} = 1 - h. \tag{37.17}$$

Solving the system of equations (37.16), (37.17), we find that

$$c = 1, \quad b = \frac{1-h}{h}.$$

Substituting these values into (37.12) and evaluating the integral, we obtain

$$w = f(z) = \ln\left[(z-1)^h \left(1 + \frac{hz}{1-h}\right)^{1-h}\right]. \ \square$$

Note that the function $\zeta = e^w$ (see Sec. 35.4) maps Π onto G, the

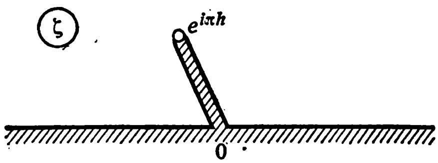


Fig. 150

half-plane $\mathrm{Im}\,\zeta > 0$ with a cut along the segment $[0, e^{i\pi h}]$ (Fig. 150). Hence, the function

$$\zeta = (z-1)^h \left(1 + \frac{hz}{1-h}\right)^{1-h}$$

maps the half-plane $\mathrm{Im}\,z > 0$ conformally onto G.

38 The Dirichlet Problem

A broad class of steady-state physical problems can be reduced to finding the harmonic functions that satisfy certain boundary conditions (e.g. see Lavrent'ev and Shabat [1] and Vladimirov [1]). In this section we will consider a method of solving such problems via conformal mappings.

38.1 Statement of the Dirichlet problem. The existence and uniqueness of a solution Suppose we know a continuous function $u_0(z)$ on the boundary Γ of a bounded domain D. The *classical Dirichlet problem* for Laplace's equation can be formulated thus: to find a function $u(z)$ that is harmonic in D, is continuous up to the boundary Γ, and assumes on Γ the values of $u_0(z)$:

$$\nabla^2 u = 0, \quad z \in D; \quad u\,|_{z\in\Gamma} = u_0(z). \tag{38.1}$$

Here and in what follows $u(z) = u(x, y)$ and $u_0(z) = u_0(x, y)$ are real functions, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is Laplace's operator.

The solution of the classical Dirichlet problem (38.1) exists and is unique. The existence proof can be found, for instance, in Vladimirov [1], while the uniqueness follows from the maximum and minimum principle for harmonic functions (Theorem 5 of Sec. 32).

Indeed, suppose $u_1(z)$ and $u_2(z)$ are two functions that are harmonic in D, are continuous up to Γ, and equal on Γ, i.e. $u_1\,|_{z\in\Gamma} = u_2\,|_{z\in\Gamma}$. Then the difference $u_1(z) - u_2(z)$ is harmonic in D, is continuous up to Γ, and is zero for $z \in \Gamma$. By Theorem 5 of Sec. 32, $u_1(z) - u_2(z) \equiv 0$ for $z \in D$, i.e. $u_1(z) \equiv u_2(z)$, $z \in D$.

Along with the classical Dirichlet problem (38.1) we will consider a more general Dirichlet problem, namely, when, $u_0(z)$ is bounded and has a finite number of discontinuities. We wish to find a function $u(z)$ that is harmonic in D, is bounded in D, is continuous up to the boundary at all points at which $u_0(z)$ is continuous, and satisfies the condition $u(z) = u_0(z)$ at such points. Note that D may not have a boundary.

The solution of this Dirichlet problem exists and is unique (e.g. see Vladimirov [1]).

The example below shows that if we lift the condition that $u(z)$ be bounded in D, the uniqueness theorem becomes invalid.

Example 1. (a) The function $u(x, y) = y$, which is harmonic in the half-plane $y > 0$, is continuous up to the boundary and is zero at $y = 0$ $(x \neq \infty)$. The function that is identically zero obviously satisfies the above-mentioned conditions, too.

(b) The function $u(x, y) = \frac{1-x^2-y^2}{(x-1)^2+y^2} = \operatorname{Re}\frac{1+z}{1-z}$, which is harmonic in the circle $x^2 + y^1 < 1$, is continuous up to the boundary $x^2 + y^2 = 1$ except at point $(1, 0)$, and is zero on $x^2 + y^2 = 1$ except at point $(1, 0)$. The function that is identically zero obviously satisfies this condition, too. □

37.2 The invariance of Laplace's equation under conformal mappings

Theorem 1 *Suppose a regular function $z = g(\zeta)$ maps a domain D conformally onto a domain D, and let $u(z)$ be a function that is harmonic in D. Then the function $\tilde{u}(\zeta) = u(g(\zeta))$ is harmonic in G.*

Proof. Let us take a simply connected domain $G_1 \subset G$. The image of G_1 obtained as a result of the conformal mapping $z = g(\zeta)$ is a simply connected domain $D_1 \subset D$. Suppose $f(z)$ is a function that is regular in D and such that $\operatorname{Re} f(z) = u(z)$ (the existence of such a function was proved earlier, in Sec. 7). Then the function $\tilde{f}(\zeta) =$

$f(g(\zeta))$ is regular in G_1 and, hence, $\tilde{u}(\zeta) = \operatorname{Re} \tilde{f}(\zeta)$ is harmonic in G_1 (see Sec. 7). Since G_1 is an arbitrary simply connected subdomain of G, we can assume that $\tilde{u}(\zeta)$ is harmonic in G.

Here is an alternative proof of Theorem 1. We introduce the functions $x(\xi, \eta) = \operatorname{Re} g(\zeta)$ and $y(\xi, \eta) = \operatorname{Im} g(\zeta)$, where $\zeta = \xi + i\eta$. Then the mapping $z = g(\zeta)$ $(z = x + iy)$ can be written thus:

$$x = x(\xi, \eta), \quad y = y(\xi, \eta). \tag{38.2}$$

Since $g(\zeta)$ is a regular function, the functions $x(\xi, \eta)$ and $y(\xi, \eta)$ satisfy the Cauchy-Riemann equations. This means that under the change of variables (38.2) we obtain

$$\frac{\partial^2 \tilde{u}}{\partial \xi^2} + \frac{\partial^2 \tilde{u}}{\partial \eta^2} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) |g'(\zeta)|^2. \tag{38.3}$$

This implies that if $u(z)$ is a harmonic function in the variables x and y, then $\tilde{u}(\zeta) = u(g(\zeta))$ is a harmonic function in the variables ξ and η, i.e. Laplace's equation is invariant under conformal mappings. This fact lies at the base of the method of solving the Dirichlet problem via conformal mappings.

Example 2. Suppose D is the domain $\operatorname{Im} z < 0$, $|z + il| > R$, where $l > R > 0$ (Fig. 125, Sec. 35). Let us solve the Dirichlet problem

$$\nabla^2 u = 0, \quad z \in D; \tag{38.4}$$

$$u\,|_{\operatorname{Im} z = 0}, \quad u\,|_{|z+il| = R} = T \equiv \text{const}. \tag{38.5}$$

Let us consider the conformal mapping $\zeta = h(z) = \frac{z + ia}{z - ia}$ of D onto the concentric annulus K: $R_1 < |\zeta| < 1$, where $a = \sqrt{l^2 - R^2}$ and $R_1 = (R + l - a)/(R + l + a)$ (Example 38 in Sec. 35). The straight line $\operatorname{Im} z = 0$ is mapped into the circle $|\zeta| = 1$, and the circle $|z + il| = R$ into the circle $|\zeta| = R_1$. Let $z = g(\zeta)$ be the inverse of the function $\zeta = h(z)$. By Theorem 1, the function $\tilde{u}(\zeta) = u(g(\zeta))$ is harmonic in K:

$$\nabla^2 \tilde{u} = 0, \quad \zeta \in K. \tag{38.6}$$

Conditions (38.5) then imply that

$$\tilde{u}\,|_{|\zeta| = 1} = 0, \quad \tilde{u}\,|_{|\zeta| = R_1} = T. \tag{38.7}$$

Thus, the Dirichlet problem (38.4), (38.5) has reduced itself to the Dirichlet problem (38.6), (38.7). Let us solve the latter.

Suppose $\zeta = \xi + i\eta = \rho e^{i\theta}$. After substituting $\xi = \rho\cos\theta$ and $\eta = \rho\sin\theta$ we can write Laplace's equation (38.6) as follows:

$$\frac{\partial^2 \tilde{u}}{\partial^2 \rho} + \frac{1}{\rho}\frac{\partial \tilde{u}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \tilde{u}}{\partial \theta^2} = 0.$$

Since the boundary functions in (38.7) do not depend on θ, it is natural to assume that the solution to (38.6), (38.7) is independent of θ, i.e. function $\tilde{u}(\zeta)$ depends only on one variable, ρ. If we find such a solution, we will have proved, in view of the fact that the Dirichlet problem can have only one solution, that the solution to (38.6), (38.7) does not depend on θ. When $\tilde{u}(\zeta)$ is independent of θ, Laplace's equation (38.6) is an ordinary differential equation:

$$\frac{d^2 \tilde{u}}{d\rho^2} + \frac{1}{\rho}\frac{d\tilde{u}}{d\rho} = 0.$$

The general solution of this equation is $\tilde{u}(\zeta) = c_1 + c_2 \ln\rho = c_1 + c_2 \ln|\zeta|$. The boundary conditions (38.7) yield $c_1 = 0$ and $c_2 = T/\ln R_1$, i.e. the function

$$\tilde{u}(\zeta) = \frac{T}{\ln R_1}\ln|\zeta|$$

is the solution to (38.6), (38.7). To find the solution to (38.4), (38.5) we need only to go over to the variables x and y ($z = x + iy$). Since $u(z) = \tilde{u}(h(z))$ and

$$|\zeta| = |h(z)| = \left|\frac{z+ia}{z-ia}\right| = \frac{|x^2+y^2-a^2+i2ax|}{x^2+(y-a)^2},$$

we conclude that the solution to (38.4), (38.5) is the function

$$u(x, y) = \frac{T}{\ln R_1}\ln\frac{\sqrt{(x^2+y^2-a^2)^2+4a^2x^2}}{x^2+(y-a)^2},$$

with $a = \sqrt{l^2 - R^2}$ and $R_1 = (R + l - a)/(R + l + a)$. □

38.3 The Dirichlet problem for Laplace's equation in a circle

Theorem 2 *Suppose a function $u(z)$ that is harmonic in the unit circle $|z| < 1$ is continuous in the closed circle $|z| \leqslant 1$. Then the f llowing Poisson integral formula is valid*:

$$u(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}\, u(e^{i\theta})\, d\theta, \qquad (38.8)$$

with $z = re^{i\varphi}$, $0 \leqslant r < 1$.

Proof. Note that at $z = 0$ this formula coincides with (10.6) (the mean value theorem for harmonic functions). Let us show that for $z \neq 0$ the value of $u(z)$ can also be found by employing the mean value theorem and a conformal mapping. We fix point $z_0 = r_0 e^{i\varphi_0}$, $0 \leqslant r_0 < 1$, and consider the conformal mapping

$$\zeta = h(z) = \frac{z - z_0}{1 - z\bar{z}_0} \tag{38.9}$$

of the circle $|z| < 1$ onto the circle $|\zeta| < 1$, $h(z_0) = 0$ (Sec. 34). From (38.9) we find that

$$z = g(\zeta) = \frac{\zeta + z_0}{1 + \zeta\bar{z}_0}. \tag{38.10}$$

The function $z = g(\zeta)$ maps the circle $|\zeta| < 1$ conformally onto the circle $|z| < 1$, so that $g(0) = z_0$.

By hypothesis, $u(z)$ is harmonic in $|z| < 1$ and continuous in the closed circle $|z| \leqslant 1$. Hence, the function $\tilde{u}(\zeta) = u(g(\zeta))$ is harmonic in the circle $|\zeta| < 1$ (Theorem 1) and continuous in the closed circle $|\zeta| \leqslant 1$. By the mean value theorem for harmonic functions (Sec. 10),

$$u(z_0) = \tilde{u}(0) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{u}(e^{i\psi})\, d\psi. \tag{38.11}$$

We return to the old variables. In (38.11) we introduce the substitution

$$e^{i\psi} = h(e^{i\theta}) = \frac{e^{i\theta} - z_0}{1 - e^{i\theta}\bar{z}_0}. \tag{38.12}$$

Then $\tilde{u}(e^{i\psi}) = \tilde{u}(h(e^{i\theta})) = u(e^{i\theta})$. From (38.12) we find that

$$d\psi = \frac{1 - |z_0|^2}{(e^{i\theta} - z_0)(e^{-i\theta} - \bar{z}_0)}\, d\theta = \frac{1 - r_0^2}{1 - 2r_0\cos(\varphi_0 - \theta) + r_0^2}\, d\theta. \tag{38.13}$$

If in (38.11)-(38.13) we substitute $z = re^{i\varphi}$ for $z_0 = r_0 e^{i\varphi_0}$, we arrive at the Poisson integral formula (38.8). The proof of the theorem is complete.

Let us transform the Poisson integral formula. We note that

$$\frac{1 - r^2}{1 - 2r\cos(\varphi - \theta) + r^2} = \frac{1 - |z|^2}{|e^{i\theta} - z|^2} = \operatorname{Re} \frac{e^{i\theta} + z}{e^{i\theta} - z},$$

which means that instead of (38.8) we can write

$$u(z) = \operatorname{Re} \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\theta, \tag{38.14}$$

since $u(e^{i\theta})$ is a real function. If in (38.14) we put $e^{i\theta} = \zeta$, and $d\theta = \partial\zeta/i\zeta$, we obtain

$$u(z) = \operatorname{Re} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} u(\zeta) \frac{\zeta+z}{\zeta-z} \frac{d\zeta}{\zeta}, \quad |z| < 1, \tag{38.15}$$

which is an alternative form of the Poisson integral formula.

Remark 1. The mean value theorem is valid for functions that are harmonic and bounded in a circle and are continuous up to the boundary of the circle everywhere except at a finite number of points. For this reason the Poisson integral formula is valid for such functions.

Corollary 1 *Let a function $f(z)$ be regular in the circle $|z| < 1$ and its real part $u(z) = \operatorname{Re} f(z)$ be continuous in the closed circle $|z| \leqslant 1$. Then the following Schwarz formula is valid:*

$$f(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} u(\zeta) \frac{\zeta+z}{\zeta-z} \frac{d\zeta}{\zeta} + i \operatorname{Im} f(0), \quad |z| < 1. \tag{38.16}$$

Proof. Let $F(z)$ be the integral on the right-hand side of (38.16). The function $F(z)$ is regular in the circle $|z| < 1$ (Theorem 1 of Sec. 16) and, by (38.15), $\operatorname{Re} F(z) = u(z) = \operatorname{Re} f(z)$. Hence, $F(z) = f(z) + iC$ (see Sec. 7.3), with C a real constant. Since

$$F(0) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} u\,|\zeta|\, \frac{d\zeta}{\zeta} = \frac{1}{2\pi} \int\limits_0^{2\pi} u(e^{i\theta})\, d\theta = u(0),$$

we conclude that $\operatorname{Im} F(0) = 0$. The fact that $\operatorname{Im} F(0) = \operatorname{Im} f(0) + C$ implies that $C = -\operatorname{Im} f(0)$, i.e. $F(z) = f(z) - i \operatorname{Im} f(0)$, from which formula (38.15) follows.

The Poisson integral formula can be used to solve the Dirichlet problem for Laplace's equation in the circle $|z| < 1$. For one, if the boundary function is a rational function of $\sin\varphi$ and $\cos\varphi$, the integral in (38.15) can be evaluated using the theory of residues.

Example 3. Let us find the solution to

$$\nabla^2 u = 0, \quad |z| < 1; \quad u|_{|z|=1} = \frac{\sin\varphi}{5+4\cos\varphi}, \tag{38.17}$$

where $z = re^{i\varphi}$. We will use formula (38.15). Suppose $\zeta = e^{i\theta}$. Then $\sin\theta = \frac{1}{2i}\left(\zeta - \frac{1}{\zeta}\right)$, $\cos\theta = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)$, and

$$u(\zeta) = \frac{\sin\theta}{5+4\cos\theta} = \frac{\zeta^2-1}{2i(2\zeta^2+5\zeta+2)}.$$

So we have to evaluate the integral

$$J = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{(\zeta^2-1)(\zeta+z)}{2i(2\zeta^2+5\zeta+2)(\zeta-z)\zeta}\, d\zeta,$$

where the circle $|\zeta| = 1$ is oriented counterclockwise. The $F(\zeta)$ in the domain $|\zeta| > 1$ has one finite singular point $\zeta = -2$, which is a first order pole, and a removable singularity at $\zeta = \infty$. Hence, by (28.19),

$$I = -\operatorname*{Res}_{\zeta=-2} F(\zeta) - \operatorname*{Res}_{\zeta=\infty} F(\zeta).$$

Then, by (28.3) and (28.12),

$$\operatorname*{Res}_{\zeta=-2} F(\zeta) = \frac{2-z}{4i(z+2)}, \quad \operatorname*{Res}_{\zeta=\infty} F(\zeta) = -\frac{1}{4i}, \qquad I = \frac{z}{2i(z+2)}.$$

Finally, from (38.15) follows the solution to (38.17):

$$u(x, y) = \operatorname{Re} \frac{z}{2i(z+2)} = \frac{y}{(x+2)^2+y^2} = \frac{r \sin \varphi}{r^2+4r\cos\varphi+4}. \ \square$$

The Dirichlet problem for Laplace's equation in an annulus $\rho < |z| < R$ (for one, in a circle or in the exterior of a circle) can also be found by separation of variables, or Fourier's method (e.g. see Vladimirov [1]). The following harmonic functions are used in this case:

$$\ln r, \ r^n \cos n\varphi, \ r^n \sin n\varphi, \ n = 0, \pm 1, \pm 2, \ \ldots . \qquad (38.18)$$

38.4 The Dirichlet problem for Laplace's equation in a half-plane

Theorem 3 *Suppose a function* $u(z)$ *that is harmonic and bounded in the half-plane* $\operatorname{Im} z > 0$ *is continuous up to the straight line* $\operatorname{Im} z = 0$ *everywhere except, perhaps, at a finite number of points. Then the following Poisson formula is valid*:

$$u(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y u(t)}{(t-x)^2+y^2}\, dt, \qquad (38.19)$$

with $z = x + iy$, *and* $y > 0$.

Proof. We fix a point $z_0 = x_0 + iy_0$, $y_0 > 0$, and consider the conformal mapping

$$\zeta = h(z) = \frac{z-z_0}{z-\bar{z}_0} \qquad (38.20)$$

of the half-plane $\operatorname{Im} z > 0$ onto the unit circle $|\zeta| < 1$, $h(z_0) = 0$ (Sec. 34). From (38.20) we find that

$$z = g(\zeta) = \frac{\zeta\bar{z}_0 - z_0}{\zeta - 1}. \qquad (38.21)$$

The function $z = g(\zeta)$ maps the circle $|\zeta| < 1$ conformally onto the half-plane $\operatorname{Im} z > 0$ in a way such that $g(0) = z_0$. By Theorem 1, $\tilde{u}(\zeta) = u(g(\zeta))$ is harmonic in $|\zeta| < 1$. By hypothesis, $\tilde{u}(\zeta)$ is bounded in $|\zeta| < 1$ and is continuous up to $|\zeta| = 1$ everywhere except at a finite number of points. By the mean value theorem (see Remark 1),

$$u(z_0) = \tilde{u}(0) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{u}(e^{i\psi})\,d\psi. \tag{38.22}$$

We return now to the old variables. In the integral in (38.22) we introduce the following substitution:

$$e^{i\psi} = h(t) = \frac{t-z_0}{t-\bar{z}_0}. \tag{38.23}$$

We have $\tilde{u}(e^{i\psi}) = \tilde{u}(h(t)) = u(t)$. From (38.23) we find that

$$d\psi = \frac{z_0-\bar{z}_0}{i\,|t-z_0|^2}\,dt = \frac{2y_0}{(t-x_0)^2+y_0^2}\,dt. \tag{38.24}$$

Substituting $z = x + iy$ for $z_0 = x_0 + iy_0$ in (38.22)-(38.24), we arrive at (38.19).

Since

$$\frac{y}{(t-x)^2+y^2} = \operatorname{Re}\frac{1}{i(t-z)},$$

we can write the Poisson formula (38.19) in the following form:

$$u(z) = \operatorname{Re}\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{u(t)}{t-z}\,dt. \tag{38.25}$$

With (38.19) or (38.25) we can solve the Dirichlet problem for Laplace's equation in the half-plane $\operatorname{Im} z > 0$. For example, let us consider the following problem:

$$\nabla^2 u = 0, \quad y > 0; \quad u|_{y=0} = R(x), \tag{38.26}$$

where the rational function $R(z)$ is real, has no poles on the real axis, and tends to zero as $z \to \infty$. By (38.25), the solution to this problem is given by the function

$$u(z) = \operatorname{Re}\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{R(t)}{t-z}\,dt,$$

with $\operatorname{Im} z > 0$. The integral can be evaluated via residues (see Sec. 29.2):

$$u(z) = -2\operatorname{Re}\sum_{\operatorname{Im}\zeta_k<0}\ \operatorname*{Res}_{\zeta=\zeta_k}\frac{R(\zeta)}{\zeta-z}. \tag{38.27}$$

Here the residues are taken over all the poles of $R(\zeta)$ that lie in the half-plane $\operatorname{Im}\zeta < 0$.

Example 4. We wish to find the solution to

$$\nabla^2 u = 0, \quad y > 0; \quad u|_{y=0} = \frac{1}{1+x^2}.$$

Formula (38.27) yields

$$u(z) = -2\operatorname{Re}\operatorname*{Res}_{\zeta=-i}\frac{1}{(1+\zeta^2)(\zeta-z)} = -2\operatorname{Re}\frac{1}{2i(z+i)}$$

$$= \frac{y+1}{x^2+(y+1)^2}. \quad \square$$

Let us take the Dirichlet problem for Laplace's equation in an arbitrary simply connected domain:

$$\nabla^2 u = 0, \quad z \in D; \quad u|_\Gamma = u_0(z), \tag{38.28}$$

where the boundary Γ of D consists of at least two boundary points. The solution to this problem can be found via a conformal mapping of D onto a circle or a half-plane and the Poisson formula.

Suppose the function $\zeta = h(z)$ maps D conformally onto the half-plane $\operatorname{Im}\zeta > 0$, with $z = g(\zeta)$ the inverse of $\zeta = h(z)$. Then $\tilde{u}(\zeta) = u(g(\zeta))$ is harmonic in the half-plane $\operatorname{Im}\zeta > 0$, and

$$\tilde{u}|_{\eta=0} = u_0(g(z))|_{z\in\Gamma} = \tilde{u}_0(\xi),$$

with $\zeta = \xi + i\eta$. By (38.25),

$$\tilde{u}(\zeta) = \operatorname{Re}\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\tilde{u}_0(t)}{t-\zeta}\,dt.$$

We now substitute $\zeta = h(z)$ and $t = h(\tau)$ into this formula. This brings us to the solution to (38.28):

$$u(z) = \operatorname{Re}\frac{1}{\pi i}\int_\Gamma \frac{u_0(\tau)\,h'(\tau)}{h(\tau)-h(z)}\,d\tau. \tag{38.29}$$

Similarly, if the function $w = f(z)$ maps D conformally onto the unit circle $|w| < 1$, then we can employ (38.15) and write the solution to (38.28) in the form

$$u(z) = \operatorname{Re}\frac{1}{2\pi i}\int_\Gamma u_0(\zeta)\,\frac{f(\zeta)+f(z)}{f(\zeta)-f(z)}\,\frac{f'(\zeta)}{f(\zeta)}\,d\zeta. \tag{38.30}$$

When solving (38.28), instead of evaluating the integral (38.29) or (38.30) it is often convenient, after we have found the conformal mapping $\zeta = h(z)$ of D onto a circle or half-plane, to evaluate Poisson's integral in (38.8) or (38.19) and then substitute $\zeta = h(z)$.

Example 5. Let us find the solution to

$$\nabla^2 u = 0, \quad 0 < y < \pi; \tag{38.31}$$

$$u|_{y=0} = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0, \end{cases} \quad u|_{y=\pi} = 0, \tag{38.32}$$

where $z = x + iy$. The function $\zeta = e^z$ $(\zeta = \xi + i\eta)$ maps the strip $0 < y < \pi$ conformally onto the half-plane $\eta > 0$ (Sec. 35). The boundary condition (38.32) then becomes

$$\tilde{u}|_{\eta=0} = \begin{cases} 1 & \text{for } \xi > 1, \\ 0 & \text{for } \xi < 1. \end{cases}$$

From (38.19) we find that

$$\tilde{u}(\zeta) = \frac{1}{\pi} \int_1^{\infty} \frac{\eta\, dt}{(\xi - t)^2 + \eta^2} = \frac{1}{2} - \frac{1}{\pi} \arctan \frac{1-\xi}{\eta}.$$

Finally, if we substitute $\xi = e^x \cos y$ and $\eta = e^x \sin y$, we arrive at the solution to the Dirichlet problem (38.31), (38.32):

$$u(z) = \frac{1}{2} - \frac{1}{\pi} \arctan \frac{e^{-x} - \cos y}{\sin y}. \quad \square$$

38.5 Green's function of the Dirichlet problem The function

$$G(z, \zeta) = \frac{1}{2\pi} \ln |z - \zeta| + g(z, \zeta), \quad z \in D, \quad \zeta \in D, \tag{38.33}$$

is known as Green's function of the Dirichlet problem for Laplace's operator in domain D if $g(z, \zeta)$ satisfies the following conditions:

(1) it is harmonic in D for each $\zeta \in D$, i.e.

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0, \quad z = x + iy \in D; \tag{38.34}$$

(2) it is continuous up to the boundary Γ of D for each $\zeta \in D$ and

$$g(z, \zeta)|_{z \in \Gamma} = -\frac{1}{2\pi} \ln |z - \zeta|_{z \in \Gamma}. \tag{38.35}$$

Note that the second condition means that

$$G(z, \zeta)_{z \in \Gamma} = 0. \tag{38.36}$$

Thus, for each $\zeta \in D$ the function $g(z, \zeta)$ is a solution of the Dirichlet problem (38.34), (38.35). From the fact that a Dirichlet problem has only one solution it follows that Green's function exists and is unique in any bounded domain with piecewise smooth boundary. Let us show that finding Green's function in a simply connected domain is equivalent to determining the function that maps this domain into the unit circle.

Theorem 4 *Suppose D is a simply connected, bounded domain and let the function $w = w(z, \zeta)$, $z \in D$, $\zeta \in D$, for each $\zeta \in D$ map D conformally onto the unit circle $|w| < 1$ in a way such that $z \in \zeta$ is mapped into point $w = 0$, and $w(\zeta, \zeta) = 0$. Then Green's function of the Dirichlet problem for Laplace's operator in D is*

$$G(z, \zeta) = \frac{1}{2\pi} \ln |w(z, \zeta)|. \tag{38.37}$$

Proof. We fix a point $\zeta \in D$. Since the mapping $w = w(z, \zeta)$ is conformal, i.e. the function $w(z, \zeta)$ is regular and univalent in D, we conclude that $dw(z, \zeta)/dz \neq 0$ for $z \in D$. The fact that $w(\zeta, \zeta) = 0$ implies that $w(z, \zeta) \neq 0$ at $z \neq \zeta$. Hence, we can write

$$w(z, \zeta) = (z - \zeta)\, \psi(z, \zeta), \tag{38.38}$$

where $\psi(z, \zeta)$ is regular in D and does not vanish for $z \in D$. From (38.38) we obtain

$$\frac{1}{2\pi} \ln |w(z, \zeta)| = \frac{1}{2\pi} \ln |z-\zeta| + g(z, \zeta), \tag{38.39}$$

where $g(z, \zeta) = \frac{1}{2\pi} \ln |\psi(z, \zeta)|$ is harmonic in D, since it is the real part of the function $\frac{1}{2\pi} \ln \psi(z, \zeta)$, which is regular in D.

Further, if $z \in \Gamma$, then $|w(z, \zeta)| = 1$ and (38.39) yields the boundary condition (38.35). Hence, (38.39) gives Green's function of the Dirichlet problem for Laplace's operator in D. The proof of the theorem is complete.

Remark 2. If $w = w(z)$ is a conformal mapping of D onto the circle $|w| < 1$, then $w(z, \zeta)$ is given by the following formula (see Sec. 34):

$$w(z, \zeta) = \frac{w(z) - w(\zeta)}{1 - w(z)\overline{w(\zeta)}} \tag{38.40}$$

Green's function $G(z, \zeta)$ possesses the following properties.

(1) It is symmetric, i.e.

$$G(z, \zeta) = G(\zeta, z). \tag{38.41}$$

(2) For each $z \in D$ it is harmonic in ξ and η ($\zeta = \xi + i\eta$) in D with the point $\zeta = z$ deleted.

(3) For each $z \in D$ it is continuous up to the boundary Γ of D and

$$G(z, \zeta)\,|_{\xi \in \Gamma} = 0. \tag{38.42}$$

Properties 2 and 3 follow from Property 1 and the definition of Green's function. Let us prove the validity of Property 1 for a simply connected domain.

From (38.40) we obtain $|w(\zeta, z)| = |w(z, \zeta)|$; whence, from (38.37) follows (38.41).

The proof of Property 1 for multiply connected domains is given in Vladimirov [1].

We can use Green's function to find the solution of the Dirichlet problem for Poisson's equation

$$\nabla^2 u = F(z), \quad z \in D, \tag{38.43}$$

with the boundary condition

$$u|_\Gamma = u_0(z). \tag{38.44}$$

Under sufficiently broad assumptions, the solution to the problem (38.43), (38.44) is given by the formula

$$u(z) = \iint_D G(z, \zeta) F(\zeta)\, d\xi\, d\eta + \int_\Gamma \frac{\partial G(z, \zeta)}{\partial n} u_0(\zeta)\, |d\zeta|, \tag{38.45}$$

where $\zeta = \xi + i\eta$, and the symbol $\partial/\partial n$ stands for differentiation along an outward normal to the boundary Γ of D with respect to ζ. A proof of (38.45) is given in Vladimirov [1].

38.6 The Neumann problem Suppose D is a bounded domain with a smooth boundary Γ and $u_1(z)$ is a function that is given on Γ. The *classical Neumann problem* can be formulated thus: to find a function $u(z)$ that is harmonic in D,

$$\nabla^2 u = 0, \quad z \in D, \tag{38.46}$$

is continuously differentiable up to the boundary Γ, and satisfies the condition

$$\left.\frac{\partial u}{\partial n}\right|_{z \in \Gamma} = u_1(z), \tag{38.47}$$

where $\partial/\partial n$ stands for the derivative along an outward normal to Γ.

The problem (38.46), (38.47) has a solution only if

$$\int_\Gamma u_1(z)\, ds = 0, \tag{38.48}$$

where $ds = |dz|$ is the element of the length of curve Γ.

Indeed, by Green's formula (see Kudryavtsev [1])

$$\iint_D \nabla^2 u\, dx\, dy = \int_\Gamma \frac{\partial u}{\partial n}\, ds,$$

from which, in view of (38.46) and (38.47), follows (38.48).

If condition (38.48) is met, the classical Neumann problem has a solution, and this solution is unique to within a constant term. The proof of the existence of a solution is given in Vladimirov [1]. Here we will prove its uniqueness.

Suppose $\tilde{u}(z)$ and $\tilde{\tilde{u}}(z)$ are two solutions to (38.46), (38.47). Then $u(z) = \tilde{u}(z) - \tilde{\tilde{u}}(z)$ is a solution to

$$\nabla^2 u = 0, \quad z \in D; \quad \left.\frac{\partial u}{\partial n}\right|_{z \in \Gamma} = 0. \tag{38.49}$$

Next we use Green's formula (see Kudryavtsev [1])

$$\iint\limits_D \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\, dy = \int\limits_\Gamma u \frac{\partial u}{\partial n}\, ds - \iint\limits_D u \nabla^2 u\, dx\, dy.$$

The right-hand side vanishes because of (38.49). On the left-hand side the integrand is nonnegative and continuous in D. Hence, $\partial u/\partial x = \partial u/\partial y = 0$, whence $u(z) \equiv \text{const}$ for $z \in D$.

Along with the classical Neumann problem we will consider the problem (38.46), (38.47) where D is not bounded. But then we must assume that both $u(z)$ and its first order partial derivatives are bounded in D.

Example 6. Let us find the solution to

$$\nabla^2 u = 0, \quad y > 0; \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = -u_1(x), \tag{38.50}$$

where $z = x + iy$, $u_1(x)$ is continuous for $-\infty < x < +\infty$, and

$$u_1(x) = O(|x|^{-1-\varepsilon}) \tag{38.51}$$

as $x \to \infty$, $\varepsilon > 0$. In this case $\partial u/\partial n|_{y=0} = -\partial u/\partial y|_{y=0}$. We will also assume that condition (38.48) is met, i.e.

$$\int\limits_{-\infty}^{+\infty} u_1(x)\, dx = 0. \tag{38.52}$$

The Neumann problem (38.50) can be reduced to the Dirichlet problem for the function $\partial u/\partial y$. Indeed, since $u(z)$ is harmonic, we conclude that $\partial u/\partial y$ is harmonic, too, for $y > 0$. We then have the following Dirichlet problem:

$$\nabla^2\left(\frac{\partial u}{\partial y}\right) = 0, \quad y > 0; \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = -u_1(x).$$

Formula (38.19) then yields

$$\frac{\partial u}{\partial y} = -\frac{1}{\pi}\int\limits_{-\infty}^{+\infty} \frac{y u_1(t)}{(t-x)^2 + y^2}\, dt,$$

whence

$$u(z) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} u_1(t) \ln[(t-x)^2 + y^2]\, dt + C(x). \qquad (38.53)$$

In this formula the integral is a harmonic function for $y > 0$ since the integrand is harmonic and condition (38.51) ensures that the integral has a finite value and partial derivatives of all orders. For this reason $C(x)$ is also a harmonic function, i.e. $C''(x) = 0$, whence $C(x) = C_1 + C_2 x$. Condition (38.52) implies that the integral in (38.53) is bounded for $y > 0$, since we can represent it in the form

$$\int_{-\infty}^{+\infty} u_1(t) \ln[(t-x)^2 + y^2]\, dt - \ln(x^2 + y^2) \int_{-\infty}^{+\infty} u_1(t)\, dt$$

$$= \int_{-\infty}^{+\infty} u_1(t) \ln \frac{(t-x)^2 + y^2}{x^2 + y^2}\, dt,$$

and for large values of $x^2 + y^2$ the integral on the right-hand side is less in absolute value than

$$\int_{-\infty}^{+\infty} |u_1(t)| \ln(|t| + 1)^2\, dt,$$

which has a finite value in view of (38.51). The function $u(z)$ is bounded and, therefore, so is $C(x)$, which means that $C_2 = 0$. Then (38.53) yields the solution to (38.50):

$$u(z) = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} u_1(t) \ln[(t-x)^2 + y^2]\, dt + C,$$

with C a constant. We can also write the solution in the form

$$u(z) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} u_1(t) \ln|t - z|\, dt + C, \qquad (38.54)$$

since $(t-x)^2 + y^2 = t - z^2$. □

The solution to the Neumann problem (38.46), (38.47) in a simply connected domain D can be found via a conformal mapping of D onto a half-plane and formula (38.54).

Suppose the function $\zeta = h(z)$, $\zeta = \xi + i\eta$, maps D conformally onto the half-plane $\operatorname{Im} \zeta > 0$, with $z = g(\zeta)$ the inverse mapping. Then the function $\tilde{u}(\zeta) = u(g(\zeta))$ is harmonic in the half-plane

$\operatorname{Im} \zeta > 0$. Find $\left.\frac{\partial \tilde{u}}{\partial \tilde{n}}\right|_{\eta=0}$, where $\tilde{n}$ is the outer normal to the boundary of the half-plane $\operatorname{Im} \zeta > 0$. Under the conformal mapping $z = g(\zeta)$ the direction of normal $\tilde{n}$ is transformed into the direction of normal n, and the stretching at points on the straight line $\operatorname{Im} \zeta = 0$ is $|g'(\zeta)|$. Hence,

$$\left.\frac{\partial \tilde{u}}{\partial \tilde{n}}\right|_{\eta=0} = \left.\frac{\partial u}{\partial n}\right|_{z=g(\xi)} |g'(\xi)| = u_1(g(\xi) \times |g'(\xi)| = \tilde{u}_1(\xi)\, |g'(\xi)|.$$

Bearing in mind that $\partial \tilde{u}/\partial n\,|_{\eta=0} = -\partial \tilde{u}/\partial \eta\,|_{\eta=0}$ and employing (38.54), we find that

$$\tilde{u}(\zeta) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \tilde{u}_1(t)\, |g'(t)| \ln |t-\zeta|\, dt + C.$$

We return to the old variables and substitute $\zeta = h(z)$ and $t = h(\tau)$. Since $g'(t) = 1/h'(\tau)$, we can write the solution to (38.46), (38.47) in the form

$$u(z) = -\frac{1}{\pi} \int_{\Gamma} u_1(\tau) \ln |h(\tau) - h(z)| \frac{h'(\tau)}{|h'(\tau)|} d\tau + C. \quad (38.55)$$

Example 7. Let us find the solution to

$$\nabla^2 u = 0, \quad r < 1; \quad \left.\frac{\partial u}{\partial r}\right|_{r=1} = u_1(e^{i\varphi}), \quad (38.56)$$

where $z = re^{i\varphi}$, $u_1(e^{i\varphi})$ is a continuous function, and $\int_0^{2\pi} u_1(e^{i\varphi})\, d\varphi = 0$. In this case $\partial u/\partial n\,|_{r=1} = \partial u/\partial r\,|_{r=1}$.

The function $z = g(\zeta) = \frac{\zeta - i}{\zeta + i}$ maps the half-plane $\operatorname{Im} \zeta > 0$ conformally onto the circle $|z| < 1$ (Sec. 34). The reverse mapping has the form $\zeta = h(z) = \frac{1+z}{1-z} i$. Since $h'(z) = \frac{2i}{(1-z)^2}$, the solution to (38.56), in view of (38.55), is given by the formula

$$u(z) = -\frac{1}{\pi} \int_{|\tau|=1} u_1(\tau) \ln \left| \frac{1+\tau}{1-\tau} i - \frac{1+z}{1-z} i \right| \frac{i\,|1-\tau|^2}{(1-\tau)^2} d\tau + C. \quad (38.57)$$

In the integral (38.57) we put $\tau = e^{i\theta}$. Then

$$\frac{i\,|1-\tau|^2}{(1-\tau)^2} = \frac{i\,(1-e^{i\theta})(1-e^{-i\theta})}{(1-e^{i\theta})^2} ie^{i\theta}\, d\theta = d\theta,$$

$$\left| \frac{1+\tau}{1-\tau} i - \frac{1+z}{1-z} i \right| = \frac{2\,|e^{i\theta} - z|}{|1-e^{i\theta}|\,|1-z|}.$$

From (38.57) we obtain

$$u(z) = -\frac{1}{\pi}\int_0^{2\pi} u_1(e^{i\theta}) \ln |e^{i\theta} - z| \, d\theta + J_1 + J_2 + C,$$

where

$$J_1 = -\frac{1}{\pi} \ln \frac{2}{|1-z|} \int_0^{2\pi} u_1(e^{i\theta}) \, d\theta = 0,$$

$$J_2 = \frac{1}{\pi}\int_0^{2\pi} u_1(e^{i\theta}) \ln |1 - e^{i\theta}| \, d\theta \equiv \text{const}$$

(is independent of z). The final solution to (38.56) is

$$u(z) = -\frac{1}{\pi}\int_0^{2\pi} u_1(e^{i\theta}) \ln |e^{i\theta} - z| \, d\theta + C, \tag{38.58}$$

where C is an arbitrary constant. □

The solution to the Neumann problem (38.46), (38.47) in a simply connected domain D can also be found via a conformal mapping of D onto a circle and formula (38.58).

The same problem can be reduced to the Dirichlet problem for the conjugate harmonic function $v(z)$. Indeed, in view of the Cauchy-Riemann equations we have

$$\left.\frac{\partial v}{\partial s}\right|_{z\in\Gamma} = \left.\frac{\partial u}{\partial n}\right|_{z\in\Gamma} = u_1(z),$$

whence

$$v(z)|_{z\in\Gamma} = \int_{z_0}^{z} u_1(\zeta) \, ds, \tag{38.59}$$

where the integral is taken along an arc of Γ. By solving the Dirichlet problem for Laplace's equation

$$\nabla^2 v = 0, \quad z \in D,$$

with the boundary condition (38.59) we can find $u(z)$ via simple integration (see Sec. 7).

39 Vector Fields in a Plane

39.1 The basic concepts Let us assign to each point $z = x + iy$ in a domain D in the complex z plane a vector $A = (A_x, A_y)$ whose components are functions of x and y, i.e. $A_x = A_x(x, y)$ and $A_y = A_y(x, y)$. Then we say that a vector field $A(x, y)$ is given in D. It is assumed that both functions, A_x and A_y, are continuously differentiable in D. A vector field in a plane can also be fixed by specifying a single complex valued function, which we will also denote by A:

$$A = A_x + iA_y. \tag{39.1}$$

In many important physical problems, A is an analytic function of z, which makes it possible to employ the methods of the theory of functions of a complex variable.

Here are the main concepts of vector analysis. (For more details the interested reader can refer to Kudryavtsev [1].)

We start with the concept of a streamline. Consider the autonomous system of differential equation

$$\frac{dx}{dt} = A_x(x, y), \quad \frac{dy}{dt} = A_y(x, y). \tag{39.2}$$

In many problems the parameter t can be interpreted as time. The phase trajectories of system (39.2), i.e. the curves $x = \varphi(t)$ and $y = \psi(t)$, $t_1 < t < t_2$, where $(\varphi(t), \psi(t))$ is a solution to (39.2), are known as the *streamlines* of the vector field (39.1). A point (x_0, y_0) at which A is zero, i.e.

$$A_x(x_0, y_0) = 0, \quad A_y(x_0, y_0) = 0,$$

is said to be a *stationary point* of system (39.2) or a *critical point* of the vector field A. A stationary point (x_0, y_0) corresponds to a phase trajectory (a streamline) that consists of only one point. The theory of ordinary differential equations makes it possible to estimate the structure of streamlines qualitatively. Precisely, three cases may occur:

(1) A streamline consists of one point (a stationary point).

(2) A streamline is a smooth closed curve.

(3) A streamline is a smooth open curve. In this case both end points of the curve lie on the boundary of D or coincide with one of the stationary points.

An important physical problem related to plane vector fields is the steady-state plane-parallel flow of a fluid. Suppose the flow is parallel to the (x, y) plane. Then the velocity of each particle of the fluid passing through point (x, y, z) is a vector of the form $v = (v_x(x, y), v_y(x, y), 0)$, and the corresponding vector field is a velocity field. Instead of a vector field in space we can now study a plane vector field, the *velocity field*.

$$v = (v_x, v_y). \tag{39.3}$$

The streamlines in this field are the curves along which the particles of the fluid move (or flow).

We turn to the concept of flux and divergence. The *flux* of a vector field A through a closed contour γ is the integral

$$N = \int_{\gamma} (A\ n)\, ds. \tag{39.4}$$

Here and in what follows $(A \cdot n)$ is the scalar product of vector A into the unit vector n normal to γ, and ds is the element of the length of curve γ. If γ is a simple closed curve oriented counterclockwise (clockwise), the normal to γ points outward (inward). An alternative formula for the flux can be written thus:

$$N = \int_{\gamma} -A_y\, dx + A_x\, dy. \tag{39.5}$$

The *divergence* of a vector field A is the quantity

$$\operatorname{div} A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}. \tag{39.6}$$

Green's formula gives the relationship between divergence and flux:

$$\int_{\gamma} (A \cdot n)\, ds = \iint_{D} \operatorname{div} A\, dx\, dy. \tag{39.7}$$

Here γ is a simple closed curve, the boundary of D.

The formula (39.6) for divergence is not invariant, since it depends on the choice of the coordinate system, while (39.7) enables us to give an invariant definition of divergence. Suppose curve γ is deformed continuously to a point (x_0, y_0), and S is the area of the surface bounded by γ. Then the divergence at this point is defined thus:

$$\operatorname{div} A = \lim_{S \to 0} \frac{\text{flux through } \gamma}{\text{area}}. \tag{39.8}$$

Thus, divergence is the density of the flux of the vector field.

Finally, let us define circulation and curl. The *circulation* of a vector field A around a closed contour γ is the integral

$$\Gamma = \int_{\gamma} (A \cdot t)\, ds, \tag{39.9}$$

where t is a unit vector tangential to γ. An alternative formula for the circulation can be written thus:

$$\Gamma = \int_{\gamma} A_x\, dx + A_y\, dy. \tag{39.10}$$

The *curl* or *vorticity* of a vector field A is the quantity

$$\operatorname{curl} A = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}. \tag{39.11}$$

Remark 1. In vector analysis the curl of a spatial vector field (at a given point) is a three-dimensional vector, while we defined the curl as a number. The relation between the two concepts is as follows. Let us use the vector field (39.1) to build a spatial plane-parallel vector field $\tilde{A} = (A_x, A_y, 0)$. Then

$$\operatorname{curl} \tilde{A} = (0, 0, \operatorname{curl} A). \tag{39.12}$$

Green's formula provides the relationship between circulation and curl:

$$\int_{\gamma} (A \cdot t)\, ds = \iint_{D} \operatorname{curl} A\, dx\, dy. \tag{39.13}$$

Here γ is a simple closed curve that is the boundary of D. This formula can be used to define the curl in an invariant manner:

$$\operatorname{curl} A = \lim_{S \to 0} \frac{\text{circulation around } \gamma}{\text{area}}. \tag{39.14}$$

Here, just as in (39.8), γ is deformed continuously to a point (x_0, y_0), and S is the area of the surface bounded by γ. Hence, curl is the density of the circulation of the vector field.

Remark 2. Let us assign to the plane vector field A a spatial plane-parallel vector field $\tilde{A}$ (see Remark 1). Vector analysis provides the following interpretation of the curl. Suppose U is an infinitely small neighborhood of point (x_0, y_0). Then the instantaneous angular velocity ω with which U rotates is $\omega = (1/2) \operatorname{curl} \tilde{A}$, where $\operatorname{curl} \tilde{A}$ is given by formula (39.12).

39.2 Solenoidal and irrotational fields A vector field A is said to be *solenoidal* (*source-free, zero-divergence*) in a domain D if its divergence is zero:

$$\operatorname{div} A = 0. \tag{39.15}$$

A solenoidal field preserves area. Precisely, suppose we take a domain D_0 in a plane and watch how each point in D_0 moves along the streamlines over the same time interval t. Then D_0 will become D_t. Condition (39.15) means that the areas of D_0 and D_t are the same. If the vector field considered is the velocity field of a fluid, condition (39.15) implies that the fluid is incompressible, provided its density is the same at each point.

Let D be a simply connected domain. Then (39.6), (39.8), and (39.15) imply that the integral

$$v(x, y) = \int_{z_0}^{z} -A_y\, dx + A_x\, dy \tag{39.16}$$

does not depend on the path of integration. This integral, therefore, defines in D a single-valued function known as the *stream function* of vector field A. The components of A can be expressed in terms of the stream function thus:

$$A_x = \frac{\partial v}{\partial y}, \quad A_y = -\frac{\partial v}{\partial x}. \tag{39.17}$$

A vector field determines the stream function uniquely, to within a constant term.

If a vector field is solenoidal, the streamlines are simply the curves along which the stream function is constant. Indeed, from (39.2) and (39.17) we find that along a streamline

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt} = -A_y A_x + A_x A_y = 0,$$

so that $v(x, y) \equiv \text{const}$ along a streamline.

If D is not simply connected, the stream function defined by (39.16) is generally not single-valued. Nevertheless, the formulas (39.17) remain valid (they are valid for all "branches" of the stream function), and locally (i.e. within any simply connected domain lying inside D) a stream function is always defined. If a vector field possesses a streamline function, i.e. there is a function $v(x, y)$ such that (39.17) is valid everywhere in D, the field is solenoidal.

A vector field A is called *irrotational* (*potential*, *nonvortical*) in a domain D if its curl is zero;

$$\operatorname{curl} A = 0 \tag{39.18}$$

everywhere in D. Suppose D is simply connected. Then the integral

$$u(x, y) = \int_{z_0}^{z} A_x\,dx + A_y\,dy \tag{39.19}$$

does not depend on the path of integration and therefore defines in D a single-valued function. This function is known as the *potential* of vector field A. The components of A can be expressed in terms of the potential thus:

$$A_x = \frac{\partial u}{\partial x}, \quad A_y = \frac{\partial u}{\partial y}. \tag{39.20}$$

Conversely, if there is a potential, i.e. a function u such that the components of the field are expressed through u via (39.20), the vector field is irrotational.

The curves along which the potential is constant, i.e. $u(x, y) \equiv \text{const}$, are known as *equipotential curves*. These curves are orthogonal to the streamlines. Indeed, the vector grad $u = (A_x, A_y)$ is orthogonal to an equipotential curve and is tangential to a streamline.

If the field is irrotational but D is multiply connected, locally (i.e. in each simply connected domain lying in D) a potential is always defined uniquely, to within a constant term. On the other hand, in the entire domain D the potential in this case may be a multiple-valued function, but (39.20) are valid for each of its branches everywhere in D.

As known from vector analysis, every vector field can be represented as the sum of a solenoidal field and an irrotational field (e.g. see Kudryavtsev [1] and Nikol'skii [1]).

39.3 Harmonic vector fields A vector field is said to be *harmonic* in a domain D if it is both solenoidal and irrotational in D, i.e.

$$\operatorname{div} A = 0, \quad \operatorname{curl} A = 0 \tag{39.21}$$

everywhere in D. A harmonic vector field possesses both a stream function and a potential. From (39.17) and (39.20) it follows that in this case these functions are related thus:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{39.22}$$

which are simply the Cauchy-Riemann equations for the function

$$f(z) = u(x, y) + iv(x, y). \tag{39.23}$$

We can therefore formulate

Theorem 1 *The stream function and the potential of a harmonic vector field constitute a pair of conjugate harmonic functions.*

The function $f(z)$ is known as the *complex potential* of the vector field A. It is analytic in D, and if D is simply connected, the complex potential is regular in D.

We can express all the characteristics of a field in terms of the complex potential. First we have

$$A = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x} = \overline{f'(z)}. \tag{39.24}$$

This implies, for one, that the derivative of the complex potential, $f'(z)$, is single-valued and hence regular in D.

Since $f'(z)\,dz = (A_x - iA_y)(dx + i\,dy)$, we can write (39.5) and (39.10) as follows:

$$N = \operatorname{Im} \int_\gamma f'(z)\,dz, \quad \Gamma = \operatorname{Re} \int_\gamma f'(z)\,dz. \tag{39.25}$$

Combining these formulas, we obtain

$$\Gamma + iN = \int_\gamma f'(z)\,dz. \tag{39.26}$$

Here are some simple examples of harmonic vector fields.

Example 1. A constant vector field. Such a field is specified by a single complex number $A = A_x + iA_y$, where A_x and A_y are both constant and real. The complex potential is $f(z) = \bar{A}z + c$, where c is a constant. The streamlines are straight lines with the direction vector being A, while the equipotential curves are straight lines orthogonal to the streamlines. The flux through any closed curve and the circulation around any closed curve are zero. □

Example 2. Suppose the complex potential is $f(z) = z^2$. Then $u(x, y) = x^2 - y^2$ and $v(x, y) = xy$. Hence, the streamlines are the hyperbolas $xy = \text{const}$, while the equipotential curves are the hyperbolas $x^2 - y^2 = \text{const}$ (Fig. 151). Point $z = 0$ is a critical point for this vector field. Indeed, from (39.24) it follows that the critical points of a harmonic vector field are those and only those at which

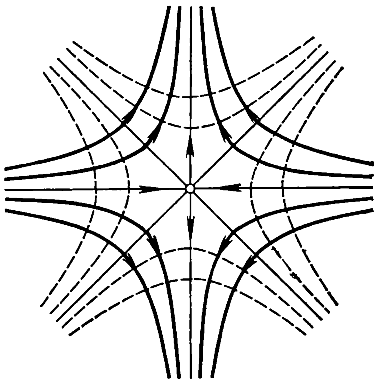


Fig. 151

$$f'(z) = 0. \tag{39.27}$$

In our example the point $z = 0$ is a streamline. Among the streamlines there are four rays: $xy = 0$, $z \neq 0$, which form right angles at point $z = 0$ (see Fig. 151). Among the equipotential curves there are also four rays: $x^2 - y^2 = 0$, $z \neq 0$. □

Remark 3. An arbitrary smooth vector field may have an extremely complex structure near a critical point. But if a vector field is harmonic, its local structure near a critical point is quite simple. Suppose z_0 is a critical point for a vector field with potential $f(z)$. Then

$$f'(z_0) = 0, \ldots, f^{(n-1)}(z_0) = 0, \quad f^{(n)}(z_0) \neq 0$$

for an $n \geqslant 2$. Then by introducing the function $z = g(\zeta)$, $z_0 = g(0)$, with $g(\zeta)$ regular and univalent at point $\zeta = 0$, we can write the complex potential near point z_0 in the form

$$f(g(\zeta)) = f(z_0) + \zeta^n \tag{39.28}$$

(Corollary 2 in Sec. 32). For this reason, the streamlines and the equipotential curves of a harmonic vector field have the same arrangement near a critical point as that of a field with the complex potential (39.28). Among the various streamlines there are $2n$ rays that emerge from or end at point $\zeta = 0$, and the angle between two adjacent rays of this kind is π/n.

Example 3. Sources and sinks. Suppose we have a field with a complex potential $f(z) = (Q/2\pi) \ln z$, where Q s a nonzero real constant. Then

$$u(x, y) = \frac{Q}{2\pi} \ln |z|, \quad v(x, y) = \frac{Q}{2\pi} \arg z,$$

and the streamlines are the rays $\arg z = \text{const}$, while the equipotential curves are the circles $|z| = \text{const}$ (Fig. 152). At $z \neq 0$ this vector field is harmonic.

Suppose γ is a simple closed curve containing the origin of coordinates in its interior and oriented counterclockwise. Since $f'(z) =$

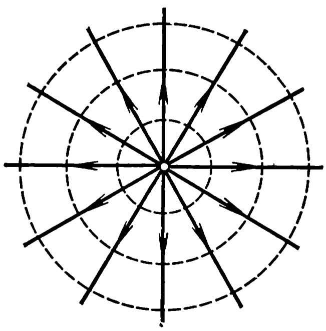


Fig. 152

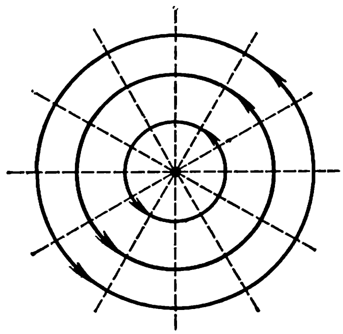


Fig. 153

$Q/(2\pi z)$, from (39.26) we find that $N = Q$ and $\Gamma = 0$. For this reason point $z = 0$ is said to be a *source of strength* Q if $Q > 0$ or a *sink of strength* $|Q|$ if $Q < 0$. □

Example 4. Vortices. Suppose we have a vector field whose complex potential is $f(z) = (\Gamma_0/2\pi i) \ln z$, with Γ_0 a nonzero real number. Then

$$u(x, y) = \frac{\Gamma_0}{2\pi} \arg z, \quad v(x, y) = -\frac{\Gamma_0}{2\pi} \ln |z|,$$

and the streamlines are the circles $|z| = \text{const}$, while the equipotential curves are the rays $\arg z = \text{const}$ (Fig. 153). If γ is a closed curve circuiting point $z = 0$ in the positive sense, then $\int_\gamma f'(z)\, dz = \Gamma_0$, with $N = 0$ and $\Gamma = \Gamma_0$. Point $z = 0$ is then called a *vortex of strength* Γ_0. □

Various combinations of the harmonic vector fields we have just discussed lead to new examples of harmonic vector fields.

Example 5. The combination of a source (sink) and a vortex has a complex potential of the form

$$f(z) = \frac{Q - i\Gamma_0}{2\pi} \ln z,$$

where Q and Γ_0 are nonzero real constants. If γ is the same curve as in Example 4, we have the following values for flux and circulation:

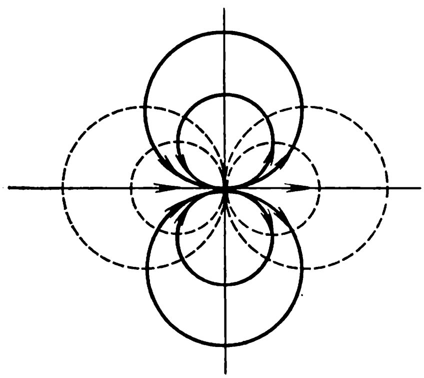


Fig. 154

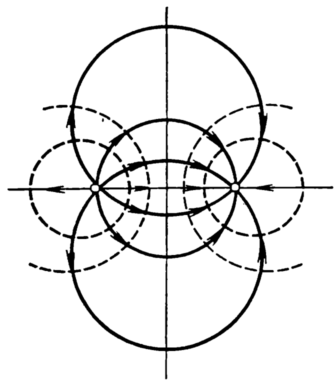


Fig. 155

$N = Q$ and $\Gamma = \Gamma_0$. The streamlines and the equipotential curves are logarithmic spirals winding up into the origin of coordinates. □

Example 6. Dipoles. In this case the complex potential is $f(z) = m/2\pi z$, with m a nonzero real constant. The streamlines and the equipotential curves are circles passing through the origin of coordinates (Fig. 154). The constant m is called the *dipole moment*, and the real axis (Ox) the *dipole axis*. □

A dipole can be obtained by combining a source and a sink (Fig. 155) of equal intensities situated at the points $z = \pm h$ and sending h to zero, with $Q \to \infty$ and $Q \times 2h \to m$. Indeed,

$$\lim \frac{Q\,2h}{2\pi} \frac{\ln(z+h) - \ln(z-h)}{2h} = \frac{m}{2\pi} \frac{d \ln z}{dz} = \frac{m}{2\pi z}.$$

Example 7. Multipoles. Suppose we have a complex potential with a pole at point $z = 0$, i.e. $f(z) = Cz^{-n}$, $n \geqslant 2$, with C a nonzero complex constant. Then we say there is a *multipole* at point $z = 0$. A multipole can also be obtained by combining sources and vortices near to the origin of coordinates and conducting an appropriate passage to the limit. □

40 Some Physical Problems from Vector Field Theory

40.1 Streamline flow around solids We take the steady-state plane-parallel flow of an ideal incompressible fluid (e.g. see Lavrent'ev and Shabat [1]). It is proved in courses of fluid mechanics that the resulting velocity field $\mathbf{V} = v_x + iv_y$ is harmonic and is characterized by a complex potential $f(z) = u(x, y) + iv(x, y)$ such that

$$\mathbf{V} = \overline{f'(z)}. \tag{40.1}$$

Suppose we have specified on a plane a simply connected domain $\widetilde{D}$ with a smooth boundary S. Let D be the exterior of S filled with the fluid. Suppose the "solid" $\widetilde{D}$ moves with a constant velocity $-\mathbf{V}_\infty$ or, which is the same, the fluid flows past the solid with a velocity $\mathbf{V}_\infty$, while the solid is at rest. (We speak of $\widetilde{D}$ as a solid because S does not vary.) Then the complex potential $f(z)$ is a function that is regular in D, with $f'(\infty) = \overline{\mathbf{V}}_\infty$. We expand $f'(z)$ about point $z = \infty$ in a Laurent series:

$$f'(z) = \overline{\mathbf{V}}_\infty + \frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \dots \, . \tag{40.2}$$

From (39.26) we find that $2\pi i c_{-1} = \Gamma + iN$, where Γ and N are the circulation and the flux of the vector field, respectively around and through any closed curve surrounding the solid $\widetilde{D}$. By definition, D contains no sources, which means that $N = 0$. Then (40.2) yields

$$f(z) = \overline{\mathbf{V}}_\infty z + c + \frac{\Gamma}{2\pi i} \ln z - \frac{c_{-2}}{z} + \dots \tag{40.3}$$

in a neighborhood of point $z = \infty$. The velocity $\mathbf{V}_\infty$ and the circulation Γ must be specified—this is the boundary condition at infinity imposed on the complex potential $f(z)$. The boundary condition at the "surface" S of the solid is as follows: at any point of S the velocity of the flow must be tangential to S. Hence, S is one of the streamlines, so that the boundary condition on S is

$$v(x, y)\,|_S \equiv \text{const.} \tag{40.4}$$

Thus, we must find a function $f(z)$ that is regular in D, has a Laurent expansion (40.3) about point $z = \infty$, with $\mathbf{V}_\infty$ and Γ fixed (complex and real) constants, and satisfies the boundary condition (40.4) on S.

Theorem 1 *The solution of the problem of streamline flow is unique.*

Proof. Suppose we have two fields with complex potentials $f_1(z)$ and $f_2(z)$ both of which are solutions. Then the difference $f_1(z) - f_2(z) = f(z)$ is regular and bounded in D. The function $v(z) = \operatorname{Im} f(z)$ is harmonic and bounded in D, assumes a constant value on S, and, by the uniqueness theorem for the solution of the Dirichlet problem, is a constant. Hence $f(z) \equiv$ const, and the potentials $f_1(z)$ and $f_2(z)$ differ by a constant, which implies that the two fields coincide. The proof of the theorem is complete.

The flow of fluid around a body is said to be *irrotational* if Γ is zero and rotational if Γ is not zero.

Theorem 2 *The potential $w = f(z)$ of an irrotational flow around a solid maps D conformally onto the exterior of a segment that is parallel to the real axis.*

Proof. Without loss of generality we can assume that $v|_S = 0$. Let us show that there is a function $w = g(z)$ that maps D conformally onto the exterior of a segment of the real axis and can be expanded about point $z = \infty$ in the series $g(z) = \mathbf{V}_\infty z + g_0 + \ldots$. Then $g(z)$ satisfies the boundary condition (40.4) and is therefore the potential; by Theorem 1, $f(z) = g(z) + $ const.

Suppose $w = h(z)$ is the function that maps D conformally onto the exterior of the segment [0, 1]. Then it has a simple pole at point $z = 0$ and can be expanded about this point in the series

$$h(z) = h_{-1}z + h_0 + \frac{h_1}{z} + \ldots .$$

The function $w = (h(z^{-1}))^{-1}$ maps D conformally onto a domain D_1, since it is a composite of two univalent functions. For small values of $|z|$ we have $w = z/(h_{-1} + h_0 z + \ldots)$, so that $w(0) = 0$ and $w'(0) = h_{-1}^{-1}$. By Riemann's theorem (see Sec. 33), for every real α there is a function $h_\alpha(z)$ that maps D conformally onto D_1 and such that $\arg h'_\alpha(0) = \alpha$. We put $\alpha = \arg \mathbf{V}_\infty$; then $g(z) = \frac{|\mathbf{V}_\infty|}{|h'_\alpha(0)|} \times \frac{1}{h_\alpha(z^{-1})}$, which is the sought-for function.

Obviously, the function $w = f(z) = u + iv$ that maps D conformally onto the exterior of a segment parallel to the u axis satisfies conditions (40.3) (at $\Gamma = 0$) and (40.4) and, therefore, is the complex potential of a flow. For this reason, the solution of the problem of irrotational flow is reduced to finding a function that maps D conformally onto the exterior of a segment of the form $u_1 \leqslant u \leqslant u_2$, $v = v_0$.

40.2 Chaplygin's and Zhukovskii's formulas Suppose an airplane wing (an airfoil) is moving in air, whose density is ρ, with a constant subsonic velocity $-\mathbf{V}_\infty$ or, which is the same, the air is traveling past the wing with a velocity $\mathbf{V}_\infty$. We can think of the wing as an infinite cylinder with generatrices orthogonal to the velocity vector. We have thus formulated a plane problem of vector field theory. Let us calculate the total force on the contour S of the wing's cross section, or the *aerodynamic lift*. Suppose $p(z)$ is the air pressure at point z. On S the pressure is directed inward along a normal, which means the force on the element dz of the length of contour S is $ip\, dz$. The total force on S is $\mathbf{P} = \int\limits_S ip\, dz$. For a steady-state irrotational flow we can use Bernoulli's formula

$$p = A - \frac{\rho}{2} v^2,$$

where A is a constant, $v = |\mathbf{V}|$, and $\mathbf{V}$ is the flow velocity. This yields

$$\mathbf{P} = -\frac{\rho i}{2} \int\limits_S v^2\, dz.$$

On S the velocity is directed along a tangent (see (40.4)), so that $\mathbf{V} = \overline{f'(z)} = ve^{i\varphi}$, where $\varphi = \arg dz$. Hence,

$$\mathbf{P} = -\frac{\rho i}{2} \int\limits_S \overline{(f'(z))^2}\, e^{-2i\varphi}\, dz = -\frac{\rho i}{2} \int\limits_S \overline{(f'(z))^2}\, d\bar{z},$$

since $e^{-2i\varphi}\, dz = d\bar{z}$. For vector $\overline{\mathbf{P}}$, which is the complex conjugate of $\mathbf{P}$, we obtain

$$\overline{\mathbf{P}} = \frac{\rho i}{2} \int\limits_S (f'(z))^2\, dz. \tag{40.5}$$

This is the classical formula obtained by the Russian mathematician S. A. Chaplygin.

From this formula and the expansion (40.2) of $\overline{f'(z)}$ about point $z = \infty$ we can find, employing the theory of residues, the following formula

$$\overline{\mathbf{P}} = 2\pi i \frac{\rho i}{2} \frac{\Gamma \mathbf{V}_\infty}{\pi i} = i\rho\Gamma\overline{\mathbf{V}}_\infty.$$

Hence,

$$\mathbf{P} = -i\rho\Gamma\mathbf{V}_\infty. \tag{40.6}$$

This constitutes the essence of the famous theorem of Zhukovskii:

The aerodynamic lift is equal in magnitude to the product of the density, the flow velocity at infinity, and the circulation. It is directed at a right angle with respect to $\mathbf{V}_\infty$ *opposite the circulation.*

40.3 Streamline flow around a circular cylinder Let us start with an irrotational flow around a circle $|z| = R$. The potential of such a flow maps the exterior of the circle conformally onto the exterior of a segment of the real axis. Due to the symmetry of the problem, we can assume that the flow is directed along the x axis, i.e. $\mathbf{V}_\infty$ is a real number. The sought-for mapping is performed by the Zhukovskii function

$$w = a\left(\frac{z}{R} + \frac{R}{z}\right),$$

where a is a real constant. From the condition that $f'(\infty) = \overline{\mathbf{V}}_\infty$ we find that $w = \overline{\mathbf{V}}_\infty z + R^2\mathbf{V}_\infty/z$. For an arbitrary flow ($\mathbf{V}_\infty$ is complex valued) we obtain a similar result: $w = \overline{\mathbf{V}}_\infty z + R^2\mathbf{V}_\infty/z$. Note that this flow is the sum of a homogeneous flow $\overline{\mathbf{V}}_\infty z$ and the flow $\mathbf{V}_\infty R^2/z$ of a dipole at point $z = 0$.

Since $\operatorname{Re} \ln z \equiv \text{const}$ for $|z| = R$, the flow $\frac{\Gamma}{2\pi i} \ln z$ also surrounds the solid. The solution to this problem is

$$f(z) = \overline{\mathbf{V}}_\infty z + \frac{\mathbf{V}_\infty R^2}{z} + \frac{\Gamma}{2\pi i} \ln z. \tag{40.7}$$

Let us find the critical points of this flow, at which $f'(z) = 0$, i.e. where the velocity of the flow is zero. The equation

$$\overline{\mathbf{V}}_\infty z^2 + \frac{\Gamma}{2\pi i} z - \mathbf{V}_\infty R^2 = 0 \tag{40.8}$$

yields

$$z_{1,2} = \frac{1}{2\overline{\mathbf{V}}_\infty}\left(\frac{i\Gamma}{2\pi} \pm \sqrt{4|\mathbf{V}_\infty|^2 R^2 - \frac{\Gamma^2}{2\pi^2}}\right). \tag{40.9}$$

For $|\Gamma| \leqslant 4\pi |\mathbf{V}_\infty| R$ the radicand is positive, so that $|z_{1,2}| = R$ and both critical points lie on this circle. In what follows we assume, for the sake of simplicity, that $\mathbf{V}_\infty$ is real. Then (40.9) yields

$$z_{1,2} = R(i\sin\alpha \pm \cos\alpha), \quad \sin\alpha = \frac{\Gamma}{4\pi\mathbf{V}_\infty R},$$

and the critical points are $z_1 = Re^{i\alpha}$ and $z_2 = Re^{i(\pi-\alpha)}$.

If $\Gamma = 0$, then $z_{1,2} = \pm R$. As circulation increases, the points move closer to each other and coincide at a critical value of circulation $\Gamma_0 = 4\pi\mathbf{V}_\infty R$.

The boundary of the circle consists of streamlines (see (40.4)). For this reason the streamline that arrives at the critical point z_2 branches into two streamlines, the upper and lower arcs of the circle (Fig. 156a). Point z_2 is known as a *branching point*. At the second critical point, z_1, the two streamlines (the arcs of the circle) converge; this point is known as a *convergence point*. Note that the streamline that arrives at point z_2 and the streamline that leaves point z_1 are orthogonal to the circle. Indeed, at point z_1 we have

$$f'(z_1) = 0, \quad f''(z_1) = \frac{1}{2\pi z_1^2}\left(\frac{2\mathbf{V}_\infty R^2}{\overline{z}_1} + i\Gamma\right) \neq 0,$$

and the local structure of the streamline is the same as for the potential $\frac{1}{2}f''(z_1)(z - z_1)^2$ (see Sec. 39). The orthogonality of these streamlines follows from Example 2 in Sec. 39.

At the critical value of circulation $\pm\Gamma$ we have

$$z_{1,2} = iR, \quad f''(iR) = 0, \quad f'''(iR) \neq 0,$$

which means that the angle between two adjacent streamlines entering point $z = iR$ is $\pi/3$ (see Sec. 39). The streamlines for this case are depicted in Fig. 156*b*.

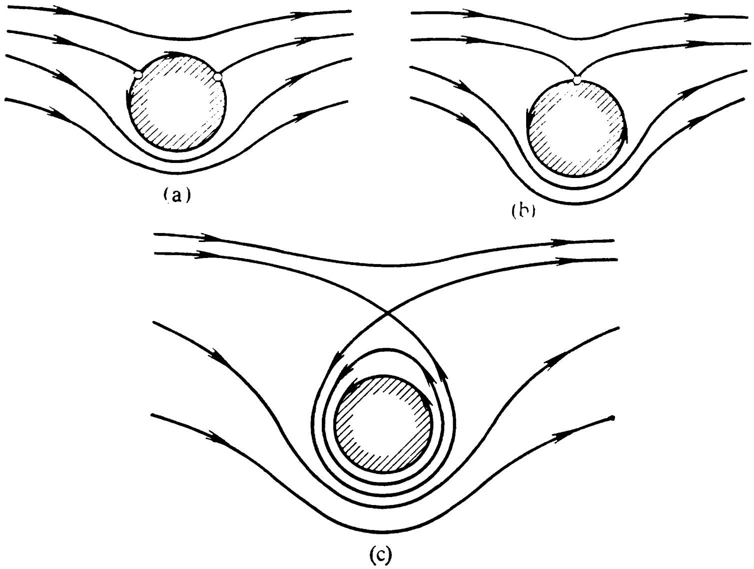


Fig. 156

If $|\Gamma| > \Gamma_0$, the radicand in (40.9) is negative and

$$|z_{1,2}| = \frac{1}{4\pi V_\infty}(\Gamma \pm \sqrt{\Gamma^2 - 16\pi^2 V_\infty^2 R^2}) \neq R.$$

From (40.8) it follows that $|z_1 z_2| = R^2$, so that one critical point lies inside the circle $|z| = R$ and the other outside the circle. This results in closed streamlines (Fig. 156*c*).

Circulation Γ can be expressed in terms of the coordinates of a convergence point thus: $\Gamma = 4\pi v_\infty R \sin\alpha$. But if $\arg \mathbf{V}_\infty = \theta$, then

$$\Gamma = 4\pi v_\infty R \sin(\alpha - \theta). \qquad (40.10)$$

40.4 Streamline flow around an ellipse and a plate Let S be the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (0<b<a).$$

Its foci lie at points $\pm c$, with $c=\sqrt{b^2-a^2}$. The solution of the problem of streamline flow around an ellipse can be reduced to the problem of streamline flow around a circle. The function that is the inverse of the Zhukovskii function, or $w(z)=z+\sqrt{z^2-c^2}$, maps the exterior of an ellipse on the exterior of a circle, $|z|>R$ (see Sec. 35.7). We select the branch of the root in the inverse of the Zhukovskii function, which is regular in the complex z plane with a cut along the segment $[-c, c]$, on which $\sqrt{z^2-c^2}$ is positive for real $z\in(c, +\infty)$. The radius R of the circle and the semiaxes of the ellipse are related through the following formulas (see Sec. 35.6): $a=\frac{1}{2}\left(R+\frac{1}{R}\right)$ and $b=\frac{1}{2}\left(R-\frac{1}{R}\right)$, from which it follows that $R=a+b$. In view of (40.7) we have

$$\begin{aligned} f(z) &= \frac{1}{2}\left[\overline{\mathbf{V}}_\infty w+\frac{\mathbf{V}_\infty (a+b)^2}{w}\right]+\frac{\Gamma}{2\pi i}\ln w \\ &= \frac{1}{2}\left[\overline{\mathbf{V}}_\infty (z+\sqrt{z^2-c^2})+\frac{\mathbf{V}_\infty (a+b)^2}{z+\sqrt{z^2-c^2}}\right]+\frac{\Gamma}{2\pi i}\ln(z+\sqrt{z^2-c^2}). \end{aligned}$$

The factor 1/2 appears because $\sqrt{z^2-c^2}\sim z\ (z\to\infty)$ and therefore $w'(\infty)=2$. Multiplying the numerator and denominator in the second term in the square brackets on the right-hand side of the above expression by $z-\sqrt{z^2-c^2}$, we finally obtain

$$\begin{aligned} f(z) = \frac{1}{2}\overline{\mathbf{V}}_\infty (z+\sqrt{z^2-c^2})+\frac{1}{2}\frac{(a+b)^2}{c^2}\mathbf{V}_\infty (z-\sqrt{z^2-c^2}) \\ +\frac{\Gamma}{2\pi i}\ln(z+\sqrt{z^2-c^2}). \quad (40.11) \end{aligned}$$

At $a=c$ and $b=0$ the ellipse becomes the segment $I=[-c, c]$, and from (40.11) we find the complex potential of streamline flow around a plate of length $2c$:

$$\begin{aligned} f(z) = \frac{1}{2}(\overline{\mathbf{V}}_\infty+\mathbf{V}_\infty)z+\frac{1}{2}(\overline{\mathbf{V}}_\infty-\mathbf{V}_\infty)\sqrt{z^2-c^2} \\ +\frac{\Gamma}{2\pi i}\ln(z+\sqrt{z^2-c^2}). \end{aligned}$$

Putting $\mathbf{V}_\infty=u_\infty+iv_\infty$, with u_∞ and v_∞ real, we obtain

$$f(z)=u_\infty z-iv_\infty\sqrt{z^2-c^2}+\frac{\Gamma}{2\pi i}\ln(z+\sqrt{z^2-c^2}). \quad (40.12)$$

The velocity of the flow can be calculated via (40.12):

$$\bar{V} = f'(z) = u_\infty + \frac{2\pi v_\infty z + \Gamma}{2\pi i \sqrt{z^2 - c^2}}, \tag{40.13}$$

and for an arbitrary value of circulation Γ the velocity becomes infinite at the edges of the plate, i.e. at the corner points of the boundary of the solid.

Here for the first time we have encountered the case where the boundary of the solid in streamline flow is not smooth. For such solids the problem of streamline flow requires additional physical assumptions. Chaplygin was the first to find such a condition. Suppose the solid has a sharp edge A. Then the flow velocity must be finite at the sharp edge of the profile. An alternative formulation of Chaplygin's condition reads as follows: *the sharp edge of the profile is a converging point*. If the profile has only one sharp edge, Chaplygin's condition uniquely determines the circulation.

Since a plate has two sharp edges $z = \pm c$, Chaplygin's condition can be met only at one of the edges. Suppose the condition is met at edge $z = c$. Then (40.13) and Chaplygin's condition gives the one possible value of the circulation:

$$\Gamma_0 = -2\pi c v_\infty, \tag{40.14}$$

while (40.13) yields the following distribution of velocities:

$$\bar{V} = u_\infty - i v_\infty \sqrt{\frac{z-c}{z+c}} \tag{40.15}$$

On edge $z = -c$ the flow velocity becomes infinite.

40.5 Streamline flow around the Zhukovskii profile Let γ be the arc of a circle passing through the points $z = \pm a$, with the middle of the arc passing through point $z = ih$, and let γ' be the circle centered at point $w = ih$ and passing through the points $w = \pm a$ (Fig. 127). Example 40 in Sec. 35 shows that the function $w = z + \sqrt{z^2 - a^2}$ maps the exterior of γ conformally onto the exterior of γ'. Note that the tangent to γ at point $z = -a$ forms an angle $\alpha = 2 \arctan (h/a)$ with the real axis, while the tangent at $z = a$ forms an angle $\beta = (\pi - \alpha)/2$.

Suppose γ_d' is a circle centered at point $w_d = ih - de^{-i\alpha/2}$, lying inside γ', and touching γ' at point $w = -a$. The radius of γ_d' is $R_d = \sqrt{a^2 + h^2} + d$. The function $w(z)$ maps the exterior of γ_d conformally onto the exterior of γ_d' (Fig. 127), which resembles the cross section of an airplane wing and is known as the *Zhukovskii profile*.

Let us solve the problem of streamline flow around the Zhukovskii profile by reducing it to the problem of streamline flow around a circle. The function

$$w=\frac{1}{2}\left(z+\sqrt{z^2-a^2}-w_d\right) \tag{40.16}$$

maps the exterior of γ_d conformally onto the exterior of the circle $|z|>R_d$, and from (40.7) we can find the complex potential:

$$f(z)=\frac{1}{2}\left[\overline{V}_\infty w+\frac{V_\infty R_d^2}{w}\right]+\frac{\Gamma}{2\pi i}\ln w,$$

where $w=w(z)$ is given by (40.16). The circulation Γ can be found from Chaplygin's condition: the sharp edge of the profile must be a convergence point. The image of point $z=-a$ is $R_0e^{-i\alpha/2}$, with $R_0>0$ (Fig. 127), and (40.10) yields the following value of the circulation:

$$\Gamma=-2\pi v_\infty\left(\sqrt{a^2+h^2}+d\right)\sin\left(\theta+\frac{\alpha}{2}\right).$$

By Zhukovskii's theorem, the aerodynamic lift on an airplane wing is

$$|\mathbf{P}|=2\pi\rho v_\infty^2\left(\sqrt{a^2+h^2}+d\right)\left(\sin\left(\theta+\frac{\alpha}{2}\right)\right).$$

Chapter VII

Simple Asymptotic Methods

41 Some Asymptotic Estimates

In this section we will consider simple asymptotic estimates of roots of transcendental equations, integrals, and series. Asymptotic estimates are relationships of the type

$$f(x) = O(g(x)), \quad f(x) = o(g(x)), \quad f(x) \sim g(x)$$

as $x \to a$. The meaning of the symbols O, o, and $\sim$ was explained in Sec. 4.

41.1 The asymptotic behavior of roots of equations We start with simple examples.

Example 1. Suppose a function $f(z)$ is regular and has a simple zero at point $z_0 \neq \infty$, i.e. $f(z_0) = 0$, $f'(z_0) \neq 0$. We consider the equation

$$f(z) = \varepsilon, \tag{41.1}$$

where ε is a small complex number. For small values of $|\varepsilon|$ (this is what we mean by ε being small), Eq. (41.1) has a root $z(\varepsilon)$ that is close to point z_0. We wish to establish the asymptotic behavior of $z(\varepsilon)$ as $\varepsilon \to 0$.

This problem can be solved via the inverse function theorem (see Sec. 13). By this theorem, in a small neighborhood of point $\varepsilon = 0$ there is a function $z(\varepsilon)$ that is the inverse of $f(z)$ (i.e. $f(z(\varepsilon)) \equiv \varepsilon$ for small values of $|\varepsilon|$). The function $z(\varepsilon)$ is regular at point $\varepsilon = 0$ and can be expanded in a Taylor series

$$z(\varepsilon) = z_0 + \sum_{n=1}^{\infty} c_n \varepsilon^n, \tag{41.2}$$

which converges in a circle $|\varepsilon| < \rho$ for a small positive ρ. The expansion coefficients of (41.2) can be calculated by the Bürmann-Lagrange formulas (31.39). In particular, $c_1 = 1/f'(z_0)$.

Expansion (41.2) yields the following asymptotic formulas:

$$z(\varepsilon) = z_0 + \sum_{k=1}^{N} c_k \varepsilon^k + O(\varepsilon^{N+1}) \quad (\varepsilon \to 0). \tag{41.3}$$

Here for $N = 1$ we have

$$z(\varepsilon) = z_0 + O(\varepsilon),$$

while for $N = 2$ we have

$$z(\varepsilon) = z_0 + \frac{\varepsilon}{f'(z_0)} + O(\varepsilon^2). \quad \square \qquad (41.4)$$

Remark 1. If we are interested only in the first few terms in expansion (41.2), we can employ the method of undetermined coefficients. Precisely, we write $z(\varepsilon)$ in the form of the expansion (41.2) and substitute it into Eq. (41.1). The function $f(z)$ can be expanded for small values of $|z - z_0|$ in a Taylor series:

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n,$$

and we write Eq. (41.1) in the form

$$\sum_{n=1}^{\infty} a_n \left(\sum_{k=1}^{\infty} c_k \varepsilon^k\right)^n - \varepsilon = 0.$$

Expanding the left-hand side of this equation in a power series in ε and nullifying the coefficients of the various power of ε, we arrive at a system of recurrence formulas for determining $c_1, c_2, \ldots$.

Example 2. Let us take Eq. (41.1) with a function $f(z)$ that is egular and has a zero of an order not less than 2 at point z_0, i.e.

$$f(z_0) = f'(z_0) = \ldots = f^{n-1}(z_0) = 0, \quad f^{(n)}(z_0) \neq 0.$$

The second inverse function theorem (see Sec. 32) implies that for small nonzero ε's Eq. (41.1) has exactly n different solutions $z_0(\varepsilon)$, $z_1(\varepsilon), \ldots, z_n(\varepsilon)$ (which are elements of a single n-valued analytic function). In this example, instead of using the inverse function theorem directly it is more convenient to transform Eq. (41.1) into an equation for which the conditions of Example 1 are applicable. Let us assume that ε varies not in the entire neighborhood of point $\varepsilon = 0$ but only in a sector S with the vertex at point $\varepsilon = 0$. For the sake of definiteness we will take the sector S: $|\varepsilon| > 0$, $|\arg \varepsilon| \leqslant \pi - \delta$ $(0 < \delta < \pi)$.

By hypothesis, in the neighborhood of point z_0 we have

$$f(z) = (z - z_0)^n g(z), \qquad (41.5)$$

where $g(z)$ is regular and nonzero at point z_0.

The equation $w^n = \varepsilon$ $(\varepsilon \neq 0)$ has exactly n different solutions

$$w_j = e^{2\pi i j/n} \sqrt[n]{\varepsilon}, \quad 0 \leqslant j \leqslant n-1, \qquad (41.6)$$

where $\sqrt[n]{\varepsilon}$ is a fixed value of the root. Suppose $\varepsilon \in S$; by $\sqrt[n]{\varepsilon}$ we denote the regular branch of the root on which $\sqrt[n]{\varepsilon}$ is positive for $\varepsilon > 0$.

The function $\sqrt[n]{g(z)}$ splits in a small neighborhood U of point z_0 into n regular branches; by $\sqrt[n]{g(z)}$ we denote one of the branches.

To isolate a branch it is sufficient to specify the value of the root $\sqrt[n]{g(z_0)}$. Since $f(z) = ((z - z_0)\sqrt[n]{g(z)})^n$, we conclude, by (41.6), that Eq. (41.1) splits in U into n independent equations:

$$(z - z_0)\sqrt[n]{g(z)} = e^{2\pi ij/n}\sqrt[n]{\varepsilon}, \quad 0 \leqslant j \leqslant n-1. \tag{41.7}$$

If $\tilde{f}(z)$ is the left-hand side of Eq. (41.7), then $\tilde{f}'(z_0) = \sqrt[n]{g(z_0)} \neq 0$, and for each equation in (41.7) the conditions of Example 1 are met. Hence, for a small $\varepsilon \in S$ Eq. (41.1) has exactly n solutions

$$z_j(\varepsilon) = z_0 + \sum_{k=1}^{\infty} c_k \left(e^{2\pi ij/n}\sqrt[n]{\varepsilon}\right)^k, \quad 0 \leqslant j \leqslant n-1. \tag{41.8}$$

The series in (41.8) converge for small ε's. The reader will recall that the symbol $\sqrt[n]{\varepsilon}$ on the right-hand side denotes a regular branch of the root in S that is positive for $\varepsilon > 0$. The coefficients c_k can be calculated via the Bürmann-Lagrange formula (31.39). For instance (see Sec. 32.1),

$$c_1 = \sqrt[n]{\frac{n!}{f^{(n)}(z_0)}}. \quad \square$$

Example 3. Let us consider the equation

$$z^3 - z^2 = \varepsilon \tag{41.9}$$

and examine the asymptotic behavior of its roots as $\varepsilon \to 0$. At $\varepsilon = 0$ Eq. (41.9) has a simple root $z_1 = 1$ and a second order root $z_2 = 0$. For small ε's Eq. (41.9) has a root $z_1(\varepsilon) \to z_1 = 1$ as $\varepsilon \to 0$ and two roots $z_{2,3}(\varepsilon) \to z_2 = 0$ as $\varepsilon \to 0$.

Let us establish the asymptotic behavior of the root $z_1(\varepsilon)$. We put $z = 1 + \varepsilon$. Then $\zeta + 2\zeta^2 + \zeta^3 = \varepsilon$. Example 1 yields $\zeta = \sum_{k=1}^{\infty} c_k \varepsilon^k$, so that

$$(c_1\varepsilon + c_2\varepsilon^2 + \ldots) + 2(c_1^2\varepsilon^2 + \ldots) + O(\varepsilon^3) - \varepsilon = 0.$$

Nullifying the coefficients of ε and ε^2, we obtain

$$c_1 - 1 = 0, \quad c_2 + 2c_1^2 = 0,$$

whence

$$z_1(\varepsilon) = 1 + \varepsilon - 2\varepsilon^2 + O(\varepsilon^3) \quad (\varepsilon \to 0).$$

Now let us establish the asymptotic behavior of $z_{2,3}(\varepsilon)$. Suppose $\varepsilon > 0$, for the sake of simplicity, and $\sqrt{\varepsilon} > 0$. Equation (41.9) splits in the neighborhood of point $z = 0$ into two equations:

$$z\sqrt{1-z} = i\sqrt{\varepsilon}, \quad z\sqrt{1-z} = -i\sqrt{\varepsilon},$$

where $\sqrt{1-z}\,|_{z=0}=1$. The first equation has the form

$$z-\frac{z^2}{2}+O(z^3)=i\sqrt{\varepsilon}.$$

We put $z=\sum_{k=1}^{\infty} c_k(\sqrt{\varepsilon})^k$. Then

$$c_1\sqrt{\varepsilon}+c_2\varepsilon-\frac{c_1^2}{2}\varepsilon+O(\varepsilon^{3/2})=i\sqrt{\varepsilon},$$

which yields $c_1=i$ and $c_2=-1/2$. Hence,

$$z_2(\varepsilon)=i\sqrt{\varepsilon}-\frac{\varepsilon}{2}+O(\varepsilon^{3/2}) \qquad (\varepsilon\to+0),$$

$$z_3(\varepsilon)=-i\sqrt{\varepsilon}-\frac{\varepsilon}{2}+O(\varepsilon^{3/2}) \quad (\varepsilon\to+0).$$

These formulas are applicable when $\varepsilon\to 0$, $\varepsilon\in S$, where S is any sector with its vertex at point $z=0$. □

Example 4. Consider the equation

$$z-\sin z=\varepsilon.$$

At $\varepsilon=0$ this equation has one root of order 3, so that for small ε's the equation has three roots lying close to $z=0$. Let us establish the asymptotic behavior of these roots as $\varepsilon\to 0$, $\varepsilon>0$. For small value of $|\varepsilon|$ we have

$$z-\sin z=\frac{z^3}{6}+O(z^5)=\varepsilon,$$

whence (41.8) yields

$$z_j(\varepsilon)=e^{2\pi ij/3}\sqrt[3]{6\varepsilon}+O(\varepsilon^{2/3}) \quad (\varepsilon\to 0), \quad j=0,\ 1,\ 2.$$

Here $\sqrt[3]{\varepsilon}>0$ for $\varepsilon>0$. □

Example 5. Let us consider Eq. (41.1) with $f(z)$ regular and having a zero at point $z=\infty$. Then for small values of $|\varepsilon|$ Eq. (41.1) has one or several solutions that tend to infinity as $\varepsilon\to 0$. We have

$$f(z)=z^{-n}\sum_{k=0}^{\infty} a_k z^{-k}, \quad a_0\neq 0,$$

where the series is convergent in the domain $|z|>R$ for large R's, and n is a positive integer. The substitution $\zeta=1/z$ transforms Eq. (41.1) into

$$\zeta^n g(\zeta)=\varepsilon, \tag{41.10}$$

where

$$g(\zeta)=\sum_{k=0}^{\infty} a_k\zeta^k, \quad g(0)\neq 0.$$

We have thus arrived at the equations discussed in Examples 1 and 2. □

Example 6. Consider the equation

$$p(z) = \lambda, \tag{41.11}$$

where $p(z)$ is a polynomial of a degree not less than two:

$$p(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n, \quad a_0 \neq 0.$$

Let us examine the behavior of the roots of this equation as $\lambda \to \infty$, $\lambda \in S$, with S the sector $|\arg \lambda| \leqslant \pi - \delta$ $(0 < \delta < \pi)$. We put $\varepsilon = 1/\lambda$ and $\zeta = 1/z$. Then we can write Eq. (41.11) in the form

$$\frac{\zeta^n}{a_0 + a_1\zeta + \ldots + a_n\zeta^n} = \varepsilon.$$

Example 2 implies that Eq. (41.11) has n roots

$$z_j(\lambda) = e^{2\pi i j/n} \sqrt[n]{\lambda} \left[\frac{1}{\sqrt[n]{a_0}} + O\left(\frac{1}{\sqrt[n]{\lambda}}\right) \right] \quad (\lambda \to \infty,\ \lambda \in S).$$

Here $0 \leqslant j \leqslant n - 1$, the value of $\sqrt[n]{a_0}$ is fixed, and $\sqrt[n]{\lambda}$ is the regular branch of the root that is positive for $\lambda > 0$. □

If $f(z)$ is a rational function, then, as $\varepsilon \to 0$, each root of Eq. (41.1) tends to one of the roots of the limiting equation $f(z) = 0$. But if $f(z)$ is not a rational function, then the roots of Eq. (41.1) behave in a much more complex way as $\varepsilon \to 0$.

Example 7. The equation $e^z = \varepsilon$ has no solutions at $\varepsilon = 0$. But if $\varepsilon \neq 0$, all the solutions of this equation are given by the formula

$$z_k(\varepsilon) = 2k\pi i + \ln \varepsilon$$

($\ln \varepsilon$ is a fixed value of the logarithm), and all roots $z_k(\varepsilon)$ tend to infinity as $\varepsilon \to 0$. □

Let us consider examples of another type. Suppose the function $f(z)$ is entire or meromorphic and let the equation

$$f(z) = 0 \tag{41.12}$$

have an infinitude of roots, $z_1, z_2, \ldots, z_n, \ldots$. By the uniqueness theorem, each bounded domain in the complex z plane can have only a finite number of roots of Eq. (41.12); hence, $z_n \to \infty$ as $n \to \infty$. Let us examine the asymptotic behavior of the roots of Eq. (41.12) for some elementary functions $f(z)$.

Example 8. The equation

$$\tan z = \frac{1}{z} \tag{41.13}$$

has an infinite number of real roots, which the graphs of the function $\tan x$ and $1/x$ clearly show. Since both $\tan x$ and $1/x$ are odd functions, the real roots of Eq. (41.13) lie symmetrically in relation to point $x = 0$. Suppose x_n is a root of Eq. (41.13) that lies in the interval $n\pi - \pi/2 < x < n\pi + \pi/2$. Let us find the asymptotic behavior of x_n as $n \to +\infty$. Assuming that $x = n\pi + y$ and $1/n\pi = \varepsilon$, we arrive at an equation for y:

$$f(y) = \varepsilon, \; f(y) = \frac{\sin y}{\cos y - y \sin y} \tag{41.14}$$

At $\varepsilon = 0$ Eq. (41.14) has a simple root $y = 0$. Let us establish the asymptotic behavior of the solution $y(\varepsilon)$ of Eq. (41.14) such that $y(\varepsilon) \to 0$ as $\varepsilon \to 0$. This equation is of type (41.1), where the function $f(y)$ is regular at point $y = 0$ and this point is a simple zero for $f(y)$. Example 1 yields

$$y(\varepsilon) = \sum_{k=1}^{\infty} c_k \varepsilon^k,$$

with $c_1 = 1$, so that

$$x_n = n\pi + \frac{1}{n\pi} + \sum_{k=2}^{\infty} \frac{c^k}{(n\pi)^k}$$

for large n's. In particular,

$$x_n = n\pi + \frac{1}{n\pi} + O\left(\frac{1}{n^2}\right) \quad (n \to +\infty).$$

Equation (41.14) also has the roots $\{-x_n\}$, $n = 1, 2, \ldots$. □

Remark 2. It can be proved that the only roots Eq. (41.13) has are real roots.

Example 9. Consider the equation

$$z - \ln z = \lambda \tag{41.15}$$

in D, with D the complex z plane with a cut along the semiaxis $(-\infty, 0]$ and without the closed circle $|z| \leqslant \rho$, $\rho > 0$. Here $\ln z$ is the regular branch of the logarithm in D that is positive for real $z = x > 0$.

Let us establish the asymptotic behavior of the roots of Eq. (41.15) as $\lambda \to +\infty$. Suppose that Eq. (41.15) has a root $z = z(\lambda)$ for large values of λ. Then $z(\lambda) \to \infty$ as $\lambda \to +\infty$. This follows from the fact that the function $z - \ln z$ is bounded in any bounded domain $\tilde{D} \subset D$.

Further, $\ln z = \ln |z| + i\varphi$, $|\varphi| < \pi$ for $z \in D$, so that $|\ln z| = o(|z|)$ as $z \to \infty$, $z \in D$. Hence, $z(\lambda) \sim \lambda$ as $\lambda \to +\infty$ and

$$z(\lambda) = \lambda(1 + \zeta(\lambda)),$$

where $\zeta(\lambda) \to 0$ as $\lambda \to +\infty$.

Let us first show that for $\lambda \geqslant \lambda_0 > 0$ and a large λ_0 Eq. (41.15) has in D only one root $z(\lambda)$, with

$$z(\lambda) = \lambda + O(\ln \lambda) \quad (\lambda \to +\infty). \tag{41.16}$$

Putting $z = \lambda(1 + \zeta)$ in (41.15), we arrive at the following equation:

$$\zeta = \frac{\ln \lambda}{\lambda} + \frac{\ln(1+\zeta)}{\lambda}. \tag{41.17}$$

Now we can use Rouche's theorem (Sec. 30). We write Eq. (41.17) in the form

$$\zeta - (\varepsilon + \delta \ln(1 + \zeta)) = 0, \tag{41.18}$$

where $\varepsilon = (\ln \lambda)/\lambda$ and $\delta = 1/\lambda$ are small parameters, with $\delta = o(\varepsilon)$ $(\lambda \to +\infty)$. Consider the circle K_ε: $|\zeta| \leqslant 2\varepsilon$. Since $\varepsilon \to 0$ as $\lambda \to +\infty$, we conclude that there is a positive λ_0 such that the circle K_ε lies within the circle K: $|\zeta| < 1/2$ for $\lambda \geqslant \lambda_0$. The function $\ln(1 + \zeta)$ is regular and bounded in K, i.e. $|\ln(1 + \zeta)| \leqslant M$, and this is true for the circle K_ε for $\lambda \geqslant \lambda_0$. At the boundary $|\zeta| = 2\varepsilon$ of K_ε we have

$$|-\varepsilon - \delta \ln(1 + \zeta)| \leqslant \varepsilon + M\delta,$$

and since $\delta = o(\varepsilon)$ as $\lambda \to +\infty$, we can write

$$|-\varepsilon - \delta \ln(1 + \zeta)| < 2\varepsilon = |\zeta|$$

for $\lambda \geqslant \lambda_1$ if λ_1 is large. By Rouche's theorem, the number of roots of Eq. (41.18) lying inside K_ε is equal to the number of roots of the equation $\zeta = 0$. Hence, for $\lambda \geqslant \lambda_1$ Eq. (41.18) has a single root $\zeta_0(\lambda)$ in the circle K_ε, with $|\zeta_0(\lambda)| < 2|\varepsilon|$, i.e. $\zeta_0(\lambda) = O((\ln \lambda)/\lambda)$ $(\lambda \to +\infty)$. We have thus proved the validity of (41.16).

Let us make (41.16) more exact. To this end we use the iteration method, i.e. we substitute the estimate $\zeta = O((\ln \lambda)/\lambda)$ into the right-hand side of Eq. (41.17). We then have

$$\zeta = \frac{\ln \lambda}{\lambda} + \frac{1}{\lambda} \ln\left(1 + O\left(\frac{\ln \lambda}{\lambda}\right)\right) = \frac{\ln \lambda}{\lambda} + O\left(\frac{\ln \lambda}{\lambda^2}\right),$$

since $\ln(1+\zeta) = O(\zeta)$ $(\zeta \to 0)$. Hence,

$$z(\lambda) = \lambda + \ln \lambda + O\left(\frac{\ln \lambda}{\lambda}\right) \quad (\lambda \to +\infty), \tag{41.16'}$$

and we have an asymptotic formula for $z(\lambda)$ that is more exact than (41.16). If we substitute the more precise formula for ζ into the right-hand side of Eq. (41.17), we obtain an asymptotic formula for $z(\lambda)$ that is still more exact. The process can be continued.

Finally, it is easy to verify that the asymptotic formula (41.16) is valid for $\lambda \in S$, $|\lambda| \to \infty$, where S is a sector of the type $|\arg \lambda| \leqslant \pi - \alpha$ $(0 < \alpha < \pi)$. In these formulas $\ln \lambda$ is the regular branch of the logarithm in S that is positive for real $\lambda > 1$. □

Example 10. Consider the equation

$$e^z = az \quad (a \neq 0). \tag{41.19}$$

We will show that this equation has an infinitude of roots and will examine their asymptotic behavior.

In any bounded domain in the complex z plane there can be only a finite number of roots of Eq. (41.19), since $e^z - az$ is an entire function. Moreover, the function $|e^z|$ exponentially increases along any ray $\arg z = \alpha$ that lies in the right half-plane, and exponentially decreases along any ray $\arg z = \alpha$ that lies in the left half-plane, while the function $|az|$ increases linearly along any ray. This implies that the roots of Eq. (41.19) group around the imaginary axis, i.e. all roots except a finite number lie in a sector that contains the imaginary axis. Here is a rigorous proof of this proposition.

In $\operatorname{Re} z \leqslant 0$, $|z| > 1/|a|$ there are no roots of Eq. (41.19) because $|e^z| \leqslant 1$, while $|az| > 1$.

Suppose $\tilde{S}_\varepsilon$ is the sector $|\arg z| \leqslant \pi/2 - \varepsilon$. Let us show that for a fixed $\varepsilon \in (0, \pi/2)$ there can be only a finite number of roots of Eq. (41.19) in $\tilde{S}_\varepsilon$. For $z \in \tilde{S}_\varepsilon$ we have $z = re^{i\varphi}$, where $|\varphi| \leqslant \pi/2 - \varepsilon$, so that $|e^z| = e^{r \cos \varphi} \geqslant e^{r \sin \varepsilon}$. Since $|az| = |a| r = o(e^{r \sin \varepsilon})$ $(r \to +\infty)$, we can write $|e^z| > |az|$ $(z \in \tilde{S}_\varepsilon, |z| > R)$ for large R's, and Eq. (41.19) has no roots in the domain $|z| > R$, $z \in \tilde{S}_\varepsilon$. Hence, Eq. (41.19) has only a finite number of roots in sector $\tilde{S}_\varepsilon$.

Now let us consider the sector S_ε: $\pi/2 - \varepsilon \leqslant \arg z \leqslant \pi/2$. If Eq. (41.19) has an infinite number of roots in the sector S_ε, the roots tend to infinity as $n \to \infty$. Suppose $z \in S_\varepsilon$ is a root of Eq. (41.19). Then there is an integer n such that

$$z = 2\pi i n + \ln a + \ln z. \tag{41.20}$$

Here $\ln a$ is a fixed value of the logarithm, and $\ln z$ is the regular branch of the logarithm in the half-plane $\operatorname{Re} z > 0$ that assumes positive values on the semiaxis $(0, \infty)$. Thus, we have arrived at the equation studied in Example 9, with $\lambda = 2\pi i n + \ln a$. To establish the asymptotic behavior of the roots we follow the same line of reasoning as in Example 9. Putting $z = 2\pi in + \zeta$, we obtain

$$\zeta = \ln (2\pi ina) + \ln \left(1 + \frac{\zeta}{2\pi in}\right),$$

so that $\zeta \sim \ln (2\pi ina)\ (n \to \infty)$, Then

$$\ln \left(1 + \frac{\zeta}{2\pi in}\right) = O\left(\frac{\zeta}{2\pi in}\right) = O\left(\frac{\ln n}{n}\right),$$

$$z_n = 2\pi in + \ln n + \ln (2\pi in) + O\left(\frac{\ln n}{n}\right) (n \to +\infty). \quad (41.21)$$

Substituting $z = z_n$ from (41.21) into Eq. (41.20), we can make this asymptotic formula more precise.

In (41.21) we have

$$\ln n > 0, \quad \operatorname{Re} \ln (2\pi in) > 0.$$

A similar collection of roots of Eq. (41.19) lies in the sector $-\pi/2 \leqslant \arg z \leqslant -\pi/2 + \varepsilon$. □

Example 11. Let us establish the asymptotic behavior of the roots of the equation

$$\sin z = z. \quad (41.22)$$

The function $|\sin z|$ grows exponentially along any ray that starts at point $z = 0$ except the two real semiaxes. For this reason the roots of Eq. (41.22) can group only around the real axis. Let us prove this proposition. Suppose $\tilde{S}_\varepsilon$ is the sector $\varepsilon \leqslant \arg z \leqslant \pi - \varepsilon$, where $0 < \varepsilon < \pi$ and ε is fixed. We put $z = re^{i\varphi}$. Then $\varepsilon \leqslant \varphi \leqslant \pi - \varepsilon$ for $z \in \tilde{S}_\varepsilon$. We have

$$|\sin z| = \frac{1}{2} |e^{iz} - e^{-iz}| \geqslant \frac{1}{2} (e^{r \sin \varepsilon} - e^{-r \sin \varepsilon}), \quad z \in \tilde{S}_\varepsilon.$$

If $z \in \tilde{S}_\varepsilon$, $|z| \geqslant R$, and R is large, then $|\sin z| > |z|$ and in this domain Eq. (41.22) has no roots. Hence, there is only a finite number of roots of Eq. (41.22) in $\tilde{S}_\varepsilon$. The same is true for the sector $-\pi + \varepsilon \leqslant \arg z \leqslant -\varepsilon$, since $f(z) = \sin z - z$ is an odd function.

The roots of Eq. (41.22) are "grouped" in fours: z_0, $\bar{z}_0$, $-z_0$, and $-\bar{z}_0$, since $f(z)$ is odd and $f(\bar{z}) = \overline{f(z)}$ (if x is a real number, $f(x)$ is a real number, too). It is therefore sufficient to examine the asymptotic behavior of the roots in the sector S_ε: $0 \leqslant \arg z \leqslant \varepsilon$. Here ε is a fixed positive number that can be chosen as small as desired.

We write Eq. (41.22) in the form

$$e^{iz} - e^{-iz} = 2iz.$$

Solving this equation for e^{iz}, we obtain

$$e^{iz}=i\,(z+\sqrt{z^2-1}). \tag{41.23}$$

The function $\sqrt{z^2-1}$ splits in sector S_ε into two regular branches $f_j(z)$, $j=1, 2$, with $f_2(z)\equiv -f_1(z)$. Let $f_1(z)$ be the branch that is positive for real $z=x>1$. Then $f_1(z)\sim z$ and $f_2(z)\sim -z$ for $z\in S_\varepsilon$. Hence, $z+f_1(z)\sim 2z$ and $z+f_2(z)\sim 1/2z$ as $z\to\infty$, $z\in S_\varepsilon$ (see Example 11 in Sec. 24). Since $\operatorname{Im} z\geqslant 0$ for $z\in S_\varepsilon$, we conclude that $|e^{iz}|\leqslant 1$ in this sector. For this reason Eq. (41.23) has the form $e^{iz}=i\,(z+f_2(z))$ for $z\in S_\varepsilon$, $|z|\geqslant 1/2$. Finding the logarithm of this relationship, we obtain

$$z=2\pi n+\frac{\pi}{2}-i\ln g(z). \tag{41.24}$$

Here $g(z)=z+f_2(z)$, and n is a positive integer; by $\ln z$ we denote the regular branch of the logarithm in S_ε that assumes real values for real z's. The reader will recall that the roots of Eq. (41.24) tend to infinity as $n\to\infty$, which means that $z_n\sim 2\pi n$ as $n\to\infty$. Moreover, for $z\in S_\varepsilon$ and $z\to\infty$ we have

$$g(z)=\frac{1}{2z}+O\left(\frac{1}{z^2}\right),$$

$$\ln g(z)=-\ln(2z)+\ln\left(1+O\left(\frac{1}{z}\right)\right)=-\ln(2z)+O\left(\frac{1}{z}\right),$$

and Eq. (41.24) takes the form

$$z=2\pi n+\frac{\pi}{2}+i\ln(2z)+O\left(\frac{1}{z}\right)$$

(here $z=z_n$). If we follow the line of reasoning developed in Example 9, we reveal the behavior of the roots

$$z_n=2\pi n+i\ln(4\pi n)+\frac{\pi}{2}+O\left(\frac{\ln n}{n}\right)\quad (n\to+\infty). \tag{41.25}$$

The other three collections of roots of Eq. (41.22) are $\{\bar z_n\}$, $\{-z_n\}$, and $\{-\bar z_n\}$. □

41.2 Simple estimates of integrals Let us consider the integral

$$F(x)=\int_a^x f(t)\,dt. \tag{41.26}$$

We are interested in the asymptotic behavior of the integral $F(x)$ as $x\to+\infty$. If the integral on the right-hand side has a finite value, then, obviously,

$$F(x)=\int_a^\infty f(t)\,dt+o(1)\quad (x\to+\infty).$$

In this case it is natural to study the behavior of the integral $G(x) = F(+\infty) - F(x)$ as $x \to +\infty$, i.e. we wish to study the function

$$G(x) = \int_x^\infty f(t)\,dt. \tag{41.27}$$

Under broad assumptions, asymptotic estimates can be integrated, i.e. if

$$f(t) \sim g(t) \quad (t \to +\infty), \tag{41.28}$$

then

$$\int_a^x f(t)\,dt \sim \int_a^x g(t)\,dt \quad (x \to +\infty). \tag{41.29}$$

Here are the corresponding sufficient conditions.

Theorem 1 *Suppose two functions, $f(t)$ and $g(t)$, are continuous for $t \geqslant a$, the function $g(t)$ is strictly positive for large values of t, and*

$$\int_a^{+\infty} g(t)\,dt = +\infty. \tag{41.30}$$

Then from (41.28) *follows* (41.29).

Proof. By L'Hospital's rule,

$$\lim_{x \to +\infty} \frac{\int_a^x f(t)\,dt}{\int_a^x g(t)\,dt} = \lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1.$$

The applicability of this rule follows from (41.30).

In exactly the same manner we can prove

Corollary 1 *Suppose the conditions of Theorem 1 are met and*

$$f(t) = o(g(t)) \quad (t \to +\infty).$$

Then

$$\int_a^x f(t)\,dt = o\left(\int_a^x g(t)\,dt\right) \quad (x \to +\infty).$$

Corollary 2 *Suppose the conditions of Theorem 1 are met and*

$$f(t) = O(g(t)) \quad (t \geqslant b \geqslant a).$$

Then

$$\int_a^x f(t)\,dt = O\left(\int_a^x g(t)\,dt\right) \quad (x \to +\infty).$$

Proof. By hypothesis, there is a positive constant C such that

$$|f(t)| \leqslant Cg(t) \quad (t \geqslant b).$$

Hence, for $x \geqslant b$ we have

$$\left|\int_b^x f(t)\,dt\right| \leqslant C \int_b^x g(t)\,dt \equiv CH(x).$$

Moreover, for $x \geqslant b$ we have

$$|F(x)| = \left|\left(\int_a^b + \int_b^x\right) f(t)\,dt\right| \leqslant C_1 + CH(x),$$

where $C_1 = \int_a^b |f(t)|\,dt$. Since $H(x) \to +\infty$ as $x \to +\infty$, we conclude that

$$C_1 + CH(x) = CH(x)(1 + o(1)) \leqslant 2CH(x)$$

for large values of x. This proves Corollary 2.

Example 12. From Theorem 1 and Corollaries 1 and 2 it follows that if $\alpha > -1$, $C \neq 0$, then the following asymptotic estimates holds as $x \to +\infty$:

$$f(x) \sim Cx^\alpha \Rightarrow \int_a^x f(t)\,dt \sim \frac{Cx^{\alpha+1}}{\alpha+1},$$

$$f(x) = o(x^\alpha) \Rightarrow \int_a^x f(t)\,dt = o(x^{\alpha+1}),$$

$$f(x) = O(x^\alpha) \Rightarrow \int_a^x f(t)\,dt = O(x^{\alpha+1}).$$

Here $f(x)$ is a continuous function for $x \geqslant a$. $\square$

Example 13. Suppose we have a function $f(x)$ that is continuous for $x \geqslant a$ and $C \neq 0$. Then the following asymptotic estimates hold as $x \to +\infty$:

$$f(x) \sim \frac{G}{x} \Rightarrow \int_a^x f(t)\,dt \sim C \ln x,$$

$$f(x) = o\left(\frac{1}{x}\right) \Rightarrow \int_a^x f(t)\,dt = o(\ln x),$$

$$f(x) = O\left(\frac{1}{x}\right) \Rightarrow \int_a^x f(t)\,dt = O(\ln x).$$

These estimates follow from Theorem 1 and Corollaries 1 and 2. $\square$

Example 14. Let us consider the integral

$$F(x) = \int_0^x \sqrt{t^2+1}\,dt.$$

Since $\sqrt{t^2+1} \sim t$ as $t \to +\infty$, we conclude that $F(x) \sim x^2/2$ $(x \to +\infty)$.

We wish to examine the asymptotic behavior of $F(x)$ as $x \to +\infty$ in greater detail. By Taylor's formula we find that for $t \geqslant 2$ we have

$$\sqrt{t^2+1} = t\sqrt{1+\frac{1}{t^2}} = t + \frac{1}{2t} + O(t^{-3}).$$

Hence, for $x \geqslant 2$ we have

$$\int_2^x \sqrt{t^2+1}\,dt = \int_2^x \left(t + \frac{1}{2t}\right) dt + \int_2^x O(t^{-3})\,dt.$$

The second integral on the right-hand side has a finite value, so that

$$F(x) = \frac{x^2}{2} + \frac{\ln x}{2} + O(1) \quad (x \to +\infty).$$

In greater detail the proof is as follows:

$$F(x) = \left(\int_0^2 + \int_2^x\right) \sqrt{t^2+1}\,dt = \frac{x^2}{2} + \frac{\ln x}{2}$$

$$+ \left[-\left(\frac{t^2}{2} + \frac{\ln l}{2}\right)\Big|_{t=2} + \int_2^x O(t^{-3})\,dt + \int_0^2 \sqrt{t^2+1}\,dt\right].$$

The expression inside the brackets has the order of $O(1)$ as $x \to +\infty$. □

Theorem and Corollaries 1 and 2 remain valid, obviously, if we replace the semiaxis $[a, +\infty)$ with a finite half-open interval $[a, b)$ and condition (41.30) by the condition

$$\int_a^b g(t)\,dt = +\infty.$$

Example 15. Let us assume that $f(x) \sim Cx^{-\alpha}$ $(x \to +0)$, where $C \neq 0$, $\alpha > 1$, and the function $f(x)$ is continuous for $0 < x \leqslant a$. Then

$$\int_a^x f(t)\,dt \sim C\frac{x^{-\alpha+1}}{-\alpha+1} \quad (x \to +0).$$

Similarly, as $\to +0$,

$$f(x) \sim \frac{C}{x} \Rightarrow \int_a^x f(t)\,dt \sim C\ln x. \ \square$$

Let us consider integrals of the type (41.27).

Example 16. Suppose two functions, $f(t)$ and $g(t)$, are continuous at $t \geqslant a$, the function $g(t)$ is positive for large values of t, and the integral $\int_a^\infty g(t)\,dt$ has a finite value. Then the following asymptotic estimates holds as $x \to +\infty$:

$$f(x) \sim g(x) \Rightarrow \int_x^\infty f(t)\,dt \sim \int_x^\infty g(t)\,dt,$$

$$f(x) = o(g(x)) \Rightarrow \int_x^\infty f(t)\,dt = o\left(\int_x^\infty g(t)\,dt\right),$$

$$f(x) = O(g(x)) \Rightarrow \int_x^\infty f(t)\,dt = O\left(\int_x^\infty g(t)\,dt\right).$$

Indeed, in all three cases the convergence of the integral of $g(t)$ along the semiaxis $t \geqslant a$ implies the convergence of the integral of $f(t)$ along the same semiaxis. After this, just as in the proof of Theorem 1, we only need to employ L'Hospital's rule. □

Example 17. Suppose $\alpha > 1$ and $C \neq 0$ are constants, and the function $f(x)$ is continuous for $x \geqslant a$. Then the following asymptotic estimates hold as $x \to +\infty$:

$$f(x) \sim Cx^{-\alpha} \Rightarrow \int_x^\infty f(t)\,dt \sim \frac{Cx^{1-\alpha}}{\alpha-1},$$

$$f(x) = o(x^{-\alpha}) \Rightarrow \int_x^\infty f(t)\,dt = o(x^{1-\alpha}),$$

$$f(x) = O(x^{-\alpha}) \Rightarrow \int_x^\infty f(t)\,dt = O(x^{1-\alpha}). \quad \square$$

Example 18. Let us consider the integral

$$F(x) = \int_0^x \sqrt{k^2+v(t)}\,dt,$$

where $v(x)$ is continuous and nonnegative for $x \geqslant 0$, $v(x) \to 0$ as $x \to +\infty$, and k is a positive constant. Let us examine the asymptotic behavior of $F(x)$ as $x \to +\infty$.

(a) Since $\sqrt{k^2+v(x)} \sim k$ as $x \to +\infty$, in view of Theorem 1 we have

$$F(x) \sim kx \quad (x \to +\infty).$$

(b) Let us also assume that $\int_0^\infty v(t)\,dt < \infty$.

Then we can have more precise estimates for $F(x)$. We have

$$F(x) = \int_0^x (\sqrt{k^2+v(t)}-k)\,dt + kx$$

$$= kx + \int_0^\infty \frac{v(t)}{\sqrt{k^2+v(t)}+k}\,dt - \int_x^\infty \frac{v(t)}{\sqrt{k^2+v(t)}+k}\,dt. \qquad (41.31)$$

The second term on the right-hand side is $o(1)$ as $x \to +\infty$. Hence,

$$F(x) = kx + C + o(1) \quad (x \to +\infty),$$

$$C = \int_0^\infty \frac{v(t)}{\sqrt{k^2+v(t)}+k}\,dt.$$

(c) If we have additional information on $v(x)$, the estimate of $F(t)$ is still more precise. Suppose, for instance, that $v(x)+k^2$ is positive for $x \geqslant 0$ and

$$v(x) \sim Ax^{-\alpha} \quad (x \to +\infty),$$

where $A \neq 0$, $\alpha > 1$. Then

$$\frac{v(x)}{\sqrt{k^2+v(x)}+k} \sim \frac{A}{2k} x^{-\alpha} \, (x \to +\infty),$$

so that formula (41.31) and Example 17 yield

$$F(x) = kx + C + \frac{Ax^{-\alpha+1}}{2k(1-\alpha)} + o(x^{-\alpha+1}) \, (x \to +\infty). \quad \square$$

Example 19. Let us examine the asymptotic behavior, as $\varepsilon \to +0$, of the integral

$$F(\varepsilon) = \int_0^1 \frac{f(t)\,dt}{t+\varepsilon}.$$

Here $f(t)$ is a continuously differentiable function for $0 \leqslant t \leqslant 1$. Note that at $\varepsilon = 0$ the integral $F(\varepsilon)$ has no finite value of $f(0) \neq 0$ and has a finite value if $f(0) = 0$. We therefore represent this integral in the form

$$F(\varepsilon) = \int_0^1 \frac{f(0)}{t+\varepsilon}\,dt + \int_0^1 \frac{f(t)-f(0)}{t+\varepsilon}\,dt \equiv F_1(\varepsilon) + F_2(\varepsilon).$$

We have

$$F_1(\varepsilon) = f(0)\,[\ln(1+\varepsilon) - \ln \varepsilon] = -f(0) \ln \varepsilon + O(\varepsilon).$$

Let us show that

$$F_2(\varepsilon) = O(1) \quad (\varepsilon \to +0).$$

The function $f(t) - f(0)$ can be represented in the form $f(t) - f(0) = t\varphi(t)$, where $\varphi(t)$ is a continuous function for $0 \leqslant t \leqslant 1$. Hence,

$$|F_2(\varepsilon)| \leqslant \int_0^1 \frac{|\varphi(t)|\,t}{t+\varepsilon}\,dt \leqslant M \int_0^1 \frac{t}{t+\varepsilon}\,dt \leqslant M,$$

where $M = \max\limits_{0 \leqslant t \leqslant 1} |\varphi(t)|$. Thus,

$$\int_0^1 \frac{f(t)\,dt}{t+\varepsilon} = -f(0) \ln \varepsilon + O(1) \quad (\varepsilon \to +0). \quad \square$$

41.3 Asymptotic estimates of some series Let us consider the sum

$$S(n)=\sum_{k=0}^{n} f(k). \tag{41.32}$$

We are interested in the asymptotic behavior of $S(n)$ as $n \to +\infty$. The general case of this problem is extremely complex. We will restrict our discussion to series with terms of constant sign. One of the main methods of obtaining asymptotic estimates for series of the type (41.32) is to replace the sum by an integral.

Theorem 2 *Suppose the function $f(x)$ is nonnegative, continuous, and monotonic for $x \geqslant 0$. Then*

$$\sum_{k=0}^{n} f(k)=\int_0^n f(x)\,dx+O(1)+O(f(n)) \quad (n\to\infty). \tag{41.33}$$

Proof. Suppose $f(x)$ does not decrease. Then

$$\int_{k-1}^{k} f(x)\,dx \leqslant f(k) \leqslant \int_{k}^{k+1} f(x)\,dx.$$

Summing the inequalities for $k=1, 2, \ldots, n-1$, we obtain

$$\int_0^{n-1} f(x)\,dx \leqslant S(n)-f(0)-f(n) \leqslant \int_1^n f(x)\,dx.$$

Consequently,

$$\left|S(n)-f(0)-f(n)-\int_0^n f(x)\,dx\right| \leqslant \int_{n-1}^{n} f(x)\,dx \leqslant f(n), \tag{41.34}$$

which proves the validity of (41.33). The case where $f(x)$ does not increase can be treated similarly.

Example 20. Let us show that

$$\sum_{k=1}^{n} \frac{1}{k}=\ln n+O(1) \quad (n\to+\infty).$$

Theorem 2 implies that, as $n\to\infty$,

$$\sum_{k=1}^{n} \frac{1}{k}=\int_1^n \frac{dx}{x}+O(1)+O\left(\frac{1}{n}\right)=\ln n+O(1). \quad \square$$

Example 21. Suppose $\alpha > -1$. Then

$$\sum_{k=1}^{n} k^{\alpha} \sim \frac{n^{\alpha+1}}{\alpha+1} \quad (n \to \infty).$$

By Theorem 2, we obtain, as $n \to \infty$,

$$\sum_{k=1}^{n} k^{\alpha} = \int_{1}^{n} x^{\alpha}\, dx + O(1) + O(n^{\alpha})$$

$$= \frac{n^{\alpha+1}}{\alpha+1}\left[1 + O\left(\frac{1}{n^{\alpha}+1}\right) + O\left(\frac{1}{n}\right)\right] \sim \frac{n^{\alpha+1}}{\alpha+1}. \quad \square$$

Here is another result concerning (41.32) being replaced by the integral $\int_{0}^{n} f(x)\, dx$.

Theorem 3 *Let the function $f(x)$ be continuously differentiable for $x \geqslant 0$. Then*

$$\left|\sum_{k=0}^{n} f(k) - \int_{0}^{n} f(x)\, dx\right| \leqslant |f(0)| + \int_{0}^{n} |f'(x)|\, dx. \qquad (41.35)$$

Proof. We have

$$f(k) = \int_{k-1}^{k} f(k)\, dx = \int_{k-1}^{k} f(x)\, dx + g(k),$$

$$g(k) = \int_{k-1}^{k} [f(k) - f(x)]\, dx.$$

Let us estimate $|g(k)|$. Since

$$f(k) - f(x) = \int_{x}^{k} f'(t)\, dt,$$

for $k-1 \leqslant x \leqslant k$ we can write

$$|f(k) - f(x)| \leqslant \int_{k-1}^{k} |f'(t)|\, dt,$$

from which follows the estimate

$$|g(k)| \leqslant \int_{k-1}^{k} |f(k) - f(x)|\, dx \leqslant \int_{k-1}^{k} \left(\int_{k-1}^{k} |f'(t)|\, dt\right) dx,$$

$$= \int_{k-1}^{k} |f'(t)|\, dt.$$

Hence,

$$\left|\sum_{k=0}^{n} f(k) - \int_0^n f(x)\,dx\right| = \left|f(0) + \sum_{k=1}^{n} g(k)\right| \leqslant |f(0)| + \sum_{k=1}^{n} \int_{k-1}^{k} |f'(t)|\,dt,$$

from which (41.35) follows.

Corollary 3 *If the conditions of Theorem 3 are met, then*

$$\left|\sum_{k=n}^{\infty} f(k) - \int_n^{\infty} f(x)\,dx\right| \leqslant |f(n)| + \int_n^{\infty} |f'(x)|\,dx, \qquad (41.36)$$

on the assumption that all the series ***and*** *integrals in this formula are convergent.*

Note that Examples 20 and 21 could be investigated via Theorem 3.

Example 22. Let us show that for $\alpha > 1$

$$\sum_{k=n}^{\infty} \frac{1}{k^{\alpha}} \sim \frac{n^{-\alpha+1}}{\alpha-1}\ (n \to \infty).$$

We use the estimate (41.36). In the case at hand $f(x) = x^{-\alpha}$ and $f'(x) = -\alpha x^{-\alpha-1}$, so that

$$\int_n^{\infty} |f'(x)|\,dx = n^{-\alpha}, \quad \int_n^{\infty} f(x)\,dx = \frac{n^{-\alpha+1}}{\alpha-1},$$

and, as $n \to +\infty$,

$$\sum_{k=n}^{\infty} \frac{1}{k^{\alpha}} = \frac{n^{-\alpha+1}}{\alpha-1} + O(n^{-\alpha}). \ \square$$

42 Asymptotic Expansions

42.1 An example of an asymptotic expansion Let us take the function

$$f(x) = \int_x^{\infty} t^{-1} e^{x-t}\,dt,$$

where $x > 0$, and study its behavior as $x \to +\infty$. Integrating by parts, we obtain

$$f(x) = -\int_x^{\infty} t^{-1} d\,(e^{x-t}) = \frac{1}{x} - \int_x^{\infty} t^{-2} e^{x-t}\,dt.$$

Repeating integration by parts, we obtain

$$f(x)=\left[\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x^3}-\ldots+\frac{(-1)^n n!}{x^{n+1}}\right]$$
$$+(-1)^{n+1}(n+1)!\int_x^\infty e^{x-t}t^{-n-2}\,dt=S_n(x)+R_n(x),$$

where $S_n(x)$ is the expression inside the square brackets. Let us estimate $R_n(x)$. Since $e^{x-t}\leqslant 1$ for $x\leqslant t$, we can write

$$|R_n(x)|\leqslant(n+1)!\int_x^\infty t^{-n-2}\,dt=\frac{n!}{x^{n+1}}.$$

Consequently, as $x\to+\infty$,

$$f(x)=\frac{1}{x}-\frac{1!}{x^2}+\frac{2!}{x^3}-\ldots+\frac{(-1)^{n-1}(n-1)!}{x^n}+O\left(\frac{1}{x^{n+1}}\right). \qquad (42.1)$$

The absolute value of the remainder term does not exceed $2n$: x^{-n-1}.

We have arrived at a sequence of asymptotic formulas each of which is more exact than the preceding one:

$$f(x)=\frac{1}{x}+O\left(\frac{1}{x^2}\right),\quad f(x)=\frac{1}{x}-\frac{1}{x^2}+O\left(\frac{1}{x^3}\right),\ \ldots,$$

etc. These formulas enable us to approximately calculate $f(x)$ for large values of x since

$$|f(x)-S_{n-1}(x)|<\frac{2n!}{x^{n+1}}. \qquad (42.2)$$

The right-hand side of (42.2) is small for large x's. For instance, for $x\geqslant 2n$ we have

$$|f(x)-S_{n-1}(x)|<\frac{1}{n2^n}.$$

For this reason, for large values of x we can calculate $f(x)$ with a high accuracy if we take a large number of terms in the asymptotic expansion (42.1).

What is striking is that the asymptotic formula (42.1) (obtained for $x\to+\infty$) remains valid for not very large values of x as well. For instance, for $x=10$ and $n=5$ we obtain $S_5(10)=0.09152$ and $0<f(10)-S_5(10)<0.00012$, and the error introduced by the approximate formula $f(10)\approx S_5(10)$ is about 0.1 percent.

Note that a direct calculation of $f(x)$ (e.g. with a computer) is the more complex the larger the value of x, while the asymptotic formula (42.1) becomes more exact as x grows. The French astronomer and mathematician Pierre Laplace once wrote that an asymptotic method is the more exact the more it is needed.

The function $f(x)$ can be assigned a series:

$$f(x) \leftrightarrow \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}.$$

This series is convergent for any value of x. Indeed, the absolute value of the ratio of the $(n+1)$th term to the nth term is $(n+1)x^{-1} \to \infty$ as $n \to \infty$. Nevertheless, this divergent series, as shown earlier, can be used to approximately calculate $f(x)$.

The example that we have just given was studied by the Swiss mathematician and physicist Leonhard Euler as early as the 18th century. At present asymptotic methods are used widely in various fields of mathematics, physics, and technology. What makes asymptotic methods important is that they enable us to arrive at simple approximations for complex objects. This, in turn, offers the possibility of obtaining an adequate qualitative description of the phenomenon being considered. The present chapter will give the reader an understanding of some of the more well-known asymptotic methods.

Let us now turn to a more rigorous description of the concept of an asymptotic expansion.

42.2 The concept of an asymptotic expansion Suppose M is a set of points (on the real axis or in the complex plane) and a is a limit point of this set. The sequence of functions $\{\varphi_n(z)\}$, $n = 0, 1, 2, \ldots$, with the functions being defined for $x \in M$ in a neighborhood of point a, is said to be an *asymptotic sequence* (as $x \to a$, $x \in M$) if for each n we have

$$\varphi_{n+1}(x) = o(\varphi_n(x)) \quad (x \to a, x \in M).$$

Here are some examples of asymptotic sequences.

Example 1. The sequence of the functions $\varphi_n(x) = x^n$ is asymptotic as $x \to 0$ (we can take a neighborhood or half-neighborhood of point $a = 0$ as M). □

Example 2. The sequence of the functions $\varphi_n(x) = x^{-n}$ is asymptotic as $x \to a$, $x \in M$ in the following cases:

(1) M is the set of points $|x| > c$, $a = \infty$;

(2) M is the semiaxis $x > c$, $a = +\infty$;

(3) M is the semiaxis $x < c$, $a = -\infty$. □

Example 3. The sequence of the functions $\varphi_n(z) = z^n$ is asymptotic as $z \to 0$, $z \in M$. For M we can use the punctured neighborhood $0 < |z| < r$ of point $z = 0$ or a sector with the vertex at this point, $0 < |z| < r$, $\alpha < \text{agr}\, z < \beta$ $(0 < \beta - \alpha \leqslant 2\pi)$. □

Example 4. The sequence of the functions $\varphi_n(z) = z^{-n}$ is asymptotic as $z \to \infty$. For M we can take a neighborhood of point $z = \infty$ $(|z| > R)$ or the sector $|z| > R$, $\alpha < \arg z < \beta$ $(0 < \beta - \alpha \leqslant 2\pi)$ in the complex plane. □

The above asymptotic sequences are known as asymptotic power sequences.

Let us now discuss the concept of an asymptotic expansion, which was introduced by the French mathematician Jules H. Poincaré.

Definition 1. Suppose the sequence $\{\varphi_n(x)\}$ is asymptotic as $x \to a$, $x \in M$. The formal series $\sum_{n=0}^{\infty} a_n \varphi_n(x)$, with the a_n being constant, is said to be an *asymptotic expansion* for $f(x)$ if for any nonnegative integer N we have

$$f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) = o(\varphi_N(x)) \quad (x \to a,\ x \in M). \tag{42.3}$$

The series $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ is said to be an *asymptotic series* for the function $f(x)$, and we use the notation

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x) \quad (x \to a,\ x \in M).$$

In Sec. 42.1 we obtained the following asymptotic expansion:

$$\int_x^{\infty} t^{-1} e^{x-t}\, dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}} \quad (x \to +\infty). \tag{42.4}$$

In what follows we will drop the reference to set M whenever this does not cause any ambiguities.

At this point we stress the important fact that an asymptotic series may be divergent. For instance, the asymptotic series (42.4) is divergent for all x's. This possibility lies in Definition 1. Indeed, let us put

$$R_N(x) = f(x) - \sum_{n=0}^{N} a_n \varphi_n(x).$$

Then, by definition,

$$\frac{R_N(x)}{\varphi_N(x)} \to 0 \quad (x \to a,\ x \in M),$$

where, however, nothing is said about the behavior of the remainder term $R_N(x)$ as $N \to \infty$ (cf. the definition of a convergent series).

Obviously, convergent series are asymptotic, e.g. $e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} (x \to 0)$. However, the term "asymptotic series" is usually reserved for series that are divergent or whose convergence cannot be established.

An important property of an asymptotic expansion is its uniqueness.

Theorem 1 *There can be only one asymptotic expansion for a given function in a given asymptotic sequence.*

Proof. Suppose there are two asymptotic expansions for a function $f(x)$:

$$f(x) \sim \sum_{n=0}^{\infty} a_n\varphi_n(x), \quad f(x) \sim \sum_{n=0}^{\infty} b_n\varphi_n(x)$$
$$(x \to a,\ x \in M).$$

Let us show that $a_n = b_n$ for all n's. By definition,

$$f(x) - a_0\varphi_0(x) = o(\varphi_0(x)),\ f(x) - b_0\varphi_0(x) = o(\varphi_0(x))$$

(we assume everywhere that $x \to a$, $x \in M$). Subtracting the first relationship from the second, we find that $(a_0 - b_0)\varphi_0(x) = o(\varphi_0(x))$. Now, if we divide the result by $\varphi_0(x)$, we obtain $a_0 - b_0 = o(1)$. Going over to the limit as $x \to a$, $x \in M$, we find that $a_0 = b_0$. Now we will show that $a_1 = b_1$. We have

$$f(x) - a_0\varphi_0(x) - a_1\varphi_1(x) = o(\varphi_1(x)),$$
$$f(x) - a_0\varphi_0(x) - b_1\varphi_1(x) = o(\varphi_1(x)),$$

from which it follows that $(a_1 - b_1)\varphi_1(x) = o(\varphi_1(x))$, and whence $a_1 = b_1$. We can similarly prove that $a_n = b_n$ for any n.

Note that different functions may have the same asymptotic expansion. For instance,

$$e^{-x} \sim 0 \times x^0 + 0 \times x^{-1} + \ldots + 0 \times x^{-n} + \ldots \quad (x \to +\infty)$$

(since e^{-x} decreases faster than any power of x as $x \to +\infty$) and

$$0 \sim 0 \times x^0 + 0 \times x^{-1} + \ldots + 0 \times x^{-n} + \ldots \quad (x \to +\infty).$$

42.3 Operations on asymptotic power series An asymptotic expansion in an asymptotic power sequence (see Examples 1-4) is called an *asymptotic power series*. These series can be subjected to the same operations as convergent power series, i.e. add, multiply, etc. them. Here are the main rules for operations on asymptotic power series. Below we assume that $z \to \infty$, $z \in S$, where S is the sector $|z| \geqslant R$, $\alpha \leqslant \arg z \leqslant \beta$ $(0 \leqslant \beta - \alpha \leqslant 2\pi)$; for one, S can be a ray or the exterior of the circle $|z| \geqslant R$.

(1) Theorem 2 *Suppose the following asymptotic expansions are valid as $z \to \infty$, $z \in S$:*

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n}, \quad g(z) \sim \sum_{n=0}^{\infty} b_n z^{-n}.$$

Then, as $z \to \infty$, $z \in S$,

(a) $$\alpha f(z) + \beta g(z) \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) z^{-n};$$

(b) $$f(z) g(z) \sim \sum_{n=0}^{\infty} c_n z^{-n};$$

(c) $$\frac{f(z)}{g(z)} \sim \sum_{n=0}^{\infty} d_n z^{-n} \quad \text{if } b_0 \neq 0.$$

The coefficients c_n and d_n are calculated by the same formulas as for convergent power series.

For example, let us prove Property (b). (The other properties can be proved similarly.) For any nonnegative integer N we have

$$f(z) = \sum_{n=0}^{N} a_n z^{-n} + O(z^{-N-1}), \quad g(z) = \sum_{n=0}^{N} b_n z^{-n} + O(z^{-N-1}),$$

whence

$$f(z) g(z) = \sum_{n=0}^{N} c_n z^{-n} + O(z^{-N-1}), \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0.$$

(2) Asymptotic power series can be integrated termwise. Precisely, we have

Theorem 3 *Suppose* $f(x)$ *is continuous for* $x > 0$ *and*

$$f(x) \sim \sum_{n=2}^{\infty} a_n x^{-n} \quad (x \to +\infty).$$

Then

$$\int_x^{\infty} f(t)\, dt \sim \sum_{n=2}^{\infty} \frac{a_n}{n-1} x^{-n+1} \quad (x \to +\infty).$$

Proof. For any integral value of N greater than unity we have

$$\int_x^{\infty} f(t)\, dt = \int_x^{\infty} \left(\sum_{n=2}^{N} a_n t^{-n} + R_N(t) \right) dt = \sum_{n=2}^{N} \frac{a_n}{n-1} x^{-n+1} + \int_x^{\infty} R_N(t)\, dt.$$

Since $|R_N(t)| \leqslant c_N t^{-N-1}$ for large values of t, where c_N is constant, we can write

$$\left| \int_x^{\infty} R_N(t)\, dt \right| \leqslant c_N \int_x^{\infty} t^{-N-1}\, dt = \frac{c_N}{N} x^{-N} = O(x^{-N}).$$

(3) The following proposition can be proved similarly:

Theorem 4 *Suppose the function $f(z)$ is regular in the sector S: $|z| \geqslant R$, $\alpha < \arg z < \beta$ $(0 < \beta - \alpha \leqslant 2\pi)$ and can be expanded in the asymptotic series*

$$f(z) \sim \sum_{n=2}^{\infty} a_n z^{-n} \quad (z \to \infty,\ z \in S).$$

Then as $z \to \infty$, $z \in \widetilde{S}$, where $\widetilde{S}$ is any closed sector lying strictly inside S, the following asymptotic expansion is valid:

$$\int_z^{\infty} f(\zeta)\, d\zeta \sim \sum_{n=2}^{\infty} \frac{a_n}{n-1} z^{-n+1}.$$

Here the integral is taken along any path lying in S.

(4) An asymptotic series generally cannot be differentiated. But if $f(z)$ is a regular function, then the corresponding asymptotic power series can be differentiated term-by-term.

Theorem 5 *Suppose the function $f(z)$ is regular in the sector S: $|z| \geqslant R$, $\alpha < \arg z < \beta$ $(0 < \beta - \alpha \leqslant 2\pi)$ and can be expanded in an asymptotic series:*

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^{-n} \quad (z \to \infty,\ z \in S).$$

Then the following asymptotic expansion is valid:

$$f'(z) \sim -\sum_{n=1}^{\infty} n a_n z^{-n-1} \quad (z \to \infty,\ z \in \widetilde{S}),$$

where $\widetilde{S}$ is any closed sector lying in S.

Proof. Suppose $\widetilde{S}$ is the sector $\alpha_1 \leqslant \arg z \leqslant \beta_1$, $\alpha < \alpha_1 < \beta_1 < \beta$. For every $z \in \widetilde{S}$ we have

$$f'(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

For γ we take the circle $|\zeta - z| = \varepsilon |z|$ lying in S, where $0 < \varepsilon < 1$. Since $z \in \widetilde{S}$, we can select ε not depending on z. By hypothesis, for any integral value of N greater than unity,

$$f(z) = \sum_{n=0}^{N} a_n z^{-n} + R_N(z), \quad |R_N(z)| \leqslant c_N |z|^{-N-1} \quad (z \in S),$$

and the function $R_N(z)$ is regular in S. We have

$$f'(z) = -\sum_{n=1}^{N} n a_n z^{-n-1} + \widetilde{R}_N(z).$$

The estimate for the remainder term is

$$|\tilde{R}_N(z)| = \frac{1}{2\pi}\left|\int\limits_{\gamma} \frac{R_N(\zeta)}{(\zeta - z)^2}\, d\zeta\right|$$

$$\leqslant c_N \varepsilon^{-1} |z|^{-1} \max_{\zeta \in \gamma} |\zeta|^{-N-1} \leqslant c'_N |z|^{-N-2},$$

since $|\zeta| \geqslant (1-\varepsilon)|z|$ for $\zeta \in \gamma$. Here c'_N is a positive constant.

43 Laplace's Method

43.1 Heuristic considerations In this section we discuss integrals of the type

$$F(\lambda) = \int\limits_a^b f(x)\, e^{\lambda S(x)}\, dx, \tag{43.1}$$

which are known as *Laplace integrals*. Here $I = [a, b]$ is a finite segment, and λ is a large parameter. We will not consider the trivial cases $f(x) \equiv 0$ and $S(x) \equiv \text{const}$. Everywhere in this section we assume that the function $S(x)$ admits only real values. The function $f(x)$, on the other hand, may be complex valued. We assume also that $f(x)$ and $S(x)$ are continuous for $x \in I$.

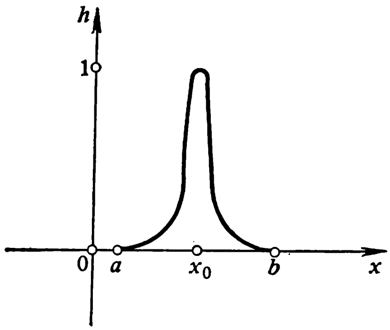


Fig. 157

We are interested in the asymptotic behavior of the integral $F(\lambda)$ as $\lambda \to +\infty$. Laplace integrals can be evaluated explicitly only in a limited number of cases, but their asymptotic behavior can be evaluated practically always. For the sake of simplicity we will assume that the peak of $S(x)$ in I is located only at one point $x_0 \in I$. Here are the two most important cases:

(1) $\max\limits_{x \in I} S(x)$ is attained at an interior point x_0 of I, and $S''(x_0) \neq 0$.

It is clear that for large positive λ's the magnitude of the integral is determined primarily by the exponential $e^{\lambda S(x)}$. Let us consider the function

$$h(x, \lambda) = e^{\lambda(S(x) - S(x_0))}.$$

By hypothesis, $h(x_0, \lambda) = 1$, while $h(x, \lambda) < 1$ at $x \neq x_0$, $\lambda > 0$. As λ grows, the maximum at x_0 becomes more and more pronounced (Fig. 157). For this reason the value of (43.1) is approximately equal to the value of the integral taken over a small neighborhood $(x_0 - \delta, x_0 + \delta)$ of point x_0. In such a neighborhood we can approximately substitute a linear function for $f(x)$ and a quadratic function for $S(x)$:

$$f(x) \approx f(x_0), \quad S(x) - S(x_0) \approx \frac{1}{2} S''(x_0)(x - x_0)^2.$$

Whence, for $f(x_0) \neq 0$ we have

$$F(\lambda) \approx e^{\lambda S(x_0)} f(x_0) \int_{x_0-\delta}^{x_0+\delta} e^{(\lambda/2)S''(x_0)(x-x_0)^2}\, dx. \tag{43.2}$$

A rigorous substantiation of these approximations will be given in Sec. 43.4.

Introducing a new variable t by the formula $x - x_0 = t/\sqrt{-\lambda S''(x_0)}$, with $S''(x_0) < 0$ since x_0 is the point of maximum, we find that the integral in (43.2) is equal to

$$\frac{1}{\sqrt{-\lambda S''(x_0)}} \int_{-\delta\sqrt{-\lambda S''(x_0)}}^{\delta\sqrt{-\lambda S''(x_0)}} e^{-t^2/2}\, dt.$$

For $\lambda \to +\infty$ the limits of integration tend to $\pm\infty$, and the integral above tends to

$$\int_{-\infty}^{\infty} e^{-t^2/2}\, dt = \sqrt{2\pi}.$$

Hence, the asymptotic behavior of the integral $F(\lambda)$ as $\lambda \to +\infty$ is the following:

$$F(\lambda) \approx f(x_0) \sqrt{-\frac{2\pi}{\lambda S''(x_0)}}\, e^{\lambda S(x_0)}. \tag{43.3}$$

(2) $\max\limits_{x \in I} S(x)$ is attained only at the end point $x = a$ of segment I, and $S'(a) \neq 0$.

The same considerations as in case (1) show that for large λ's the integral $F(\lambda)$ is approximately equal to the value of the integral taken over a small segment $[a, a + \delta]$. On this segment we can replace $f(x)$ and $S(x)$ by linear functions:

$$f(x) \approx f(a), \quad S(x) \approx S(a) + (x - a) S'(a).$$

Then

$$F(\lambda) \approx e^{\lambda S(a)} f(a) \int_{a}^{a+\delta} e^{\lambda(x-a)S'(a)}\, dx.$$

The integral here is equal to

$$-\frac{1}{\lambda S'(a)}+\frac{e^{\lambda\delta S'(a)}}{\lambda S'(a)}\sim-\frac{1}{|\lambda S'(a)|},$$

since $S'(a)<0$. Hence, as $\lambda\to+\infty$,

$$F(\lambda)\approx\frac{f(a)}{-\lambda S'(a)}e^{\lambda S(a)}. \tag{43.4}$$

Formulas (43.3) and (43.4) are the basic asymptotic formulas for Laplace integrals. Let us now derive them rigorously.

43.2 The maximum of $S(x)$ at an end point of the interval Let us first obtain a rough estimate for Laplace integrals.

Lemma 1 *Let $I=(a, b)$ be a finite or infinite interval,*

$$S(x)\leqslant C, \quad x\in I, \tag{43.5}$$

and the integral (43.1) *be absolutely convergent for a $\lambda_0>0$. Then for* $\operatorname{Re}\lambda\geqslant\lambda_0$ *we have*

$$|F(\lambda)|\leqslant C_1e^{C\operatorname{Re}\lambda}, \tag{43.6}$$

where C_1 is a constant.

Proof. In view of (43.5) we have, for $\operatorname{Re}\lambda\geqslant\lambda_0$,

$$|e^{(\lambda-\lambda_0)S(x)}|\leqslant C_0e^{C\operatorname{Re}\lambda}, \quad x\in I,$$

where $C_0=e^{-\lambda_0C}$. Hence,

$$|F(\lambda)|=\left|\int_a^b f(x)\,e^{\lambda_0S(x)}e^{(\lambda-\lambda_0)S(x)}\,dx\right|\leqslant C_0e^{C\operatorname{Re}\lambda}\int_a^b |f(x)|\,e^{\lambda_0S(x)}\,dx.$$

By hypothesis, the last integral has a finite value, which proves the validity of estimate (43.6).

From now on the interval $I=[a, b]$ is a finite segment, and the functions $f(x)$ and $S(x)$ are continuous for $x\in I$.

The asymptotic formulas for Laplace integrals, as we will show below, can be used not only when $\lambda\to+\infty$ but when $\lambda\to\infty$, $\lambda\in S_\varepsilon$, where S_ε is the sector $|\arg\lambda|\leqslant\pi/2-\varepsilon$ in the complex λ plane. Here $0<\varepsilon<\pi/2$. Note that if $\lambda\in S_\varepsilon$, then

$$|\lambda|\geqslant\operatorname{Re}\lambda\geqslant|\lambda|\sin\varepsilon.$$

For this reason, as $\lambda\to\infty$, $\lambda\in S_\varepsilon$, we have

$$(\operatorname{Re}\lambda)^{-n}=O(|\lambda|^{-n}), \quad n\geqslant 0, \quad |e^{-C\lambda}|=O(|\lambda|^{-N}),$$

where N is any positive number.

Theorem 1 *Suppose*

$$S(x)<S(a), \quad x\neq a; \quad S'(a)\neq 0, \tag{43.7}$$

and let us assume that the functions $f(x)$ and $S(x)$ can be differentiated any number of times in a neighborhood of point $x = a$. Then, as $\lambda \to \infty$, $\lambda \in S_\varepsilon$, the following asymptotic behavior is valid:

$$F(\lambda) \sim e^{\lambda S(a)} \sum_{n=0}^{\infty} c_n \lambda^{-n-1}. \tag{43.8}$$

This expansion can be differentiated termwise any number of times. The coefficients c_n can be calculated by the formula

$$c_n = (-1)^{n+1} \left(\frac{1}{S'(x)} \frac{d}{dx}\right)^n \left(\frac{f(x)}{S'(x)}\right)\bigg|_{x=a}. \tag{43.9}$$

The principal term in the asymptotic expansion has the form (43.4) *or, more exactly,*

$$F(\lambda) = \frac{e^{\lambda S(a)}}{-\lambda S'(a)} \left[f(a) + O\left(\frac{1}{\lambda}\right)\right]. \tag{43.10}$$

Proof. Since $S'(a) \neq 0$, we can select a positive δ such that $S'(x) \neq 0$ for $a \leqslant x \leqslant a + \delta$. We split the integral (43.1) into two:

$$F(\lambda) = F_1(\lambda) + F_2(\lambda),$$

where $F_1(\lambda)$ is the integral taken along the segment $[a, a + \delta]$. Since $S(x)$ attains its maximum on I only at point a, we conclude that $S(x) \leqslant S(a) - c$ for $a + \delta \leqslant x \leqslant b$, where c is a positive constant. By Lemma 1, for $\lambda \in S_\varepsilon$ we have

$$|F_2(\lambda)| \leqslant c_0 \, |e^{\lambda(S(a)-c)}|. \tag{43.11}$$

For this reason, the integral $F_2(\lambda)$ is exponentially small compared with $e^{\lambda S(a)}$ and, in particular, compared with any term $c_n\lambda^{-n-1}e^{\lambda S(a)}$ in the asymptotic series (43.8).

The integral $F_1(\lambda)$, which is taken along the segment $[a, a + \delta]$, will be integrated by parts:

$$F_1(\lambda) = \int_a^{a+\delta} \frac{f(x)}{\lambda S'(x)} \, d\left(e^{\lambda S(x)}\right) = \frac{f(x)\, e^{\lambda S(x)}}{\lambda S'(x)}\bigg|_a^{a+\delta} + \frac{1}{\lambda} F_{11}(\lambda), \tag{43.12}$$

$$F_{11}(\lambda) = \int_a^{a+\delta} e^{\lambda S(x)} f_1(x)\, dx, \quad f_1(x) = -\frac{d}{dx}\left(\frac{f(x)}{S'(x)}\right).$$

The upper limit of the first term on the right-hand side of (43.12) is exponentially small compared with $e^{\lambda S(a)}$ (as $\lambda \to \infty$, $\lambda \in S_\varepsilon$) since $S(a + \delta) - S(a) < 0$.

Let us estimate the value of $F_{11}(\lambda)$. We have $S'(x) < 0$ on the segment $I_1 = [a, a+\delta]$, whereby there is a positive constant S_1 such that $S'(x) \leqslant -S_1$ for $x \in I_1$. By Lagrange's formula,

$$S(x) - S(a) = (x-a)\, S'(\xi),$$

where $\xi \in (a, a+\delta)$, and

$$S(x) - S(a) \leqslant -S_1(x-a), \quad S_1 > 0$$

on I_1. Since $f_1(x)$ is continuous, we have $|f_1(x)| \leqslant M$ for $x \in I_1$. Hence,

$$|F_{11}(\lambda)\, e^{-\lambda S(a)}| \leqslant \int_a^{a+\delta} |f_1(x)|\, |e^{\lambda(S(x)-S(a))}|\, dx$$

$$\leqslant M \int_a^{a+\delta} e^{-S_1(x-a)\operatorname{Re}\lambda}\, dx < \frac{M}{S_1 \operatorname{Re}\lambda} \leqslant \frac{c}{|\lambda|} \quad (\lambda \in S_2).$$

With this estimate in mind, we can write (43.12) thus:

$$F_1(\lambda) = e^{\lambda S(a)}\left[\frac{f(a)}{-\lambda S'(a)} + O(\lambda^{-2})\right]. \tag{43.13}$$

The last relationship, the estimate (43.11), and the fact that $F(\lambda) = F_1(\lambda) + F_2(\lambda)$ yields formula (43.10) for the principal term in the asymptotic expansion.

The integral $F_{11}(\lambda)$ has exactly the same form as $F_1(\lambda)$, and

$$F_{11}(\lambda) = \left.\frac{f_1(x)\, e^{\lambda S(x)}}{\lambda S'(x)}\right|_a^{a+\delta} + \frac{1}{\lambda} F_{12}(\lambda).$$

The integral $F_{12}(\lambda)$ has the same form as $F_1(\lambda)$, the only difference being that

$$f(x) \to f_2(x) = -\frac{d}{dx}\left(\frac{f_1(x)}{S'(x)}\right).$$

If we substitute f_1 for f in (43.13), the integral $F_{11}(\lambda)$ obeys (43.13), which means that

$$F(\lambda) = e^{\lambda S(a)}\left[\frac{f(a)}{-\lambda S'(a)} + \frac{f_1(a)}{-\lambda^2 S'(a)} + O(\lambda^{-3})\right].$$

If we proceed with this process still further, we arrive at the expansion (43.8) and formula (43.9).

What remains to be proved is that the series (43.8) can be differentiated term-by-term. The function $F(\lambda)$ is an entire function of λ (Theorem 1 of Sec. 16), and the asymptotic series (43.8) can be differentiated term-by-term in view of Theorem 5 of Sec. 42.

Example 1. Consider the Laplace transform of $f(x)$, i.e.

$$F(\lambda) = \int_0^\infty f(x) e^{-\lambda x} \, dx. \tag{43.14}$$

Let us assume that $f(x)$ is piecewise continuous for nonnegative x's, can be differentiated any number of times in a neighborhood of point $x = 0$, and satisfies the estimate

$$|f(x)| \leqslant Me^{Cx}$$

for $x \geqslant 0$. Let us show that

$$F(\lambda) \sim \sum_{n=0}^{\infty} f^{(n)}(0) \lambda^{-n-1} \tag{43.15}$$

as $\lambda \to \infty$, $\lambda \in S_\varepsilon$.

In the case at hand, $S(x) = -x$, so that $\max\limits_{x \geqslant 0} S(x) = S(0) = 0$, and $S'(0) \neq 0$. But we cannot apply Theorem 1 directly to this integral because the domain of integration is unlimited.

We split the integral $F(\lambda)$ into two: $F(\lambda) = F_1(\lambda) + F_2(\lambda)$, where $F_1(\lambda)$ is the integral taken along the segment $[0, 1]$. Since $S(x) = -x \leqslant -1$ for $x \geqslant 1$, by Lemma 1 we have

$$|F_2(\lambda)| \leqslant C |e^{-\lambda}| \quad (\lambda \in S_\varepsilon),$$

which is exponentially small as $\lambda \to \infty$, $\lambda \in S_\varepsilon$. Applying Theorem 1 to $F_1(\lambda)$, we arrive at (43.15). □

Example 2. Let us take the error integral

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt$$

and establish its asymptotic behavior as $x \to +\infty$. Since

$$\int_0^\infty e^{-t^2} \, dt = \frac{1}{2} \sqrt{\pi},$$

we conclude that

$$\Phi(x) = 1 - \frac{2}{\sqrt{\pi}} F(x), \quad F(x) = \int_x^\infty e^{-t^2} \, dt.$$

Let us now represent $F(x)$ in the form of the Laplace transform (43.14).

Changing the variable according to $t = x\tau$ and putting $\tau^2 = 1 + u$, we obtain

$$F(x) = \frac{1}{2} x e^{-x^2} \int_0^\infty e^{-x^2 u} (1+u)^{-1/2}\, du.$$

The integral on the right-hand side has the form (43.14), where $\lambda = x^2$, and $f = (1 + u)^{-1/2}$, so that $f^{(k)}(0) = (-1/2)^k (2k - 1)!!$. Applying formula (43.15), we find that, as $x \to +\infty$,

$$\Phi(x) \sim 1 - \frac{1}{x\sqrt{\pi}} e^{-x^2} - \frac{1}{x\sqrt{\pi}} e^{-x^2} \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)!!}{2^k x^{2k}}. \tag{43.16}$$

The same formula holds for complex valued x's, $|x| \to \infty$, $|\arg x| \leqslant \pi/4 - \varepsilon$ $(0 < \varepsilon < \pi/4)$. Indeed, if x lies in this sector, then $\lambda = x^2$ lies in the sector $S_{2\varepsilon}$: $|\arg \lambda| \leqslant \pi/2 - 2\varepsilon$, in which formula (43.15) is valid. □

43.3 Watson's lemma The asymptotic behavior of many Laplace integrals can be examined by establishing the asymptotic behavior of the "typical" integral

$$\Phi(\lambda) = \int_0^a e^{-\lambda t^\alpha} t^{\beta-1} f(t)\, dt. \tag{43.17}$$

Lemma 2 (Watson's lemma) *Suppose α and β are positive and let $f(t)$ be continuous for $0 \leqslant t \leqslant a$ and infinitely differentiable in a neighborhood of point $t = 0$. Then, as $\lambda \to \infty$, $\lambda \in S_\varepsilon$, the following asymptotic expansion holds*:

$$\Phi(\lambda) \sim \frac{1}{\alpha} \sum_{n=0}^{\infty} \lambda^{-(n+\beta)/\alpha} \Gamma\left(\frac{n+\beta}{\alpha}\right) \frac{f^{(n)}(0)}{n!}. \tag{43.18}$$

This expansion can be differentiated with respect to λ any number of times.

Before proving this lemma, we will prove the validity of the formula

$$\int_0^\infty e^{-\lambda t^\alpha} t^{\beta-1}\, dt = \frac{1}{\alpha} \lambda^{-\beta/\alpha} \Gamma\left(\frac{\beta}{\alpha}\right) \tag{43.19}$$

for Re λ positive. Here $\lambda^{-\beta/\alpha}$ is the regular branch in half-plane $\operatorname{Re} \lambda > 0$ that is positive for positive values of λ.

Suppose λ is positive. Substituting $\lambda t^{\alpha} = y$, we find that the integral on the left-hand side of (43.19) is equal to

$$\lambda^{-\beta/\alpha} \frac{1}{\alpha} \int_0^{\infty} e^{-y} y^{(\beta/\alpha)-1}\, dy = \frac{1}{\alpha} \lambda^{-\beta/\alpha} \Gamma\left(\frac{\beta}{\alpha}\right).$$

This integral is a regular function in the half-plane $\operatorname{Re} \lambda > 0$. The right-hand side of (43.19) is analytically continuable from the semiaxis $(0, +\infty)$ into the half-plane $\operatorname{Re} \lambda > 0$. Since both functions coincide on the semiaxis $(0, +\infty)$, by the principle of analytic continuation, they coincide in the half-plane $\operatorname{Re} \lambda > 0$.

Proof of Lemma 2. Let us split the integral $\Phi(\lambda)$ into two:

$$\Phi(\lambda) = \Phi_1(\lambda) + \Phi_2(\lambda),$$

where the integral Φ_1 is taken along the segment $[0, \delta]$, with δ positive and small. Since $-t^{\alpha} \leqslant -\delta^{\alpha} < 0$ for $\delta \leqslant t \leqslant a$, we have, according to Lemma 1, the following estimate for $\Phi_2(\lambda)$:

$$|\Phi_2(\lambda)| \leqslant C\, |e^{-\delta^{\alpha}\lambda}|$$

for $\lambda \in S_{\varepsilon}$, $|\lambda| \geqslant 1$, and the right-hand side is exponentially small. The following expansion is valid on the segment $[0, \delta]$:

$$f(t) = \sum_{n=0}^{N} f_n t^n + \psi_N(t),$$

where $f_n = \frac{f^{(n)}(0)}{n!}$ and $|\psi_N(t)| \leqslant C_N t^{N+1}$, $0 \leqslant t \leqslant \delta$. Whence $\Phi_1(\lambda)$ is equal to the following sum:

$$\Phi_1(\lambda) = \sum_{n=0}^{N} f_n \Phi_{1n}(\lambda) + R_N(\lambda),$$

where we have introduced the notations

$$\Phi_{1n}(\lambda) = \int_0^{a} t^{n+\beta-1} e^{-\lambda t^{\alpha}}\, dt, \quad R_N(\lambda) = \int_0^{a} \psi_N(t)\, e^{-\lambda t^{\alpha}}\, dt.$$

Let us write $\Phi_{1n}(\lambda)$ in the form of the difference between respective integrals along the semiaxes $(0, +\infty)$ and $(a, +\infty)$. The first integral in this difference can be evaluated via (43.19), while the second does not exceed, by absolute value, the quantity $C\,|e^{-a^{\alpha}\lambda}|$, since $t^{\alpha} \leqslant -a^{\alpha}$ for $t \geqslant a$. The final result is

$$\Phi_{1n}(\lambda) = \frac{1}{\alpha} \lambda^{-(n+\beta)/\alpha} \Gamma\left(\frac{n+\beta}{\alpha}\right) + O(e^{-a^{\alpha}\lambda}).$$

Finally, the absolute value of the integrand in $R_N(\lambda)$ does not exceed $C_N t^{N+1} \mid e^{-\lambda t^\alpha} \mid$, whence

$$|R_N(\lambda)| \leqslant C_N \int_0^\infty t^{N+\beta} e^{-t^\alpha \operatorname{Re}\lambda}\, dt$$
$$= C'_N (\operatorname{Re}\lambda)^{-(N+1+\beta)/\alpha} = O(|\lambda|^{-(N+\beta+1)/\alpha})$$

for $\lambda \in S_\varepsilon$. The final result is, for $\lambda \in S_\varepsilon$,

$$\Phi(\lambda) = \frac{1}{\alpha} \sum_{n=0}^{N} f_n \Gamma\left(\frac{n+\beta}{\alpha}\right) \lambda^{-(n+\beta)/\alpha} + O\left(|\lambda|^{-\frac{N+\beta+1}{\alpha}}\right)$$

(the sum of all exponentially small terms is included in the remainder term). Hence we have proved the validity of the asymptotic

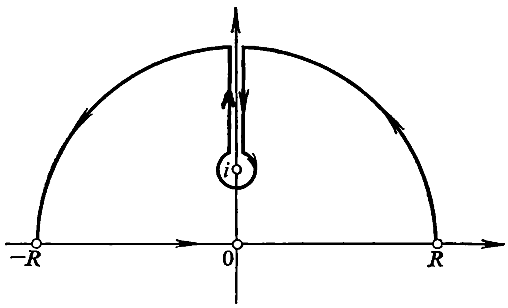


Fig. 158

expansion (43.18). The possibility of term-by-term differentiation of expansion (43.18) can be proved in the same way as in Theorem 1.

Example 3. Let us establish the asymptotic behavior, as $x \to +\infty$, of the integral

$$K_0(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{itx}}{\sqrt{1+t^2}}\, dt.$$

The integral $K_0(x)$ is known as a *modified Hankel function*, or the *MacDonald function*.

We cannot apply Watson's lemma to this integral directly. We then deform the integration contour. We cut the complex t plane along the ray $l = [i, +i\infty)$. Then the function $f(t) = (1+t^2)^{-1/2}$, $f(0) > 0$, becomes regular in the half-plane $\operatorname{Im} t > 0$ with a cut along l. For $x > 0$ the integration contour can be deformed into the cut along l. To prove this, let us consider the contour $\Gamma_{\rho, R}$ (Fig. 158). It consists of the segment $[-R, R]$, the arcs $C_R^{\pm}$ of the circle $|t| = R$, the circle C_ρ: $|t-i| = \rho$, and the intervals along the banks of the cut. Since $|f(t)| \leqslant c|t|^{-1}$ as $|t| \to \infty$, the integrals taken along the arcs of circle $|t| = R$ tend to zero as $R \to \infty$, according to Jordan's lemma. Further, for $t \in C_\rho$ we have

$$|t^2+1| = |(t-i)(t+i)| = \rho\,|2i + O(\rho)| \geqslant \rho \qquad (\rho \to 0);$$

thus, the integrand has the order of $O(1/\sqrt{\rho})$ as $\rho \to 0$, $t \in C_\rho$. Whence, the integral along C_ρ tends to zero as $\rho \to 0$, and the integral $K_0(x)$ is equal to the integral along the banks of the cut. Let us show that for $\tau > 1$ we have

$$f(i\tau+0) = \frac{-i}{\sqrt{\tau^2-1}}$$

(this is the value of $f(t)$ on the right bank of the cut). We have

$$f(i\tau+0) = \left|\frac{1}{\sqrt{\tau^2-1}}\right| e^{-i(1/2)(\varphi_1+\varphi_2)},$$

$$\varphi_1 = \Delta_\gamma \arg(t+i), \quad \varphi_2 = \Delta_\gamma \arg(t-i).$$

Here curve γ lies in the upper half-plane and connects points 0 and $i\tau+0$, so that $\varphi_1 = 0$ and $\varphi_2 = +\pi$. Similarly,

$$f(i\tau-0) = \frac{i}{\sqrt{\tau^2-1}}.$$

Hence,

$$K_0(x) = -i\int_1^\infty \frac{e^{-x\tau}}{\sqrt{\tau^2-1}}\,d(i\tau) = \int_1^\infty \frac{e^{-x\tau}}{\sqrt{\tau^2-1}}\,d\tau = e^{-x}\int_0^\infty \frac{e^{-xt}\,dt}{\sqrt{t(t+2)}}$$

(we have substituted $t+1$ for τ). We can now apply Watson's lemma to the integral on the right-hand side (with $\alpha = 1$ and $\beta = 1/2$). Hence,

$$K_0(x) = e^{-x}\sqrt{\frac{\pi}{2x}}\left(1+O\left(\frac{1}{x}\right)\right) \quad (x \to +\infty). \ \square$$

43.4 The maximum of $S(x)$ inside the interval

Theorem 2 *Suppose*

$$S(x) < S(x_0), \quad x \neq x_0, \quad a < x_0 < b, \quad S''(x_0) \neq 0, \tag{43.20}$$

and let the functions $f(x)$ and $S(x)$ be infinitely differentiable in a neighborhood of point x_0. Then, as $\lambda \to \infty$, $\lambda \in S_\varepsilon$, the following asymptotic expansion is valid:

$$F(\lambda) \sim e^{\lambda S(x_0)} \sum_{n=0}^{\infty} c_n \lambda^{-n-1/2}. \tag{43.21}$$

This expansion can be differentiated term-by-term any number of times, and the principal term in the expansion is expressed by (43.3) *or, precisely,*

$$F(\lambda) = \sqrt{-\frac{2\pi}{\lambda S''(x_0)}}\, e^{\lambda S(x_0)}\,[f(x_0) + O(\lambda^{-1})]. \tag{43.22}$$

We will need following

Lemma 3 *Suppose* $S(x)$ *can be differentiated any number of times in a neighborhood of point* x_0, *and*

$$S'(x_0) = 0, \quad S''(x_0) < 0. \tag{43.23}$$

Then there are neighborhoods U *and* V *of points* $x = x_0$ *and* $y = 0$, *respectively, and a function* $\varphi(y)$, such that

$$S(\varphi(y)) - S(x_0) = -y^2, \quad y \in V, \tag{43.24}$$

the function $\varphi(y)$ *can be differentiated any number of times for* $y \in V$,

$$\varphi'(0) = \sqrt{-\frac{2}{S''(x_0)}}, \tag{43.25}$$

and the function $x = \varphi(y)$ *maps* V *onto* U *in a one-to-one manner.*

Proof of Lemma 3 Without loss of generality, we can assume that $x_0 = 0$ and $S(x_0) = 0$. By Taylor's formula,

$$S(x) = \int_0^x (x-t)\, S''(t)\, dt = x^2 \int_0^1 (1-t)\, S''(xt)\, dt \equiv -x^2 h(x).$$

If U_0 is a small neighborhood of point $x = 0$, then $S''(xt)$ is negative for $x \in U_0$, $0 \leqslant t \leqslant 1$, since $S''(0)$ is negative. For this reason, the function

$$h(x) = \int_0^1 (t-1)\, S''(xt)\, dt$$

is positive for $x \in U_0$ and can be differentiated any number of times. We put

$$x\sqrt{h(x)} = y, \tag{43.26}$$

i.e. $x^2 h(x) = y^2$ or, which is the same, $S(x) = -y^2$. Here $\sqrt{h(x)}$ is positive. Since

$$\frac{d}{dx}\left(x\sqrt{h(x)}\right)\Big|_{x=0} = \sqrt{h(0)} = \sqrt{-\frac{S''(0)}{2}} \neq 0,$$

by the inverse function theorem we conclude that Eq. (43.26) has a solution $x = \varphi(y)$, $\varphi(0) = 0$ with the required properties. The proof of Lemma 3 is complete.

Proof of Theorem 2. Let $x_0 = 0$ and $S(x_0) = 0$. We select a small neighborhood $[-\delta_1, \delta_2]$ of point $x = 0$ and split the integral $F(\lambda)$ into three integrals:

$$F(\lambda) = F_1(\lambda) + F_2(\lambda) + F_3(\lambda).$$

Here $F_1(\lambda)$ is the integral taken along the segment $[a, -\delta_1]$, $F_2(\lambda)$ the integral taken along the segment $[-\delta_1, \delta_2]$, and $F_3(\lambda)$ the integral taken along the segment $[\delta_2, b]$. Since $S(x) < S(0)$ for $x \in I$, $x \neq 0$, we conclude that the integrals $F_1(\lambda)$ and $F_3(\lambda)$ are exponentially small as $\lambda \to \infty$, $\lambda \in S_\varepsilon$, i.e.

$$F_j(\lambda) = O(e^{-\lambda c}) \quad (c > 0), \quad j = 1, 3.$$

The proof is the same as in Theorem 1.

Now let us select δ_1 and δ_2 in a way such that $S(-\delta_2) = S(\delta_1)$. We have $S(\delta_1) = -\varepsilon^2$, with ε positive, since $x = 0$ is the point at which $S(x)$ attains its maximum. We substitute $\varphi(y)$ for x in the integral $F_2(\lambda)$, i.e.

$$S(\varphi(y)) = -y^2,$$

which is justified in view of Lemma 2. Then

$$F_2(\lambda) = \int_{-\varepsilon}^{\varepsilon} e^{-\lambda y^2} h(y)\, dy, \quad h(y) = f(\varphi(y))\, \varphi'(y).$$

Further,

$$F_2(\lambda) = \int_0^{\varepsilon} e^{-\lambda y^2} g(y)\, dy,$$

where $g(y) = h(y) + h(-y)$.

Now we need only to apply Watson's lemma to the integral $F_2(\lambda)$. Here $\alpha = 2$ and $\beta = 1$, besides, $g(y)$ is an even function, so that $g^{(k)}(0) = 0$ for all odd values of k. The final result is the expansion (43.21) for $F_2(\lambda)$, where the coefficients c_n are given by the formula

$$c_n = \Gamma\left(n + \frac{1}{2}\right) \frac{h^{(2n)}(0)}{(2n)!}. \tag{43.27}$$

Here we took into account that $g^{(2n)}(0) = 2h^{(2n)}(0)$. The coefficient c_0 is given by the formula

$$c_0 = \Gamma\left(\frac{1}{2}\right) h(0) = \Gamma\left(\frac{1}{2}\right) f(x_0)\, \varphi'(0) = \sqrt{\frac{2\pi}{-S''(x_0)}}\, f(x_0),$$

since $\Gamma(1/2) = \sqrt{\pi}$, and $\varphi'(0)$ is given by (43.25).

The proof of Theorem 2 leads to

Corollary 1 *Suppose* $\max_{x \in I} S(x)$ *is attained at the end point* $x = a$ *of the segment* I, *and* $S'(a) = 0$ *and* $S''(a) \neq 0$. *Then, as* $\lambda \to \infty$, $\lambda \in S_\varepsilon$, *the following asymptotic expansion is valid*:

$$F(\lambda) \sim e^{\lambda S(a)} \sum_{n=0}^{\infty} d_n \lambda^{-(n+1)/2}. \qquad (43.28)$$

The principal term in the asymptotic expansion is

$$F(\lambda) = \frac{1}{2} e^{\lambda S(a)} \sqrt{-\frac{2\pi}{\lambda S''(a)}} \left[f(a) + O\left(\frac{1}{\sqrt{\lambda}}\right)\right]. \qquad (43.29)$$

Example 4. Consider the gamma function

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt.$$

Let us prove Stirling's formula

$$\Gamma(x+1) = x^x e^{-x} \sqrt{2\pi x} \left(1 + O\left(\frac{1}{x}\right)\right) \quad (x \to +\infty). \qquad (43.30)$$

This integral is not of the (43.1) type, but we will transform it into such a type. The integrand $t^x e^{-t}$ attains the maximal value on the semiaxis $t > 0$ at point $t_0(x) = x$, which tends to infinity as $x \to +\infty$. To restrict its variation, we substitute xt' for t. Then

$$\Gamma(x) = x^{x+1} \int_0^\infty e^{x(\ln t - t)}\, dt.$$

This integral has the form (43.1): $\lambda \to x$, $S = \ln t - t$, and $f(t) \equiv 1$. The point of maximum is $t_0 = 1$, with $S(t_0) = -1$ and $S''(t_0) = -1$.

To apply Theorem 2, we partition the region of integration into three parts: (0, 1/2), (1/2, 3/2), and (3/2, ∞). The integrals taken over the first and third parts are exponentially small compared with $e^{-x} = e^{xS(t_0)}$, according to Lemma 1. The asymptotic behavior of the integral taken along the segment [1/2, 3/2] is determined from (43.22), and we arrive at (43.30).

Formula (43.30) yields Stirling's formula for factorial $n!$:

$$n! \sim n^n e^{-n} \sqrt{2\pi n} \quad (n \to +\infty).$$

The asymptotic formula (43.30) is also valid for complex values of z, with $z \to \infty$, $z \in S_\varepsilon$, where S_ε is the sector $|\arg z| \leqslant \pi - \varepsilon$ (see Evgrafov [2]). Here ε is fixed and $0 < \varepsilon < \pi$. There is also a more exact asymptotic expansion for the logarithm of the gamma function (see Fedoryuk [1]):

$$\ln \Gamma(z) \sim \left(z-\frac{1}{2}\right) \ln z - z + \frac{1}{2} \ln(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_n}{2n(2n-1) z^{2n-1}}$$

$$(z \in S_\varepsilon, z \to \infty),$$

where the B_n are the Bernoulli number (see Example 4 in Sec. 12). For the remainder term in (43.30) the following estimate has been found:

$$|O(1/x)| \leqslant 1/12x. \ \square$$

Example 5. Let us establish the asymptotic behavior of the sum

$$F(n) = \sum_{k=0}^{n} \binom{n}{k} k! n^{-k}$$

as $n \to \infty$. Let us transform this sum into an integral. Using the identity $k! n^{-k-1} = \int_0^\infty e^{-nx} x^k \, dx$, we obtain

$$F(n) = n \int_0^\infty e^{-nx} (1+x)^n \, dx,$$

whence

$$F(n) = n \int_0^\infty e^{nS(x)} \, dx,$$

where $S(x) = -x + \ln(1+x)$. The function $S(x)$ attains its maximum on the semiaxis $x > 0$ only at point $x = 0$, with $S(0) = 0$ and $S''(0) = -1$. Applying Corollary 1 of Theorem 2, we find that

$$F(n) = \sqrt{\frac{\pi n}{2}} \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \quad (n \to \infty). \ \square$$

44 The Method of Stationary Phase

44.1 Statement of the problem Let us take the integral

$$F(\lambda) = \int_a^b f(x) e^{i\lambda S(x)} \, dx. \tag{44.1}$$

Here $I = [a, b]$ is a finite segment, the function $S(x)$ admits only real values, and λ is a large positive parameter. Integrals of the type (44.1) are known as *Fourier integrals*, and $S(x)$ is known as the *phase*, or *phase function*. We are interested in the asymptotic behavior of $F(\lambda)$ as $\lambda \to +\infty$. We will not consider the trivial cases with $f(x) \equiv 0$ or $S(x) \equiv \text{const}$.

A particular case of Fourier integrals is the Fourier transform

$$F(\lambda) = \int_a^b f(x)\, e^{i\lambda x}\, dx. \tag{44.2}$$

Let the function $f(x)$ be continuous for $a \leqslant x \leqslant b$. Then $F(\lambda)$ tends to zero as $\lambda \to +\infty$. Indeed, for large λ's the function $\operatorname{Re}(f(x)\, e^{i\lambda x})$ rapidly oscillates, and two neighboring half-waves have areas that are approximately equal in absolute value but are opposite in sign. For this reason the sum of these areas constitute a small quantity, in view of which the entire integral

$$\int_a^b \operatorname{Re}(f(x)\, e^{i\lambda x})\, dx$$

is small, too.

The following proposition provides the most general result concerning the asymptotic behavior of integrals of type (43.2):

The Riemann-Lebesgue lemma (see Nikol'skii [1]) *Suppose the integral* $\int_a^b |f(x)|\, dx$ *has a finite value. Then*

$$\int_a^b f(x)\, e^{i\lambda x}\, dx \to 0 \quad (\lambda \to +\infty).$$

The Riemann-Lebesgue lemma says nothing of the rate with which the integral $F(\lambda)$ tends to zero. The fact is that this rate depends essentially on the properties of the derivatives of $f(x)$ and can be very small. It was found that asymptotic expansions for Fourier integrals can be found only when $f(x)$ and $S(x)$ are smooth functions. We will consider only the case where both functions can be differentiated any number of times on I.

Theorem 1 *Suppose the functions $f(x)$ and $S(x)$ are infinitely differentiable and $S'(x) \neq 0$ for $x \in I$. Then, as $\lambda \to +\infty$, the integral* (44.1) *possesses the following asymptotic expansion*:

$$F(\lambda) \sim \frac{1}{i\lambda} e^{i\lambda S(b)} \sum_{n=0}^{\infty} b_n (i\lambda)^{-n} - \frac{1}{i\lambda} e^{i\lambda S(a)} \sum_{n=0}^{\infty} a_n (i\lambda)^{-n}. \tag{44.3}$$

This expansion can be differentiated with respect to λ *any number of times. The principal term in the asymptotic expansion has the form*

$$F(\lambda) = \frac{f(b)}{i\lambda S'(b)} e^{i\lambda S(b)} - \frac{f(a)}{i\lambda S'(a)} e^{i\lambda S(a)} + O(\lambda^{-2}). \qquad (44.4)$$

The coefficients a_n *and* b_n *are calculated by the following formulas:*

$$a_n = (-1)^n M^n \left(\frac{f(x)}{S'(x)}\right)\Big|_{x=a}, \quad b_n = (-1)^n M^n \left(\frac{f(x)}{S'(x)}\right)\Big|_{x=b}, \qquad (44.5)$$
$$M = \frac{1}{S'(x)} \frac{d}{dx}.$$

Note that the formulas for the a_n coincide with those for the c_n, (44.9).

Proof of Theorem 1 We integrate (44.1) by parts in the same way as we did in the proof of Theorem 1 of Sec. 43:

$$F(\lambda) = \frac{1}{i\lambda} \int_a^b \frac{f(x)}{S'(x)} d(e^{i\lambda S(x)}) = \frac{1}{i\lambda} e^{i\lambda S(x)} \frac{f(x)}{S'(x)}\Big|_a^b + \frac{1}{i\lambda} F_1(\lambda),$$

$$F_1(\lambda) = -\int_a^b e^{i\lambda S(x)} \frac{d}{dx}\left(\frac{f(x)}{S'(x)}\right) dx.$$

By the Riemann-Lebesgue lemma, $F_1(\lambda) = o(1)$ $(\lambda \to +\infty)$, and we have thus proved the validity of (44.4) with the remainder term of the order of $o(1/\lambda)$. The integral $F_1(\lambda)$ has exactly the same form as $F(\lambda)$; integration by parts once more yields

$$F_1(\lambda) = \frac{f_1(x)}{i\lambda S'(x)} e^{i\lambda S(x)}\Big|_a^b + \frac{1}{i\lambda} F_2(\lambda).$$

Here $f_1(x) = -(f(x)/S'(x))'$, and F_2 is obtained from F_1 by substituting f_1 for f. Since $F_2(\lambda) = o(1)$ $(\lambda \to +\infty)$ in view of the Riemann-Lebesgue lemma, we conclude that $F_1(\lambda) = O(\lambda^{-1})$, which proves the validity of (44.4) completely. We have also proved that

$$F(\lambda) = \frac{1}{i\lambda}\left[\left(b_0 + \frac{b_1}{i\lambda}\right) e^{i\lambda S(b)} - \left(a_0 + \frac{a_1}{i\lambda}\right) e^{i\lambda S(a)}\right] + \frac{1}{(i\lambda)^2} F_2(\lambda),$$

where the coefficients a_j and b_j have the form (44.5). Continuing the process of integration by parts, we arrive at the expansion (44.3).

Since the integral $F(\lambda)$ has a finite value for all complex valued λ's, we conclude that $F(\lambda)$ is an entire function of λ (see Theorem 1 of Sec. 16). The possibility of term-by-term differentiation of (44.3) follows from Theorem 5 of Sec. 42.

The proof of Theorem 1 leads to

Corollary 1 *Suppose $f(x)$ and $S(x)$ can be differentiated k and $k+1$ times, respectively (with k a positive integer) on the segment $[a, b]$. Then, as $\lambda \to +\infty$,*

$$F(\lambda) \frac{1}{i\lambda} e^{i\lambda S(b)} \sum_{n=0}^{k-1} b_n (i\lambda)^{-n} - \frac{1}{i\lambda} e^{i\lambda S(a)} \sum_{n=0}^{k-1} a_n (i\lambda)^{-n} + o(\lambda^{-k}). \tag{44.6}$$

A particular case of this corollary is the asymptotic estimate for Fourier coefficients, which is known from mathematical analysis.

Corollary 2 *Suppose the function $f(x)$ is k times continuously differentiable on the segment $[0, 2\pi]$ and*

$$f^{(j)}(0) = f^{(j)}(2\pi), \quad 0 \leqslant j \leqslant k. \tag{44.7}$$

Then

$$c_m = \int_0^{2\pi} e^{imx} f(x)\, dx = o(m^{-k}) \tag{44.8}$$

as $m \to +\infty$.

Indeed, since $e^{i2m\pi} = 1$ when m is an integer and condition (44.7) is met, all terms in (44.6) except the remainder term cancel out.

Integration by parts enables us to find the asymptotic behavior of some other classes of integrals of rapidly oscillating functions.

Example 1. Let us consider the integral

$$\Phi(x) = \int_x^{\infty} e^{it^2}\, dt$$

and establish its asymptotic behavior as $x \to +\infty$. Integration by parts yields

$$\Phi(x) = \int_x^{\infty} \frac{1}{2it}\, d(e^{it^2}) = -\frac{e^{ix^2}}{2xi} + \frac{1}{2i} \int_x^{\infty} e^{it^2} \frac{dt}{t^2}.$$

Let us estimate the value of the integral on the right-hand side. We have

$$\left| \int_x^{\infty} e^{it^2} \frac{dt}{t^2} \right| \leqslant \int_x^{\infty} \frac{dt}{t^2} = \frac{1}{x}.$$

We have therefore found that

$$\Phi(x) = -\frac{e^{ix^2}}{2ix} + O\left(\frac{1}{x}\right)$$

as $x \to +\infty$. Both terms on the right-hand side are of the same order of magnitude. Hence,

$$\Phi(x) = O\left(\frac{1}{x}\right) \quad (x \to +\infty).$$

To obtain a more precise estimate, we integrate by parts once more:

$$\int_x^\infty t^{-2} e^{it^2}\, dt = \frac{1}{2i} \int_x^\infty t^{-3} d\,(e^{it^2}) = -\frac{1}{2ix^3} e^{ix^2} + \frac{3}{2i} \int_x^\infty t^{-4} e^{it^2}\, dt.$$

The absolute value of the integral on the right-hand side does not exceed

$$\int_x^\infty t^{-4}\, dt = O(x^{-3}) \quad (x \to +\infty).$$

This yields

$$\Phi(x) = \frac{ie^{ix^2}}{2x} + O\left(\frac{1}{x^3}\right) \quad (x \to +\infty).$$

Continuing the process of integration by parts, we arrive at the asymptotic expansion for $\Phi(x)$ as $x \to +\infty$. We give this expansion with the first two terms:

$$\Phi(x) = e^{ix^2}\left(\frac{i}{2x} + \frac{1}{4x^3}\right) + O\left(\frac{1}{x^5}\right). \quad \square$$

44.2 The contribution from a nondegenerate stationary point The hypothesis of Theorem 1 contains one important restriction, namely, $S'(x) \neq 0$ for $x \in I$, i.e. $S(x)$ (the phase) does not have a stationary point within the interval. But if such stationary points exist, then the asymptotic behavior of $F(\lambda)$ has a different nature than that stated in Theorem 1. The phase $S(x) = x^2$ has a stationary point $x = 0$. In the neighborhood of this point (over an interval of the order of $1/\sqrt{\lambda}$) the function $\cos \lambda x^2$ does not oscillate, while the sum of the areas of the other waves in the cosine are of the order of $O(\lambda^{-1})$, i.e. is considerably less than the area about the stationary point. For this reason the value of the integral $F(\lambda)$ will be of the order of $1/\sqrt{\lambda}$. Let us give a rigorous foundation to these heuristic considerations.

Lemma 1 *Suppose a function $f(x)$ is infinitely differentiable on the segment $[0, a]$ and $\alpha \neq 0$. Then, as $\lambda \to +\infty$,*

$$\Phi(\lambda) = \int_0^a f(x)\, e^{(i/2)\alpha\lambda x^2}\, dx = \frac{1}{2}\sqrt{\frac{2\pi}{|\alpha|\lambda}}\, e^{i(\pi/4)\delta(\alpha)} f(0) + O\left(\frac{1}{\lambda}\right), \tag{44.9}$$

$$\delta(\alpha) = \operatorname{sign} \alpha.$$

Proof. Let $\alpha > 0$ and $f(x) \equiv 1$. Introducing the variable $t = x\sqrt{\alpha\lambda}$, we have

$$\int_0^a e^{(i/2)\alpha\lambda x^2}\,dx = \frac{1}{\sqrt{\alpha\lambda}}\int_0^{a\sqrt{\alpha\lambda}} e^{it^2/2}\,dt$$

$$= \frac{1}{\sqrt{\alpha\lambda}}\left[\int_0^\infty e^{it^2/2}\,dt - \int_{a\sqrt{\alpha\lambda}}^\infty e^{it^2/2}\,dt\right].$$

The first integral in the brackets is a Fresnel integral and is equal to $(1/2)\, e^{i\pi/4}\sqrt{2\pi}$ (see Sec. 29). The other is an integral whose value is of the order of $O(1/\sqrt{\lambda})$ as $\lambda \to +\infty$, in view of Example 1, so that

$$\int_0^a e^{(i/2)\alpha\lambda x^2}\,dx = \frac{1}{2}\sqrt{\frac{2\pi}{\alpha\lambda}}\, e^{i\pi/4} O\left(\frac{1}{\lambda}\right) \quad (\lambda \to +\infty). \tag{44.10}$$

Now suppose α is negative. Then

$$\overline{\int_0^a e^{i\alpha x^2}\,dx} = \int_0^a e^{i\beta x^2}\,dx,$$

with $\beta = -\alpha > 0$. Hence, formula (44.10) remains valid for $\alpha < 0$ if we substitute $-\alpha = |\alpha|$ for α and $e^{-i\pi/4}$ for $e^{i\pi/4}$.

Let us represent $f(x)$ in the form

$$f(x) = f(0) + [f(x) - f(0)] = f(0) + xg(x),$$

where $g(x) = \dfrac{f(x) - f(0)}{x}$ is a function that is infinitely differentiable for $0 \leqslant x \leqslant a$. Then

$$\Phi(\lambda) = \frac{1}{2} f(0)\sqrt{\frac{2\pi}{|\alpha|\lambda}}\, e^{i\frac{\pi}{4}\delta(\alpha)} + O\left(\frac{1}{\lambda}\right) + \Phi_1(\lambda), \tag{44.11}$$

$$\Phi_1(\lambda) = \int_0^a e^{\frac{i}{2}\alpha\lambda x^2} xg(x)\,dx.$$

Let us estimate $\Phi_1(\lambda)$. We have

$$|\Phi_1(\lambda)| = \left|\frac{1}{i\alpha\lambda}\int_0^a g(x)\,d\left(e^{(i/2)\alpha\lambda x^2}\right)\right|$$

$$= \frac{1}{|\alpha|\lambda}\left|g(a)\, e^{(i/2)\alpha\lambda a^2} - g(0) - \int_0^a e^{(i/2)\alpha\lambda x^2} g'(x)\,dx\right|$$

$$\leqslant \frac{1}{|\alpha|\lambda}\left[|g(a)| + |g(0)| + \int_0^a |g'(x)|\,dx\right] = O\left(\frac{1}{\lambda}\right)$$

as $\lambda \to +\infty$. Substituting this estimate into (44.11), we arrive at (44.9). The proof of the lemma is complete.

Remark 1. The proof of Lemma 1 implies that (44.9) is valid if $f(x)$ is doubly differentiable on $[0, a]$.

Theorem 2 *Suppose $f(x)$ and $S(x)$ are infinitely differentiable on the segment $[a, b]$ and the function $S(x)$ has only one stationary point $x_0 \in [a, b]$, with $a < x_0 < b$. If $S''(x_0) \neq 0$, then the integral* (44.1) *possesses the following formula*:

$$F(\lambda) = e^{i\lambda S(x_0)} e^{i(\pi/4)\delta_0} \sqrt{\frac{2\pi}{\lambda\,|S''(x_0)|}}\, f(x_0) + O\left(\frac{1}{\lambda}\right) \quad (\lambda \to +\infty). \tag{44.12}$$

Here $\delta_0 = \operatorname{sign} S''(x_0)$.

Proof. Let us split the integration domain into two segments, $[a, x_0]$ and $[x_0, b]$, and the integral $F(\lambda)$ into two integrals, $F_1(\lambda)$ and $F_2(\lambda)$, respectively. Suppose $S''(x_0)$ is positive, for the sake of definiteness. Then $S'(x_0)$ is positive for $x_0 < x \leqslant b$, and the function $S(x)$ monotonically increases for $x_0 < x \leqslant b$, i.e. $S(x) > S(x_0)$ in this interval. In the integral $F_2(\lambda)$ (taken along the segment $[x_0, b]$) we introduce a new variable t by the relationship $x = \varphi(t)$ in a way such that $S(x) - S(x_0) = t^2$ (see Sec. 43). Then

$$F_2(\lambda) = e^{i\lambda S(x_0)} \int_0^{b'} e^{i\lambda t^2} g(t)\, dt.$$

Here

$$g(t) = f(\varphi(t))\, \varphi'(t), \quad b' = \sqrt{S(b) - S(x_0)} > 0.$$

By Lemma 1, we have

$$F_2(\lambda) = \frac{1}{2} e^{i\lambda S(x_0)} e^{i\pi/4} \sqrt{\frac{\pi}{\lambda}}\, g(0) + O\left(\frac{1}{\lambda}\right)$$

as $\lambda \to +\infty$, with $g(0) = f(x_0)\sqrt{\dfrac{2}{S''(x)_0}}$. A similar formula exists for $F_1(\lambda)$. Combining the two, we arrive at (44.12). The case with $S''(x_0) < 0$ can be reduced to the case with $S''(x_0) > 0$:

$$\overline{F(\lambda)} = \int_a^b e^{i\lambda \tilde{S}(x)} \overline{f(x)}\, dx, \quad \tilde{S}(x) = -S(x),$$

and $\tilde{S}''(x_0) > 0$. The proof of the theorem is complete.

Example 2. Let us calculate the asymptotic behavior of the Bessel function

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i(x \sin\varphi - n\varphi)}\, d\varphi$$

as $x \to +\infty$, with n a nonnegative integer. In the case at hand the phase $S(\varphi) = \sin\varphi$, and there are two stationary points, $\varphi_1 = \pi/2$ and $\varphi_2 = 3\pi/2$, with

$$S(\varphi_1) = 1, \quad S''(\varphi_1) = -1, \quad S(\varphi_2) = -1, \quad S''(\varphi_2) = 1.$$

The asymptotic behavior of $J_n(x)$ is given by the sum of the contributions from the points φ_1 and φ_2 (i.e. expressions of the type (44.12)) and a term of the order of $O(1/x)$, i.e.

$$J_n(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{x}\right) \quad (x \to +\infty). \quad \Box$$

44.3 Poisson's summation formula This formula enables us to replace a series of the form $\sum_{n=-\infty}^{\infty} f(n)$ with another series, namely

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi ikx} f(x)\, dx. \tag{44.13}$$

This formula is valid if

(a) $f(x)$ is continuously differentiable for $-\infty < x < \infty$,

(b) the series $\sum_{n=-\infty}^{\infty} f(n)$ is convergent,

(c) the series $\sum_{n=-\infty}^{\infty} f'(n+x)$ is uniformly convergent for $0 \leqslant x \leqslant 1$.

The proof of the validity of (44.13) under these or other conditions is given in Evgrafov [2]. We will restrict our discussion to a formal derivation of (44.13). Consider the function $\varphi(x) = \sum_{n=-\infty}^{\infty} f(n+x)$. This function is periodic with a period equal to unity. Let us expand $\varphi(x)$ in a Fourier series:

$$\varphi(x) = \sum_{k=-\infty}^{\infty} \varphi_k e^{2\pi ikx},$$

whence

$$\sum_{k=-\infty}^{\infty} \varphi_k = \sum_{n=-\infty}^{\infty} f(n). \tag{44.14}$$

Let us show that formula (44.14) leads to Poisson's summation formula. We have

$$\varphi_k=\int_0^1 e^{-2\pi ikx}\varphi(x)\,dx=\int_0^1\sum_{n=-\infty}^{\infty} f(n+x)\,e^{-2\pi ikx}\,dx$$

$$=\sum_{n=-\infty}^{\infty}\int_n^{n+1} f(x)\,e^{-2\pi ikx}\,dx=\int_{-\infty}^{\infty} e^{-2\pi ikx}f(x)\,dx,$$

and, substituting the φ_k into (44.14), we arrive at (44.13).

Formula (44.13) proves to be convenient when the integrals

$$\varphi_n=\int_{-\infty}^{\infty} e^{-2\pi inx}f(x)\,dx$$

decrease as $n\to\infty$ faster than $f(n)$ (i.e. when the Fourier transform of $f(x)$ decreases as $|x|\to\infty$ faster than $f(x)$). For one, this is the case for a rapidly oscillating $f(x)$.

Example 3. Let us consider the series

$$F(t)=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{\sqrt{n^2+t^2}}$$

and establish the asymptotic behavior of $F(t)$ as $t\to\pm\infty$. Here

$$f(x,\ t)=\frac{e^{i\pi x}}{\sqrt{x^2+t^2}}.$$

We apply Poisson's summation formula. Conditions (a) and (b) are met. Let us see whether condition (c) is (for a fixed $t>0$). We have

$$f'_x(x,\ t)=\pi ie^{i\pi x}(x^2+t^2)^{-1/2}-xe^{i\pi x}(x^2+t^2)^{-3/2}.$$

Consider the series

$$S_1=\sum_{k=1}^{\infty}(-1)^k a_k,\quad a_k=[(x+k)^2+t^2]^{-1/2}.$$

Since the functions $a_k(x)$ decrease monotonically in k for each fixed $x\in[0,\ t]$, and the partial sums of the series $\sum\limits_{k=1}^{\infty}(-1)^k$ are limited, we conclude, from Dirichlet's test (see Kudryavtsev [1]), that the series S_1 is uniformly convergent on $[0,\ 1]$. The uniform convergence of the series

$$S_2=\sum_{k=1}^{\infty}(-1)^k(x+k)[(x+k)^2+t^2]^{-3/2}$$

for $x \in [0, 1]$ can be proved similarly. The same is true for the series S_1 and S_2 when the summation is carried out from $-\infty$ to -1. Hence, conditions (a), (b), and (c) are met. Applying (44.13), we find that

$$F(t)=\sum_{k=-\infty}^{\infty}\varphi_k(t),\quad \varphi_k(t)=\int_{-\infty}^{\infty}e^{-2\pi ikx+\pi ix}(x^2+t^2)^{-1/2}\,dx.$$

Substituting ty for x, we obtain

$$\varphi_k(t)=\int_{-\infty}^{\infty}e^{-it\pi(2k-1)y}(y^2+1)^{-1/2}\,dy,$$

so that $\varphi_k(t)=2K_0((2k-1)\pi t)$ (see Example 3 in Sec. 43). In Example 3 in Sec. 43 it was shown that $K_0(b)$ is an even function and that

$$K_0(b)=\sqrt{\frac{\pi}{2b}}\,e^{-b}[1+O(b^{-1}]\quad (b\to+\infty).$$

Hence, for $|b|\geqslant 1$ we have

$$|K_0(b)|\leqslant Ce^{-|b|},$$

where C does not depend on b, and

$$|F(t)-\varphi_0(t)-\varphi_1(t)|\leqslant 4C\sum_{k=2}^{\infty}|K_0((2k-1)\pi t)$$

$$\leqslant 4C\sum_{k=2}^{\infty}e^{-(2k-1)\pi t}\leqslant 8Ce^{-3\pi t}.$$

We finally obtain

$$F(t)=2\varphi_0(t)+O(e^{-3\pi t})=2\sqrt{\frac{1}{2t}}\,e^{-\pi t}[1+O(t^{-1})]\quad (t\to+\infty).\ \square$$

45 The Saddle-Point Method

45.1 Preliminary considerations Let us take the integral

$$F(\lambda)=\int_{\gamma}f(z)\,e^{\lambda S(z)}\,dz,\tag{45.1}$$

where γ is a piecewise smooth curve in the complex z plane, and the functions $f(z)$ and $S(z)$ are regular in a domain D that contains γ. We are interested in the asymptotic behavior of $F(\lambda)$ as $\lambda\to+\infty$. The trivial cases $f(z)\equiv 0$ or $S(z)\equiv$ const are of no interest to us.

In Sec. 43 it was found that if γ is a segment and $S(z)$ admits real values on γ, the asymptotic behavior of (45.1) can be established by applying Laplace's method. We will try to transform the integral in (45.1) in such a way as to make it possible to apply Laplace's method. Since $f(z)$ and $S(z)$ are regular in D, we can deform γ in D (with the end points remaining fixed) without changing the value of $F(\lambda)$. Suppose we can deform γ into a contour $\tilde{\gamma}$ such that

(1) $\max\limits_{z\in\tilde{\gamma}} |e^{\lambda S(\lambda)}|$ is attained only at one point $z_0 \in \tilde{\gamma}$ (z_0 an interior point of the contour),

(2) $\operatorname{Im} S(z) \equiv \text{const}$ for $z \in \tilde{\gamma}$ in a neighborhood of point z_0.

Suppose $\tilde{\gamma}_0$ is a small arc of $\tilde{\gamma}$ that contains point z_0. Then $\operatorname{Re} S(z) \leqslant \operatorname{Re} S(z_0) - \delta$, where $\delta > 0$, for $z \overline{\in} \tilde{\gamma}_0$, $z \in \tilde{\gamma}$. This follows from the fact that $\max\limits_{z\in\tilde{\gamma}} \operatorname{Re} S(z)$ is attained only at point z_0, according to condition (1). For this reason the integral taken along the arc $\tilde{\gamma} - \tilde{\gamma}_0$ is of the order of $O(|e^{\lambda(S(z_0)-\delta)}|)$ as $\lambda \to +\infty$ (see Lemma 1 of Sec. 43). Let us consider the integral taken along the arc $\tilde{\gamma}_0$; suppose $z = \varphi(t)$, with $-t_0 \leqslant t \leqslant t_0$ and $\varphi(0) = z_0$, is the equation of this arc. By condition (2), $\operatorname{Im} S(z) \equiv \operatorname{Im} S(z_0)$ on $\tilde{\gamma}_0$, so that the integral taken along this arc is

$$F_1(\lambda) = e^{i\lambda \operatorname{Im} S(z_0)} \int\limits_{-t_0}^{t_0} \tilde{f}(t)\, e^{\lambda \tilde{S}(t)}\, dt,$$

where $\tilde{f}(t) = f(\varphi(t))\,\varphi'(t)$ and $\tilde{S}(t) = \operatorname{Re} S(\varphi(t))$. In $F_1(\lambda)$ the function $\tilde{S}(t)$ assumes only real values; hence, $F_1(\lambda)$ belongs to the class of integrals discussed in Sec. 43 and its asymptotic behavior can be established by Laplace's method.

Note, in addition, that $S'(z_0) = 0$. Indeed, $\frac{d}{dt} \operatorname{Im} S(z)\,|_{t=0} = 0$, by condition (2), and since $\max\limits_{z\in\tilde{\gamma}} \operatorname{Re} S(z)$ is attained at point z_0 (condition (1), we conclude that $\frac{d}{dt} \operatorname{Re} S(z)\,|_{t=0} = 0$. Hence, $\frac{d}{dt} S(z)\,|_{t=0} = 0$, so that $S'(z_0) = 0$.

A point z_0 at which $S'(z_0) = 0$ is said to be a *saddle point*, and a contour that obeys conditions (1) and (2) must pass through the saddle point of $S(z)$.

In a similar manner we can establish the asymptotic behavior of the integral (45.1) when $\max\limits_{z\in\gamma} \operatorname{Re} S(z)$ is attained only at one end point of γ. In this case the point, z_0, may not be a saddle point.

Thus, if the function $\operatorname{Re} S(z)$ on contour γ attains its maximum only at a finite number of points, which are either saddle points or end points of the contour (we call such a contour a *saddle contour*), then the asymptotic behavior of integral (45.1) can be established by Laplace's method. The most difficult problem in applying the saddle-point method is finding a saddle contour $\tilde{\gamma}$ equivalent to the initial contour γ (the equivalence of γ and $\tilde{\gamma}$ means that the integrals of type (45.1) are equal along these contours). Many problems have been solved by the saddle-point method (e.g. see Evgrafov [2], Fedoryuk [1], Lavrent'ev and Shabat [1], Morse and Feshbach [1], and Whittaker and Watson [1]), but there is not a single general technique that would enable us to find an equivalent saddle contour $\tilde{\gamma}$ from given functions $f(z)$ and $S(z)$ and a given contour γ.

We will now give a rigorous derivation of the asymptotic formulas for integral (45.1) taken along a saddle contour. But first we will study the local structure of the curves along which $\operatorname{Re} S(z)$ or $\operatorname{Im} S(z)$ remain constant (the *level curves*).

45.2 The structure of level curves of harmonic functions Suppose $S(z)$ is regular in a neighborhood of point z_0. Let us study the level curves of $\operatorname{Re} S(z) = \operatorname{Re} S(z_0) + \varepsilon$ and $\operatorname{Im} S(z) = \operatorname{Im} S(z_0) + \varepsilon$ for small ε's in a neighborhood of point z_0.

Lemma 1 *Suppose $S'(z_0)$ is not zero. Then in a small neighborhood of point z_0 the level curves* $\operatorname{Re} S(z) = \text{const}$ *and* $\operatorname{Im} S(z) = \text{const}$ *are smooth curves.*

Proof. The function $S(z)$ is univalent at point z_0, since $S'(z_0) \neq 0$. For this reason the function $w = S(z)$ maps a small neighborhood U of point z_0 conformally and in a one-to-one manner onto a small neighborhood V of point $w_0 = S(z_0)$. We select U in a way such that V is the square $|u - u_0| < \delta$, $|v - v_0| < \delta$, where $w = u + iv$ and $w_0 = u_0 + iv_0$. Under such a mapping the level curves of the functions $\operatorname{Re} S(z)$ and $\operatorname{Im} S(z)$ lying in U are mapped into segments of straight lines $u = \text{const}$ and $v = \text{const}$ lying in V. These segments are, obviously, smooth curves (lines), and their preimages are smooth curves, too, since $z = S^{-1}(w)$, which is the inverse of $S(z)$, is regular at point w_0 (see Theorem 1 of Sec. 13). The proof of the lemma is complete.

We have thus established that the local structure of the level curves of $\operatorname{Re} S(z)$ and $\operatorname{Im} S(z)$ in a neighborhood of a point that is not a saddle point is exactly the same as that of the function $S(z) = z$ (Fig. 159).

Now let us study the structure of the level curves of $\operatorname{Re} S(z)$ and $\operatorname{Im} S(z)$ in a neighborhood of a saddle point. But first let us consider a simple case.

Example 1. Let us study the level curves of the real and imaginary parts of the function $S(z) = -z^2$. Point $z = 0$ is a saddle point. Assuming that $z = x + iy$ and $S = u + iv$, we find that $u =$

$y^2 - x^2$ and $v = -2xy$. The family of level curves has the form

$$x^2 - y^2 = C_1, \quad 2xy = C_2,$$

where C_1 and C_2 are constants. If both C_1 and C_2 are not zeros, each of the curves $\operatorname{Re} S = C_1$ and $\operatorname{Im} S = C_2$ is a hyperbola, while the curve $u = 0$ consists of the two straight lines $x - y = 0$ and $x + y = 0$ and the curve $v = 0$ of the two straight lines $x = 0$ and $y = 0$ (Fig. 160). The level curves $\operatorname{Re} S(z) = \operatorname{Re} S(0)$ (i.e. the straight lines $x \pm y = 0$) divide the complex z plane into four sectors, with the signs of $\operatorname{Re}(S(z) - S(0))$ in two neighboring sectors being different (Fig. 160). Suppose D_0 is the sector $|\arg z| < \pi/4$ and D_1 is the sector $|\arg(-z)| < \pi/4$; in these sectors $\operatorname{Re}(-z^2) < 0$. The level curve that passes through the saddle poin

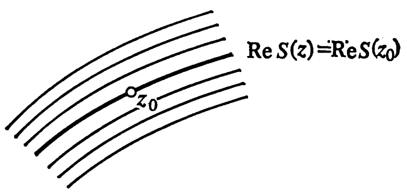


Fig. 159

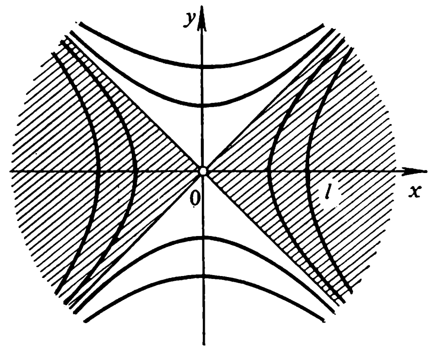


Fig. 160

$z = 0$ is $\operatorname{Im} S(z) = \operatorname{Im} S(0)$, which is the straight line l: $y = 0$. Along this line we have $\operatorname{Re} S(z) = -x^2$, i.e. the function $\operatorname{Re} S(z)$ decreases strictly monotonically as point z moves along l away from the saddle point $z = 0$. Line (in general a curve) l is known as the *path of steepest descent*. □

Take a three-dimensional space with coordinates x, y, and $\operatorname{Re} S$ and the surface $\operatorname{Re} S = \operatorname{Re}(-z^2)$, i.e. $\operatorname{Re} S = y^2 - x^2$. This surface is a hyperbolic paraboloid (Fig. 161), and the origin of coordinates is the saddle point. A mountain pass or a saddle resemble such a surface; hence the names "saddle-point method" and "saddle point". The path of steepest descent from the saddle point is projected onto the (x, y) plane into l.

Let us now show that if z_0 is a simple saddle point for $S(z)$, i.e. if $S''(z_0) \neq 0$, then in a neighborhood of this point the level curves of $\operatorname{Re} S(z)$ and $\operatorname{Im} S(z)$ have the same structure as those in the case of $S(z) = -z^2$.

Lemma 2 *Suppose point z_0 is a simple saddle point of $S(z)$, i.e. $S'(z_0) = 0$ and $S''(z_0) \neq 0$. Then in a small neighborhood U the* *level curve* Re $S(z) = \text{Re}\, S(z_0)$ *consists of two smooth curves* l_1 *and* l_2 *that are orthogonal to each other at point* z_0 *and divide* U *into four sectors. In neighboring sectors the signs of* $\text{Re}\,(S(z) - S(z_0))$ *are different. This situation is depicted in Fig. 162.*

Proof. Suppose U is a small neighborhood of point z_0. Then there is a function $\varphi(\zeta)$ that is regular in a neighborhood V of point $\zeta = 0$ and such that

$$S(\varphi(\zeta)) = S(z_0) - \zeta^2, \quad \zeta \in V \tag{45.2}$$

(see Corollary 2 in Sec. 32). Moreover, $\varphi'(0) \neq 0$, and the function $z = \varphi(\zeta)$ maps V onto U in a one-to-one manner. The level curves of $\text{Re}\, S(z)$ and $\text{Im}\, S(z)$ are mapped by $\zeta = \varphi^{-1}(z)$ into the curves

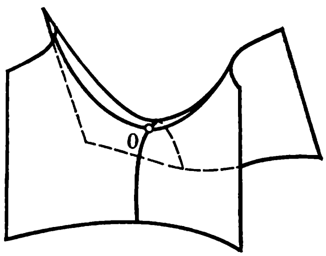


Fig. 161

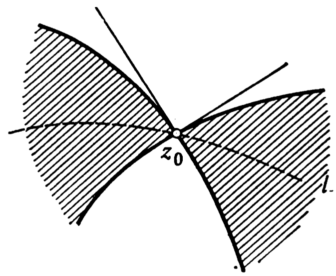


Fig. 162

$\text{Re}\,\zeta^2 = \text{const}$ and $\text{Im}\,\zeta^2 = \text{const}$, whose structure was studied earlier. Returning to the variable z, we complete the proof of the lemma.

Corollary 1 *Through the sectors in which* $\text{Re}\, S(z) < \text{Re}\, S(z_0)$ *there passes a smooth curve* l *such that* $\text{Im}\, S(z) = \text{Im}\, S(z_0)$ *for* $z \in l$. *The function* $\text{Re}\, S(z)$ *decreases strictly monotonically as* z *moves along* l *away from* z_0.

The curve l is the path of steepest descent (Fig. 162; the dashed line).

45.3 The contribution from an end point of the integration path
In what follows we will always assume that γ is a finite curve and that the functions $f(z)$ and $S(z)$ are regular in a domain D containing γ.

Theorem 1 *Suppose* $\max\limits_{z \in \gamma} \text{Re}\, S(z)$ *is attained only at the initial point* a *of* γ *and* $S'(a) \neq 0$. *Then, as* $\lambda \to +\infty$, *the following asymptotic expansion is valid:*

$$F(\lambda) \equiv \int_\gamma f(z)\, e^{\lambda S(z)}\, dz \sim \lambda^{-1} e^{\lambda S(a)} \sum_{n=0}^{\infty} c_n \lambda^{-n}. \tag{45.3}$$

This expansion can be differentiated with respect to λ *term-by-term any number of times. The principal term of the asymptotic expansion has the form*

$$F(\lambda)=\frac{1}{-\lambda S'(a)}e^{\lambda S(a)}\left[f(a)+O\left(\frac{1}{\lambda}\right)\right]. \qquad (45.4)$$

The expansion coefficients in (45.3) are given by the formula

$$c_n=(-1)^n M^n\left(\frac{f(z)}{S'(z)}\right)\Big|_{z=a}, \quad M=\frac{1}{S'(z)}\frac{d}{dz}. \qquad (45.5)$$

Note that for Laplace integrals (45.3)-(45.5) coincide with formulas (43.8) and (43.9). The proof is similar to the proof of Theorem 1 of Sec. 43.

45.4 The contribution from a simple saddle point Let us establish the asymptotic behavior of (45.1) in the case where $\max_{z\in\gamma}$ Re $S(z)$ is attained at an interior point of the integration path. Precisely, let the following conditions be met:

(a) $\max_{z\in\gamma}$ Re $S(z)$ is attained only at a point z_0 that is an interior point of γ and a simple saddle point (i.e. $S'(z_0)=0$ and $S''(z_0)\neq 0$),

(b) in a neighborhood of point z_0 the contour γ passes through both sectors in which Re $S(z) <$ Re $S(z_0)$ (Fig. 162).

Theorem 2 *Suppose conditions* (a) *and* (b) *are met. Then, as* $\lambda\to+\infty$, *the following asymptotic expansion is valid*:

$$F(\lambda)\equiv\int_{\lambda} f(z)\,e^{\lambda S(z)}\,dz\sim e^{\lambda S(z_0)}\sum_{n=0}^{\infty}c_n\lambda^{-n-1/2}. \qquad (45.6)$$

This expansion can be differentiated with respect to λ *term-by-term any number of times. The principal term in the asymptotic expansion has the form*

$$F(\lambda)=\sqrt{-\frac{2\pi}{\lambda S''(z_0)}}\,e^{\lambda S(z_0)}\left[f(z_0)+O\,\frac{1}{\lambda}\right], \quad (\lambda\to+\infty). \qquad (45.7)$$

The choice of the branch of the root in (45.7) and the formula for the expansion coefficients in (45.6) are given below.

Proof of Theorem 2. Suppose U is a small neighborhood of point z_0, $\gamma_0=\gamma\cap U$, and γ_1 and γ_2 are the remaining arcs of γ. Let us split the integral $F(\lambda)$ into three terms: $F(\lambda)=F_0(\lambda)+F_1(\lambda)+F_2(\lambda)$, where $F_j(\lambda)$ is an integral of the type (45.1) taken along arc γ_j, $j=0, 1, 2$. Since $\max_{z\in\gamma}$ Re $S(z)$ is attained only at point

$z_0 \in \gamma_0$, we can show, just as we did in the proof of Theorem 2 of Sec. 43, that the following estimate holds for $F_1(\lambda)$ and $F_2(\lambda)$:

$$|F_j(\lambda)| \leqslant c\,|e^{\lambda(S(z_0)-\delta)}| \quad (\lambda > 0), \; j = 1, 2, \tag{45.8}$$

where c and δ are positive constants.

Let us find the asymptotic expansion for $F_0(\lambda)$. If U is small, then there is a neighborhood V of point $\zeta = 0$ and a function $z = \varphi(\zeta)$ such that (i) $S(\varphi(\zeta)) = S(z_0) - \zeta^2$, $\zeta \in V$, and (ii) the function $\varphi(\zeta)$ is regular in V and maps V onto U in a one-to-one manner, with $\varphi(0) = z_0$.

This follows from Corollary 2 in Sec. 32. Substituting $\varphi(\zeta)$ for z in $F_0(\lambda)$, we obtain

$$F_0(\lambda) = e^{\lambda S(z_0)} \int_{\tilde{\gamma}} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta. \tag{45.9}$$

Here $g(\zeta) = f(\varphi(\zeta))\varphi'(\zeta)$, and $\tilde{\gamma}$ is the image of contour γ_0. For V we can take the circle $|\zeta| < \rho$ of a small radius $\rho > 0$; we can also assume that $\varphi(\zeta)$ is regular in the closed circle $|\zeta| \leqslant \rho$.

The level curve $\operatorname{Re}(-\zeta^2) = 0$ consists of two straight lines $\xi \pm \eta = 0$ ($\zeta = \xi + i\eta$) and divides V into four sectors. Suppose D_1 is the sector containing the interval l_1: $(0, \rho)$ and D_2 is the sector containing the interval l_2: $(-\rho, 0)$. Curve γ_0, by hypothesis, consists of two curves γ_{01} and γ_{02} (with a common point z_0); these curves lie in the different sectors in which $\operatorname{Re} S(z) < \operatorname{Re} S(z_0)$. Hence, point $\xi = 0$ partitions curve $\tilde{\gamma}$ into two curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ that lie in sectors D_1 and D_2, respectively. Suppose C_1 is the arc of circle $|\zeta| = \rho$ that lies in D_1 and connects the end points of the curves l_1 and $\tilde{\gamma}_1$. By Cauchy's theorem,

$$\int_{\tilde{\gamma}_1} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta = \int_0^{\rho} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta + \int_{C_1} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta. \tag{45.10}$$

Since $\operatorname{Re}(-\zeta^2) < 0$ on C_1, there is a positive constant δ_1 such that $\operatorname{Re}(-\zeta^2) \leqslant -\delta_1$ on C_1, and the integral taken along $\tilde{\gamma}_1$ is equal to the sum of the integral taken along the segment $[0, \rho]$ and a term of the order of $O(e^{-\lambda\delta_1})$ ($\lambda \to +\infty$). Applying the same line of reasoning to the integral taken along the arc $\tilde{\gamma}_2$, we find that

$$e^{-\lambda S(z_0)} F_0(\lambda) = \int_{-\rho}^{\rho} e^{-\lambda\zeta^2} g(\zeta)\, d\zeta + O(e^{-\lambda\delta'}) \quad (\lambda \to +\infty), \tag{45.11}$$

where δ' is a positive constant. On the right-hand side of (45.11) we have an integral taken along a segment, i.e. a Laplace integral, (43.1), with $S = -\zeta^2$. Further $\max\limits_{-\rho\leqslant\zeta\leqslant\rho} S(\zeta)$ is attained only at point $\zeta = 0$, with $S''(0) \neq 0$. Applying Theorem of Sec. 43, we arrive at expansion (45.6). The proof of the theorem is complete.

Let us select a branch of the root in (45.7) (the choice depends, of course, on the orientation of γ). In proving Theorem 2 it was found that γ can be deformed into a contour γ' which in a neighborhood of the saddle point z_0 coincides with the path of steepest descent l: $\operatorname{Im} S(z) = \operatorname{Im} S(z_0)$ on l and $\operatorname{Re} S(z) < \operatorname{Re} S(z_0)$ for $z \in l$, $z \neq z_0$. Let us show that

$$\arg \sqrt{\frac{1}{-S''(z_0)}} = \varphi_0, \qquad (45.12)$$

where φ_0 is the angle between the tangent to l at point z_0 and the positive direction of the real axis.

It is sufficient to consider the case with $f(z) \equiv 1$ and $S(z) = az^2/2$, since the principal term in the asymptotic expansion is expressed only in terms of $f(z)$, $S(z)$, and $S''(z)$ at the saddle point. The path of steepest descent l, which passes through the saddle point $z = 0$, is the straight line (see Sec. 45.2) on which $\operatorname{Im} S(z) \equiv 0$ and while $\operatorname{Re} S(z)$ is negative for $z \neq 0$. We write its equation in the form $z = e^{i\varphi_0}\rho$, $-\infty < \rho < \infty$. Then $S(z) = -|a|\rho^2/2$ for $z \in l$. The integral along l is equal to

$$\int_l e^{\lambda a z^2/2}\, dz = e^{i\varphi_0} \int_{-\infty}^{\infty} e^{-\lambda|a|\rho^2/2}\, d\rho = e^{i\varphi_0} \sqrt{\frac{2\pi}{|a|}},$$

which proves formula (45.12).

The proof of Theorem 2 leads to

Theorem 3 *Suppose* $\max\limits_{z\in\gamma} \operatorname{Re} S(z)$ *is attained only at the initial point* a *of contour* γ, *with* $S'(a) = 0$ *and* $S''(a) \neq 0$. *Then, as* $\lambda \to +\infty$, *the following asymptotic expansion is valid*:

$$F(\lambda) \equiv \int_\gamma f(z)\, e^{\lambda S(z)}\, dz \sim e^{\lambda S(a)} \sum_{n=0}^{\infty} a_n \lambda^{-\frac{n+1}{2}}. \qquad (45.13)$$

This expansion can be differentiated with respect to λ *term-by-term any number of times. The principal term in the asymptotic expansion has the form*

$$F(\lambda) = \frac{1}{2} \sqrt{-\frac{2\pi}{\lambda S''(a)}}\, e^{\lambda S(a)} [f(a) + O(\lambda^{-1})] \quad (\lambda \to +\infty). \qquad (45.14)$$

The choice of the branch of the root in (45.14) is the same as in (45.10).

Corollary 2 *Suppose* $\max\limits_{z\in\gamma} \operatorname{Re} S(z)$ *is attained at a finite number of points* $z_1, z_2, \ldots, z_m$, *which are either the end points of the path of integration or saddle points on the contour that obey condition (b) of Theorem 2. Then the asymptotic behavior of* (45.1) *as* $\lambda \to +\infty$ *is determined by the sum of the contributions from all points* $z_1, z_2, \ldots, z_m$.

Remark 1. If all the points z_j at which $S'(z_j) = 0$ are simple saddle points, the asymptotic behavior of (45.1) is determined by formulas (45.3), (45.6) and (45.13), with the principal term in the asymptotic expansion determined by formulas (45.4), (45.7) and (45.14). The asymptotic behavior can also be determined when there are saddle points of multiplicity greater than unity among the z_j (e.g. see Evgrafov [2], Fedoryuk [1], and Lavrent'ev and Shabat [1]).

45.5 Examples

Example 2. Let us establish the asymptotic behavior of the Airy-Fock function

$$\operatorname{Ai}(x) = \frac{1}{\pi}\int\limits_0^\infty \cos\left(\frac{t^3}{3} + tx\right) dt$$

as $x \to +\infty$. We first transform this integral into

$$\operatorname{Ai}(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{i\left(\frac{t^3}{3} + tx\right)} dt \tag{45.15}$$

(the function $\sin(t^3/3 + tx)$ is odd in t and the integral of this function taken along the real axis is zero). The integral in (45.15) is conditionally convergent; we will transform it so as to make it absolutely convergent.

Let us take the straight line l_{η_0}: $-\infty < \xi < \infty$, $\eta = \eta_0$ in the complex $\zeta = \xi + i\eta$ plane, which line is parallel to the real axis. On this line

$$\operatorname{Re} S(\zeta, x) = -\xi^2\eta_0 + \frac{\eta_0^3}{3} - x\eta_0, \tag{45.16}$$

where $S(\zeta, x) = i(\zeta^3/3 + x\zeta)$. Hence, $\int\limits_{l_{\eta_0}} e^{S(\zeta, x)}\, d\zeta$ is absolutely convergent if $\eta_0 > 0$. It was found that the integral (45.15) is equal to the integral taken along the straight line l_{η_0} for any positive η_0, i.e.

$$\operatorname{Ai}(x) = \frac{1}{2\pi}\int\limits_{l_{\eta_0}} e^{S(\zeta, x)}\, d\xi. \tag{45.17}$$

For every fixed $x > 0$ the function $S(\zeta, x)$ has two saddle points, $\zeta_1(x) = i\sqrt{x}$ and $\zeta_2(x) = -i\sqrt{x}$. For l_{η_0} we select the straight line that passes through the saddle point $\zeta_1(x)$, i.e. we put $\eta_0 = \sqrt{x}$. We substitute $\xi\sqrt{x}$ for ξ in the integral in (45.17), so as to bring it to the (45.1) type. Then

$$\operatorname{Ai}(x) = \frac{\sqrt{x}}{2\pi}\int_{-\infty}^{\infty} e^{x^{3/2}\tilde{S}(\tilde{\xi})}\,d\tilde{\xi},\quad \tilde{S}(\tilde{\xi}) = i\left[\frac{(\tilde{\xi}+i)^3}{3} + \tilde{\xi} + i\right]. \quad (45.18)$$

The path of integration contains the saddle point $\tilde{\xi} = 0$ of $\tilde{S}(\tilde{\xi})$. Further, for real $\tilde{\xi}$'s we have

$$\operatorname{Re}\tilde{S}(\tilde{\xi}) = -\tilde{\xi}^2 - \frac{2}{3}, \quad (45.19)$$

so that max Re $\tilde{S}(\tilde{\xi})$ is attained in the integration path only at the saddle point $\tilde{\xi} = 0$. This is a simple saddle point since $\tilde{S}''(0) = -2 \neq 0$.

Thus, the integral in (45.18) meets all the conditions of Theorem 2 except one, namely, the path of integration is an infinite straight line. Let us partition this straight line into three parts: the rays $(-\infty, -1)$ and $(1, \infty)$ and the segment $[-1, 1]$. In view of (45.19) we have

$$\left|\int_1^{\infty} e^{x^{3/2}\tilde{S}(\tilde{\xi})}\,d\tilde{\xi}\right| \leqslant e^{-(2/3)x^{3/2}}\int_1^{\infty} e^{-x^{3/2}\tilde{\xi}^2}\,d\tilde{\xi}.$$

By Lemma 1 of Sec. 43, the integral on the right-hand side is of the order of $O(e^{-x^{3/2}})$ $(x \to +\infty)$ since $-\tilde{\xi}^2 \leqslant -1$ for $\tilde{\xi} \geqslant 1$, so that the integral taken along the ray $1 \leqslant \tilde{\xi} < \infty$ is exponentially small compared with $e^{-2/3x^{3/2}}$ as $x \to +\infty$. The integral taken along the ray $-\infty \leqslant \tilde{\xi} \leqslant 1$ can be estimated in the same manner.

The asymptotic behavior of the integral taken along the segment $[-1, 1]$ can be established via Theorem 2; the principal term in the asymptotic expansion is calculated by formula (45.7). We have $\tilde{S}(0) = -2/3$ and $\tilde{S}''(0) = -2$. What is left is to select a branch of the root in (45.7). We have

$$\tilde{S}(\tilde{\zeta}) - \tilde{S}(0) \sim -\tilde{\zeta}^2 \quad (\tilde{\zeta} \to 0),$$

where $\tilde{\zeta} = \tilde{\xi} + i\tilde{\eta}$. For this reason the path of steepest descent l, which passes through the saddle point $\tilde{\zeta} = 0$, has the same tangent as the path of steepest descent l_0 corresponding to $-\tilde{\zeta}^2$. The equation of l_0 has the form $\tilde{\zeta} = \rho$, $-\infty < \rho < \infty$, i.e. $\varphi_0 = 0$ in (45.12). The final asymptotic formula is

$$\mathrm{Ai}\,(x) = \frac{1}{2\sqrt{\pi}}\, x^{-1/4} e^{-\frac{2}{3}x^{3/2}} [1 + O\,(x^{-3/2})] \quad (x \to +\infty). \tag{45.20}$$

In the case at hand we can calculate all the coefficients in the asymptotic series (see Fedoryuk [1]). The asymptotic behavior of the Airy-Fock function as $x \to -\infty$ will be established in Example 4. □

Example 3. Let us establish the asymptotic behavior for real $x \to \infty$ of the integral

$$F\,(x) = \int_{-\infty}^{\infty} e^{-\frac{t^{2n}}{2n} + ixt}\, dt, \tag{45.21}$$

where n is a positive integer. Since

$$F\,(-x) = \overline{F\,(x)} \tag{45.22}$$

for real x's, it is sufficient to establish the asymptotic behavior of the integral (45.21) as $x \to +\infty$. By changing the variable from $x^{-1/(2n-1)}\, t$ to t, we transform the integral $F\,(x)$ to the form (45.1):

$$F\,(x) = x^{1/(2n-1)}\, \Phi\,(\lambda), \quad \Phi\,(\lambda) = \int_{-\infty}^{\infty} e^{\lambda S(t)}\, dt, \tag{45.23}$$

where

$$\lambda = x^{2n/(2n-1)}, \quad S\,(t) = -\frac{t^{2n}}{2n} + it. \tag{45.24}$$

The saddle points of $S\,(t)$ are determined from the equation $t^{2n-1} = i$ and are

$$t_k = e^{i\varphi_k}, \quad \varphi_k = \frac{(\pi/2) + 2k\pi}{2n-1}, \quad 0 \leqslant k \leqslant 2n-2. \tag{45.25}$$

Hence,

$$S\,(t_k) = \left(1 - \frac{1}{2n}\right) it_k, \quad \mathrm{Re}\, S\,(t_k) = \left(\frac{1}{2n} - 1\right) \sin \varphi_k. \tag{45.26}$$

For this reason $\mathrm{Re}\, S\,(t_k) < 0$ if point t_k lies in the upper half-plane of the complex t plane and $\mathrm{Re}\, S\,(t_k) > 0$ if t_k lies in the lower half-plane.

Integral (45.21) tends to zero as $x \to +\infty$ by the Riemann-Lebesgue lemma (see Sec. 44). Hence, the points t_k lying in the lower half-plane contribute nothing to the asymptotic behavior of $\Phi\,(\lambda)$, since the absolute value of the integrand, $|e^{\lambda S(t_k)}| = e^{\lambda \operatorname{Re} S(t_k)}$ at such a point exponentially grows as $x \to +\infty$. For this reason the asymptotic behavior of $\Phi(\lambda)$ is determined only by those points t_k that lie in the upper half-plane $\operatorname{Im} t > 0$.

Since there are no saddle points of $S(t)$ in the integration path, we must deform this contour into a saddle contour. For $|t| \to \infty$ we have $S(t) \sim -t^{2n}/2n$, i.e. $\operatorname{Re} S(t) \to -\infty$ as t tends to ∞ in the sectors $|\arg t| < \pi/2n$ and $|\arg t - \pi| < \pi/2n$, which contain the real axis. Moreover, on each straight line $\operatorname{Im} t = c$ (c is a constant) we have $\operatorname{Re} S(t) \sim -\frac{1}{2n} (\operatorname{Re} t)^{2n}$ ($\operatorname{Re} t \to \pm\infty$). Hence the integral of the type (45.23) taken along the straight line $\operatorname{Im} t = c$ is absolutely convergent. It is easy to show that

$$\Phi(\lambda) = \int\limits_{\operatorname{Im} t = c} e^{\lambda S(t)}\, dt$$

for any value of c. Of course, there are other contours γ equivalent to the real axis besides the straight lines $\operatorname{Im} t = c$; for instance, for γ we can take any simple infinite curve with the rays $\arg t = \alpha$, $|\alpha| < \pi/2n$, and $\arg(-t) = \beta$, $|\beta| < \pi/2n$, as asymptotes. But the saddle contour is among the straight lines parallel to the real axis.

We select the straight line $\operatorname{Im} t = \operatorname{Im} t_0$ that passes through the saddle point $t_0 = e^{i\pi/2(2n-1)}$ as the integration path n $\Phi(\lambda)$. There is also another saddle point that lies on this line, namely, $t = -\bar{t}_0$. We wish to show that $\max \operatorname{Re} S(t)$ is attained on the straight line l: $\operatorname{Im} t = \operatorname{Im} t_0$ only at the saddle points t_0 and $-\bar{t}_0$. We have $t = \xi + i\eta_0$ and $\eta_0 = \operatorname{Im} t_0$ on l, so that $\operatorname{Re}(it) = -\eta_0 = \text{const}$. The points at which $\operatorname{Re}[(\xi + i\eta_0)^{2n}]$ is extremal can be determined from the equation

$$0 = \frac{d}{d\xi} \operatorname{Re}[(\xi + i\eta_0)^{2n}] = 2n \operatorname{Re}(\xi + i\eta_0)^{2n-1},$$

whence

$$(\xi + i\eta_0)^{2n-1} = iy \tag{45.27}$$

at a point of extremum, with y a real number. Suppose $y > 0$. Then

$$\xi + i\eta_0 = y^{1/(2n-1)} e^{i\varphi_k}, \quad e^{i\varphi_k} = \sqrt[2n-1]{i}.$$

This implies that at points of extremum we have

$$\xi = \eta_0 \cot \varphi_k, \quad \xi + i\eta_0 = \frac{\eta_0}{\sin \varphi_k} e^{i\varphi_k}$$

$$-\operatorname{Re}(\xi + i\eta_0)^{2n} = -\operatorname{Re}\left[\frac{\eta_0^{2n}}{(\sin \varphi_k)^{2n}} e^{i2n\varphi_k}\right] = \frac{\eta_0^{2n}}{(\sin \varphi_k)^{2n-1}},$$

since $e^{i(2n-1)\varphi_k}=i$. But since $\varphi_k=\frac{\pi/2+2k\pi}{2n-1}$, the $\max\limits_k (\sin\varphi_k)^{-2n+1}$ is attained at $k=0$. This value of k corresponds to the point of extremum $\xi+i\eta_0=\frac{\eta_0}{\sin\varphi_0}e^{i\varphi_0}=t_0$.

If $y<0$, then we have

$$(-\xi+i\eta_0)^{2n-1}=-iy,$$

and the point $-\xi+i\eta_0$ lies on the straight line l. Just as in the case with $y>0$, we can prove that $\max\limits_{t\in l}\operatorname{Re} S(t)$ is attained at the point $-\bar{t}_0$.

We have therefore established that $\max\limits_{t\in l}\operatorname{Re} S(t)$ is attained only at the saddle points t_0 and $-\bar{t}_0$. Following the line of reasoning developed in Example 1, we can easily show that the asymptotic behavior of $\Phi(\lambda)$ is determined by the sum of the contributions from these two saddle points, in spite of the fact that the curve is infinite. From (45.25) and (45.26) we find that

$$\begin{gathered}S(t_0)=i\left(1-\frac{1}{2n}\right)e^{i\varphi_0},\quad S''(t_0)=-i(2n-1)e^{-i\varphi_0};\\ S(-\bar{t}_0)=\overline{S(t_0)},\quad S''(-\bar{t}_0)=\overline{S''(t_0)};\\ \varphi_0=\frac{\pi}{2(2n-1)}.\end{gathered}\tag{45.28}$$

Asymptotically the integral $\Phi(\lambda)$ is equal to a sum of expressions of the (45.7) type; what remains to be specified is the branches of $\sqrt{-1/S''(t)}$ in these formulas.

For small values of $|t-t_0|$ we have

$$S(t)-S(t_0)\sim\frac{1}{2}S''(t_0)(t-t_0),$$

so that the equation of the path of steepest descent l_0, which passes through point t_0, has the form

$$t=t_0+\rho e^{i\psi_0}+O(\rho^2)\ (\rho\to 0),\ \psi_0=-\frac{\pi}{4}+\frac{\varphi_0}{2},$$

which follows from (45.28). Hence, in (45.7) we must have

$$\sqrt{-\frac{1}{S''(t_0)}}=\frac{1}{|\sqrt{S''(t_0)}|}e^{i\psi_0},$$

so that the contribution from point t_0 to the integral $\Phi(\lambda)$ is

$$V(t_0)=e^{i\lambda S(t_0)}\sqrt{\frac{2\pi}{\lambda(2n-1)}}e^{i\psi_0}[1+O(\lambda^{-1})].$$

Similar considerations can be applied to the saddle point at $\bar{t}_0$, but a more simple approach is to employ the fact that $F(x)$ assumes real values for real x's. Indeed, the function $e^{-t^{2n}/2n} \sin tx$ is odd in t, and the integral of this function taken along the real axis is equal to zero. Hence,

$$F(x) = \int_{-\infty}^{\infty} e^{-t^{2n}/(2n)} \cos tx \, dt$$

for real x's.

The principal term in the asymptotic expansion is of the form $\Phi(\lambda) \approx V(t_0) + V(-\bar{t}_0)$, and, since the values of $\Phi(\lambda)$ must be real, we conclude that $V(-\bar{t}_0) \sim \overline{V(t_0)}$. Hence, the contribution from the saddle point $-\bar{t}_0$ to the asymptotic behavior of $\Phi(\lambda)$ is equal to

$$V(-\bar{t}_0) = \overline{V(t_0)}\,[1 + O(\lambda^{-1})].$$

The final result is, as $x \to +\infty$,

$$F(x) = Ae^{-ax^{2n/(2n-1)}} x^{-(n-1)/(2n-1)} \times \left[\cos\left(bx^{2n/(2n-1)} - \frac{\pi}{4} + \frac{\varphi_0}{2}\right) + O\left(x^{-2n/(2n-1)}\right)\right]. \qquad (45.29)$$

Here

$$A = 2\sqrt{\frac{2\pi}{2n-1}}, \quad a = \left(1 - \frac{1}{2n}\right)\sin\varphi_0,$$

$$b = \left(1 - \frac{1}{2n}\right)\cos\varphi_0, \text{ and } \varphi_0 = \frac{\pi}{2(2n-1)}.$$

We have found that the integral (45.21) exponentially decreases as $x \to \pm\infty$ and has an infinitude of real zeros.

45.6 The saddle-point method and the method of stationary phase In Sec. 44 we considered integrals of the type

$$F(\lambda) = \int_a^b f(x)\, e^{i\lambda S(x)}\, dx$$

taken along a finite segment $[a, b]$ on which $S(x)$ assumes only real values. We found the principal term in the asymptotic expansion for the case where $S(x)$ has only one stationary point x_0, $a < x_0 < b$, $S''(x_0) \neq 0$.

Suppose that now the functions $f(x)$ and $S(x)$ are the values of two functions $f(z)$ and $S(z)$ regular in a neighborhood of the segment $[a, b]$. Then we can show that the following asymptotic expansion is valid:

$$F(\lambda) \sim e^{i\lambda S(b)} \sum_{n=0}^{\infty} b_n (i\lambda)^{-n-1}$$

$$- e^{i\lambda S(a)} \sum_{n=0}^{\infty} a_n (i\lambda)^{-n-1} + e^{i\lambda S(x_0)} \sum_{n=0}^{\infty} c_n \lambda^{-n-\frac{1}{2}} \quad (\lambda \to +\infty). \qquad (45.30)$$

Here the coefficients a_n and b_n are determined via (44.5). In other words, the asymptotic behavior of the integral $F(\lambda)$ as $\lambda \to +\infty$ is determined by the contributions from the saddle point and the end points a and b of the path of integration.

To prove this proposition, it is sufficient to deform the segment $[a, b]$, the path of integration, into a contour γ such that $\max\limits_{z \in \gamma} \operatorname{Re}(iS(z))$ is attained only at the points $z = a$, $z = b$, and

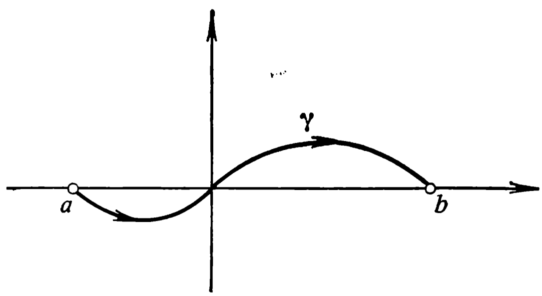


Fig. 163

$z = x_0$. The asymptotic behavior of the integral taken along γ is described by (45.30), in view of Corollary 2 and Theorems 1 and 2.

Let us restrict our discussion to the case of a quadratic phase, $S(x) = x^2$; the general case can be investigated similarly. We have $\operatorname{Re}(iz^2) < 0$ in the first and third quadrants. We replace the initial path of integration, $[a, b]$, by the contour γ depicted in Fig. 163. Then $\operatorname{Re}(iz^2) < 0$ everywhere on γ except the end points a and b and the saddle point $z = 0$. The contour satisfies the conditions of Corollary 2.

Thus, the method of stationary phase is a particular case of the saddle-point method if the integrand is a regular function.

Remark 2. Formula (45.30) is also valid if $f(x)$ and $S(x)$ are infinitely differentiable on the segment $[a, b]$ (see Fedoryuk [1]).

Example 4. Let us establish the asymptotic behavior of the Airy-Fock function (45.15) as $x \to -\infty$. Substituting t for $t\sqrt{|x|}$, we obtain

$$\mathrm{Ai}\,(x) = \frac{\sqrt{|x|}}{2\pi} F(\lambda), \quad F(\lambda) = \int_{-\infty}^{\infty} e^{\lambda S(t)}\,dt, \tag{45.31}$$

where $\lambda = |x|^{3/2}$ and $S(t) = i(t^3/3 - t)$. The saddle points $t_{1,2} = \pm 1$ of $S(t)$ lie in the path of integration. Let us deform the integration contour in a way such that Re $S(t)$ is attained only at the saddle points t_1 and t_2.

On the real axis in the complex t plane we have Re $S(t) = 0$. Let us see where Re $S(t)$ is negative. Putting $t = \xi + i\eta$, we find

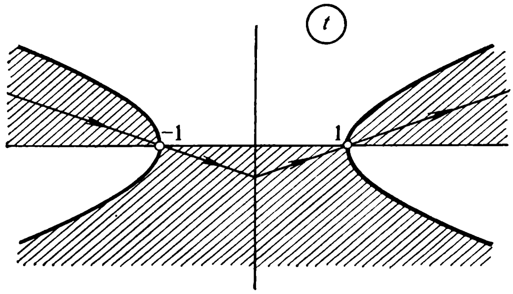


Fig. 164

that the equation Re $S(t) = 0$ has the form $\eta(\xi^2 - \eta^2/3 - 1) = 0$. For this reason the curve representing Re $S(t) = 0$ consists of the real axis ($\eta = 0$) and the hyperbola $\xi^2 - \eta^2/3 - 1 = 0$. This curve is depicted in Fig. 164 (the region in which Re $S(t)$ is negative is hatched). We replace the path of integration by γ (Fig. 164). Since $t \sim |t|\,e^{i\pi/6}$ on γ as Re $t \to +\infty$, we conclude that Re $S(t) \sim -(1/3)\,|t|^3$ and the function $|e^{\lambda S(t)}|$ decreases exponentially as Re $t \to +\infty$, $t \in \gamma$. The same is true when Re $t \to -\infty$, $t \in \gamma$. It was found that the integral (45.31) is equal to the integral taken along γ:

$$F(\lambda) = \int_{\gamma} e^{\lambda S(t)}\,dt.$$

On γ we have Re $S(t) < 0$ everywhere except at the saddle points t_1 and t_2, at which Re $S(t) = 0$. In view of Corollary 2, the asymptotic behavior of $F(\lambda)$ is determined by the sum of the contributions from the saddle points t_1 and t_2. We have

$$S(t_{1,2}) = \mp \frac{2}{3} i, \quad S''(t_{1,2}) = \pm 2i,$$

and the final result is

$$\mathrm{Ai}(x)=\frac{1}{\sqrt{\pi}}\,|\,x\,|^{-1/4}\left[\cos\left(\frac{2}{3}\,|\,x\,|^{3/2}+\frac{\pi}{4}\right)+O\left(|\,x\,|^{-3/2}\right)\right]$$
$$(x\to-\infty).\ \square$$

46 Laplace's Method of Contour Integration

46.1 Laplace's contour transformation Let us start with a second-order linear homogeneous differential equation with linear coefficients

$$(a_0z+a_1)\,w''+(b_0z+b_1)\,w'+(c_0z+c_1)\,w=0, \qquad (46.1)$$

where the a_j, b_j, and c_j are constants. We will look for the solution to this equation in the form

$$w(z)=\int\limits_C e^{\zeta z}v(\zeta)\,d\zeta, \qquad (46.2)$$

where $v(\zeta)$ is the unknown function, and C is an integration contour that is independent of z. Below we give a formal derivation of the solution. Differentiation under the integral sign yields

$$w'(z)=\int\limits_C e^{\zeta z}\zeta v(\zeta)\,d\zeta,\quad w''(z)=\int\limits_C e^{\zeta z}\zeta^2 v(\zeta)\,d\zeta.$$

If we now integrate by parts, we find that

$$zw(z)=\int\limits_C v(\zeta)\,de^{\zeta z}=v(\zeta)\,e^{\zeta z}\,\Big|_C-\int\limits_C e^{\zeta z}v'(\zeta)\,d\zeta,$$

where the first term on the right-hand side is taken at the end points of contour C. Similarly,

$$zw'(z)=\zeta v(\zeta)\,e^{\zeta z}\,\Big|_C-\int\limits_C e^{\zeta z}(\zeta v(\zeta))'\,d\zeta,$$

$$zw''(z)=\zeta^2 v(\zeta)\,e^{\zeta z}\,\Big|_C-\int\limits_C e^{\zeta z}(\zeta^2 v(\zeta))'\,d\zeta.$$

Suppose contour C and the function $v(\zeta)$ are selected in a way such that the sum of the terms without the integrals vanishes:

$$(a_0\zeta^2+b_0\zeta+c_0)\,v(\zeta)\,e^{\zeta z}\,|_C=0. \qquad (46.3)$$

Then Eq. (46.1) takes the form

$$\int\limits_C e^{\zeta z}\left[(a_1\zeta^2+b_1\zeta+c_1)\,v(\zeta)-a_0(\zeta^2 v(\zeta))'-b_0(\zeta v(\zeta))'-c_0v'(\zeta)\right]d\zeta=0.$$

We select the function $v(\zeta)$ in a way such that it satisfies the equation

$$\frac{d}{d\zeta}(a_0\zeta^2 v(\zeta) = b_0\zeta v(\zeta) + c_0 v(\zeta)) - (a_1\zeta^2 + b_1\zeta + c_1)\,v(\zeta) = 0, \qquad (46.4)$$

which means that $w(z)$ is a solution to Eq. (46.1).

Equation (46.4) is a first-order linear homogeneous differential equation and can easily be integrated. We have

$$\frac{v'(\zeta)}{v(\zeta)} = \frac{a_1\zeta^2 + (b_1 - 2a_0)\,\zeta + (c_1 - b_0)}{a_0\zeta^2 + b_0\zeta + c_0}.$$

Let us expand the right-hand side of this equation into partial fractions. Two cases are possible.

(1) $a_0 \neq 0$. We will assume that $a_0 = 1$ and $a_1 = 0$ (a linear change of variable easily transforms Eq. (46.1) into this form). Suppose the equation

$$\zeta^2 + b_0\zeta + c_0 = 0 \qquad (46.5)$$

has two different roots ζ_1 and ζ_2. Then

$$\frac{v'(\zeta)}{v(\zeta)} = \frac{p-1}{\zeta - \zeta_1} + \frac{q-1}{\zeta - \zeta_2}, \qquad (46.6)$$

$$p = \frac{(b_1 - 2)\,\zeta_1 + (c_1 - b_0) + \zeta_1 - \zeta_2}{\zeta_1 - \zeta_2},$$

$$q = \frac{(b_1 - 2)\,\zeta_2 + (c_1 - b_0) + \zeta_2 - \zeta_1}{\zeta_2 - \zeta_1}.$$

Integration of Eq. (46.6) yields

$$v(\zeta) = A\,(\zeta - \zeta_1)^{p-1}\,(\zeta - \zeta_2)^{q-1},$$

so that

$$w(z) = A\int_C (\zeta - \zeta_1)^{p-1}\,(\zeta - \zeta_2)^{q-1}\,e^{\zeta z}\,d\zeta. \qquad (46.7)$$

Here A is an arbitrary constant and the following condition must be met:

$$(\zeta - \zeta_1)^p\,(\zeta - \zeta_2)^q e^{\zeta z}\,\big|_C = 0. \qquad (46.8)$$

(2) $a_0 = 0$. Let us assume that $a_1 = 1$ and $b_0 \neq 0$. Then

$$\frac{v'(\zeta)}{v(\zeta)} = A\zeta + B + \frac{p}{\zeta + c_0/b_0},$$

$$A = \frac{1}{b_0}, \quad B = \frac{b_1}{b_0} - \frac{c_0}{b_0^2}, \quad p = \frac{c_0^2}{b_0^3} - \frac{b_1 c_0}{b_0^2} + \frac{c_1}{b_0} - 1.$$

Integration yields $v(\zeta) = ce^{\frac{1}{2}A\zeta^2+B\zeta}\left(\zeta+\frac{c_0}{b_0}\right)^p$, so that

$$w(z) = c\int\limits_C \left(\zeta+\frac{c_0}{b_0}\right)^p e^{\frac{1}{2}A\zeta^2+B\zeta+\zeta z}\,d\zeta. \tag{46.9}$$

If $a_0 = b_0 = 0$ and $a_1 = 1$, then, similarly,

$$w(z) = c\int\limits_C \exp\left[c_0^{-1}\left(\frac{\zeta^3}{3}+\frac{b_1\zeta^2}{2}+c_1\zeta\right)+\zeta z\right]d\zeta, \tag{46.10}$$

where c is an arbitrary constant.

To solve Eq. (46.1) completely, we must specify contour C. Two linearly independent solutions of Eq. (46.1) are constructed by selecting two different contours C. But first we will study one of the more simple equations of type (46.1).

46.2 Airy's equation This is the equation

$$w'' - zw = 0. \tag{46.11}$$

Its solution, in view of (46.10), has the form

$$w(z) = \int\limits_C e^{\zeta z-\zeta^3/3}\,d\zeta$$

if

$$e^{\zeta z-\zeta^3/3}\,\Big|_C = 0. \tag{46.12}$$

Obviously, for C we must not take a closed integration contour, since otherwise $w(z) \equiv 0$. Further, C must be an infinite contour, since otherwise (46.12) will not be valid for all z's. Let us investigate the behavior of $e^{-\zeta^3/3}$ as $\zeta \to \infty$. This function tends to zero along any ray that starts at point $\zeta = 0$ and lies in one of the sectors

$$S_0:\ |\arg\zeta| < \pi/6,\quad S_1:\ \pi/2 < \arg\zeta < 5\pi/6,$$
$$S_{-1}: \frac{-5\pi}{6} < \arg\zeta < -\frac{\pi}{2}.$$

Suppose l_0 and $l_{\pm 1}$ are the rays $\arg\zeta = 0$ and $\arg\zeta = \pm 2\pi/3$, respectively. Consider the contours $C_1 = l_1 - l_{-1}$, $C_2 = l_0 - l_1$, and $C_3 = l_{-1} - l_0$. Any one of these satisfies (46.12), and we arrive at three solutions to Eq. (46.11):

$$w_j(z) = \int\limits_{C_j} e^{\zeta z-\zeta^3/3}\,d\zeta,\quad j = 1,\ 2,\ 3. \tag{46.13}$$

These solutions, of course, are not linearly independent: the construction of the C_j implies that

$$w_1(z) + w_2(z) + w_3(z) \equiv 0. \tag{46.14}$$

But any two of these solutions are, as we show below, linearly independent.

For many applications the most interesting solution is $w_1(z)$. Let us transform it to the "standard" form. Using Jordan's lemma, we can show that C_1 can be deformed (for Im $z > 0$) into the imaginary axis, so that

$$w_1(z) = i \int_{-\infty}^{\infty} e^{i(tz + t^3/3)}\, dt.$$

The solution $w_1(z)$ differs from Airy's function Ai (z) (Sec. 41) only by a constant factor.

The solutions $w_2(z)$ and $w_3(z)$ can be expressed in terms of $w_1(z)$. Since the contours C_2 and C_3 are obtained from C_1 by rotating C_1 through an angle of $-2\pi/3$ and $+2\pi/3$, respectively, and since $(\zeta e^{\pm i2\pi/3})^3 = \zeta^3$, we have

$$w_2(z) = e^{-2i\pi/3} w_1(e^{-2\pi i/3} z), \quad w_3(z) = e^{2i\pi/3} w_1(e^{2\pi i/3} z).$$

46.3 Equation (46.1) with $a_0 \neq 0$ Consider the equation

$$zw'' + (b_0 z + b_1) w' + (c_0 z + c_1) w = 0. \qquad (46.15)$$

Suppose the roots ζ_1 and ζ_2 of Eq. (46.5) are distinct. Then the integral (46.7) is a solution to Eq. (46.15) if condition (46.8) is met. Let us select a contour C in a way such that this condition is indeed met. The points ζ_1 and ζ_2 are the branch points of the integrand if p and q are not integers. We fix a point ζ_0 that is neither ζ_1 nor ζ_2. We then proceed with the following traversals: we circuit (1) point ζ_1 in the positive sense, (2) point ζ_2 in the positive sense, (3) point ζ_1 in the negative sense, and (4) point ζ_2 in the negative sense. This results in a closed contour C (Fig. 165). After the first traversal the initial value of the function $(\zeta - \zeta_1)^{p-1} (\zeta - \zeta_2)^{q-1}$ at point ζ_0 is multiplied by $e^{2\pi i p}$, after the second by $e^{2\pi i q}$, after the third by $e^{-2\pi i p}$, and after the fourth by $e^{-2\pi i q}$, so that this function proves to be single-valued on C.

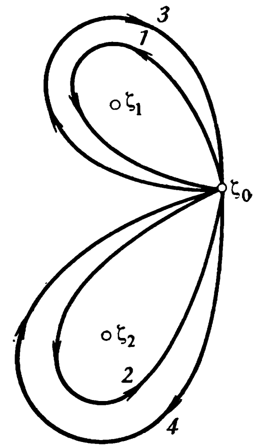


Fig. 165

The reader will recall that one solution of Eq. (46.15) is regular at point $z=0$ (see Sec. 27). This is the solution we have just constructed. The second linear independent solution will have to be constructed using another contour C. Let us cut the complex ζ plane along rays that start at points ζ_1 and ζ_2 and go to the left parallel to the real axis (cuts l_1 and l_2; Fig. 166). For the sake of simplicity we assume that $\operatorname{Im}\zeta_1 \neq \operatorname{Im}\zeta_2$. For C_1 and C_2 we take contours that traverse the cuts in the positive sense (Fig. 166). By the monodromy theorem, the function $(\zeta-\zeta_1)^{p-1}(\zeta-\zeta_2)^{q-1}$ splits in the complex ζ plane with the cuts l_1 and l_2 into regular branches, one of which we fix. We put

$$w_j(z)=\int\limits_{C_j}(\zeta-\zeta_1)^{p-1}(\zeta-\zeta_2)^{q-1}e^{\zeta z}\,d\zeta,\quad j=1,\ 2. \tag{46.17}$$

Let us show that at $\operatorname{Re} z>0$ both integrals converge and condition (46.8) is met. On C_1 we have $\zeta=\zeta_1-t$, $0<t<\infty$, so that $|e^{\zeta z}|=|e^{\zeta_1 z}|\,e^{-t\operatorname{Re} z}$; whence $|e^{\zeta z}|$ decreases exponentially as $t\to+\infty$. The function $|(\zeta-\zeta_1)^{p-1}(\zeta-\zeta_2)^{q-1}|$ can increase as $t\to+\infty$ only as a power function, and from this follows the convergence of the integral for $w_1(z)$ at $\operatorname{Re} z>0$. We can easily show that this integral is uniformly convergent in any half-plane of the type $\operatorname{Re} z\geqslant a>0$. Hence (see Corollary 1 in Sec. 16), the function $w_1(z)$ (and therefore $w_2(z)$) is regular in the half-plane $\operatorname{Re} z>0$.

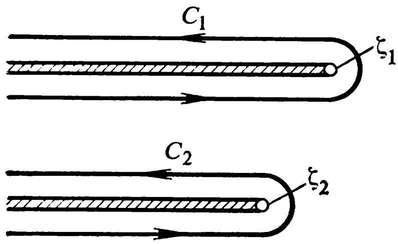


Fig. 166

In Sec. 27 it was demonstrated that every solution to Eq. (46.15) is an analytic function in the complex z plane with points ζ_1 and ζ_2 deleted, whereas we have just proved that the functions $w_1(z)$ and $w_2(z)$ are analytic only in the right half-plane. Rotating the contour of integration C_2 in the same way as we did in Sec. 16 (Theorem 6), we can show that the solution $w_1(z)$ (or $w_2(z)$) can be continued analytically into the plane with a cut along a ray that starts at point ζ_1 (or ζ_2) and passes through point ζ_2 (or ζ_1).

46.4 The asymptotic behavior of the solutions Let us restrict our discussion to the case where the exponents p and q are real and not integral numbers. We select the branches of $(\zeta-\zeta_1)^{p-1}$ and $(\zeta-\zeta_2)^{q-1}$ in a way such that they are positive for $\zeta-\zeta_1$ and $\zeta-\zeta_2$ positive, i.e. on the continuations of the cuts. Let us show that, as $x\to+\infty$, the following asymptotic expansions are valid:

$$\begin{aligned} w_1(x) &\sim e^{\zeta_1 x}x^{-p}\sin\pi p\sum_{n=0}^{\infty}a_n x^{-n},\\ w_2(x) &\sim e^{\zeta_2 x}x^{-q}\sin\pi q\sum_{n=0}^{\infty}b_n x^{-n}. \end{aligned} \tag{46.18}$$

The coefficients a_n and b_n have the form

$$a_n = 2i(-1)^n \Gamma(n+p)(\zeta_1-\zeta_2)^{q-n-1}\frac{(q-1)\ldots(q-k)}{k!},$$
$$b_n = 2i(-1)^n \Gamma(n+q)(\zeta_2-\zeta_1)^{p-n-1}\frac{(p-1)\ldots(p-k)}{k!}. \tag{46.19}$$

The choice of the branches is as follows: $|\arg(\zeta_1-\zeta_2)|<\pi$ in the first formula in (46.19) and $|\arg(\zeta_2-\zeta_1)|<\pi$ in the second.

To establish the asymptotic behavior of the solutions we employ Watson's lemma (Sec. 43). Take solution $w_1(x)$. We substitute t for $\zeta-\zeta_1$. This yields

$$w_1(x) = e^{\zeta_1 x}\int_{(0+)} t^{p-1}(t+\zeta_1-\zeta_2)^{q-1}e^{xt}\,dt.$$

The integration contour traverses the cut along $(-\infty, 0)$ in the positive sense. Suppose p positive. Then the integration contour can be deformed in a way such that it follows the banks of the cut. Due to the choice of branch, $t^{p-1}=e^{i\pi(p-1)}|t|^{p-1}$ on the upper bank of the cut and $t^{p-1}=e^{-i\pi(p-1)}|t|^{(p-1)}$ on the lower. Hence,

$$w_1(x) = 2i\sin\pi p e^{\zeta_1 x}I(x),$$
$$I(x)=\int_0^\infty t^{p-1}(\zeta_1-\zeta_2-t)^{q-1}e^{-tx}\,dt.$$

The asymptotic behavior of the integral $I(x)$ as $x\to+\infty$ is established via Watson's lemma (here $\alpha=1$, $\beta=p$ and $f(t)=(\zeta_1-\zeta_2-t)^{q-1}$). This results in (46.18) and (46.19).

Suppose p is negative. Integration by parts yields

$$w_1(x)e^{-\zeta_1 x} = \frac{1}{p}\int_{(0+)}\varphi(t)\,dt^p = -\frac{1}{p}\int_{(0+)} t^p\varphi'(t)\,dt,$$
$$\varphi(t)=(t+\zeta_1-\zeta_2)^{q-1}e^{xt}.$$

If $p>-1$, the asymptotic behavior of the integral on the right-hand side can be established by the same method as in the case with $p>0$. But if $p>-m$, where m is a positive integer not less than 2, we integrate by parts $m-1$ times more. This results in the integral

$$w_1(x)=e^{\zeta_1 x}(-1)^m\frac{1}{p(p+1)\ldots(p+m-1)}\int_{(0+)} t^{p+m-1}\varphi^{(m)}(t)\,dt,$$

whose asymptotic behavior can be established in the same manner as for the case with $p>0$. Now we only have to show that (46.18) and (46.19) are valid for $p < 0$. Suppose $p > 0$. Then we can also integrate m times by parts. For the integral that we obtain the asymptotic expansion (46.18) is valid, since it coincides with the initial integral and the asymptotic expansion is unique. Hence, for $p < 0$ formulas (46.18) and (46.19) are valid. The asymptotic behavior of the solution $w_2(x)$ can be established in a similar manner.

Remark 1. Formulas (46.18) and (46.19) hold not only as $x \to +\infty$ but as $|z| \to \infty$, $\operatorname{Re} z > 0$, uniformly in $\arg z$ in every sector of the type $|\arg z| \leqslant \pi/2 - \varepsilon$, $\varepsilon > 0$. This follows from Watson's lemma.

Remark 2. The asymptotic expansion (46.18) can be differentiated termwise any number of times.

We take Bessel's differential equation as an example:

$$z^2w'' + zw' + (z^2 - n^2)\,w = 0. \tag{46.20}$$

We introduce the substitution

$$w = z^n u. \tag{46.21}$$

Then for $u(z)$ we have an equation of the (46.1) type:

$$zu'' + (2n + 1)\,u' + zu = 0.$$

In this case $\zeta_{1,2} = \pm i$, $p = q = n + 1/2$, and we have two linearly independent solutions of Bessel's equation:

$$\begin{aligned} w_1(z) &= z^n \int_{C_1} (\zeta^2 + 1)^{n-1/2}\, e^{\zeta z}\, d\zeta, \\ w_2(z) &= z^n \int_{C_2} (\zeta^2 + 1)^{n-1/2}\, e^{\zeta z}\, d\zeta. \end{aligned} \tag{46.22}$$

The contours C_1 and C_2 are depicted in Fig. 166, where $\zeta_1 = i$ and $\zeta_2 = -i$.

The asymptotic formulas (46.18) and (46.19) take the form

$$w_1(z) \sim \frac{1}{\sqrt{z}}\, e^{iz} e^{-\frac{\pi}{2}\left(n-\frac{1}{2}\right)i} \left(1 + e^{2\pi n i}\right) 2^{n-\frac{1}{2}} \times \sum_{k=0}^{\infty} \binom{n-1/2}{k} \Gamma(n+k+1/2)\,(i/2z)^k,$$

$$w_2(z) \sim \frac{1}{\sqrt{z}}\, e^{-iz} e^{-\frac{3\pi}{2} ni + \frac{3\pi i}{4}} \left(1 + e^{2\pi n i}\right) 2^{n-1/2} \times \sum_{k=0}^{\infty} \binom{n-1/2}{k} \Gamma(n+k+1/2) \left(-\frac{i}{2z}\right)^k.$$

These asymptotic expansions are valid for $|z| \to \infty$, $|\arg z| \leqslant \pi/2 - \varepsilon$ $(\varepsilon > 0)$.

Laplace's method of contour integration makes it possible to solve a linear ordinary differential equation with linear coefficients and of arbitrary order. The solution of

$$\sum_{k=0}^{n} (a_{k0}z + a_{k1})\, w^{(n-k)}(z) = 0$$

is sought in the form (46.2), and the unknown function $v(\zeta)$ obeys a first-order linear differential equation, which can easily be solved.

46.5 Difference equations with linear coefficients We start with a second-order homogeneous differential equation

$$A_n y_{n+2} + B_n y_{n+1} + C_n y_n = 0, \quad n = 0, 1, 2, \ldots, \tag{46.23}$$

where the A_n, B_n, and C_n are given numbers. The numbers y_0 and y_1 are assumed to be fixed, so that the recurrence relations (46.23) enable us to find y_2, then y_3, etc. The problem we are considering here is to find an explicit formula for the term y_n in the sequence $y_0, y_1, y_2, y_3, \ldots$. Such formulas have been found (e.g. see Smirnov [1]) when the A_n, B_n, and C_n are constant numbers, i.e. do not depend on n. It so happens that explicit formulas, although not as simple, can be obtained for the case where the coefficients in Eq. (46.23) are linear functions, precisely, for equations of the type

$$[a_0(n+2) + a_1]\, y_{n+2} + [b_0(n+1) + b_1]\, y_{n+1} + (c_0 n + c_1)\, y_n = 0. \tag{46.24}$$

Here the a_j, b_j, and c_j are complex valued constants and $(a_0, a_1) \neq (0, 0)$.

A simple example of Eq. (46.24) is the binomial equation

$$[a_0(n+2) + a_1]\, y_{n+2} + (c_0 n + c_1)\, y_n = 0. \tag{46.25}$$

We put $y_1 = 0$ and $y_0 \neq 0$. Then all the terms in the sequence $\{y_n\}$ with odd numbers vanish: $y_{2m+1} = 0$, $m = 0, 1, 2, \ldots$. Then Eq. (46.25) yields

$$y_{2n} = -\frac{(2n-2)c_0 + c_1}{2na_0 + a_1}\, y_{2n-2},$$

from which we find that

$$y_{2n} = \frac{(-1)^n [(2n-2)c_0 + c_1][(2n-4)c_0 + c_1] \ldots c_1}{(2na_0 + a_1)(2(n-1)a_0 + a_1) \ldots (2a_0 + a_1)}\, y_0.$$

Let us transform the denominator of this fraction. We have

$$(2na_0 + a_1) \ldots (2a_0 + a_1) = 2^n a_0^n \left(n + \frac{a_1}{2a_0}\right) \ldots \left(1 + \frac{a_1}{2a_0}\right)$$
$$= 2^n a_0^n \frac{\Gamma\left(n + 1 + \frac{a_1}{2a_0}\right)}{\Gamma\left(1 + \frac{a_1}{2a_0}\right)}.$$

If we transform the numerator in a similar manner, we obtain

$$y_{2n} = \frac{(-1)^n \Gamma\left(n+\frac{c_1}{2c_0}\right)\Gamma\left(1+\frac{a_1}{2a_0}\right)}{\Gamma\left(n+1+\frac{a_1}{2a_0}\right)\Gamma\left(\frac{c_1}{2c_0}\right)}\left(\frac{c_0}{a_0}\right)^n y_0. \qquad (46.26)$$

Here we assumed that $a_0 \neq 0$ and $c_0 \neq 0$ and that $a_1/2a_0$ and $c_1/2c_0 + 1$ are not negative integers. If we put $y_0 = 0$ and $y_1 \neq 0$, we arrive at a sequence in which the terms with even numbers vanish, which results in a formula similar to (46.26). If $\{y_n^1\}$ is the sequence (46.26) with $y_0 = 1$ and $y_1 = 0$ and $\{y_n^2\}$ the sequence with $y_0 = 0$ and $y_1 = 1$, then every solution of Eq. (46.25) has the form $\{c_1 y_n^1 + c_2 y_n^2\}$, where c_1 and c_2 are arbitrary constants.

Employing formula (46.26), we can find the asymptotic behavior of y_{2n} as $n \to \infty$. We will use Stirling's asymptotic formula for the gamma function (see Sec. 43):

$$\Gamma\left(n+1+\frac{a_1}{2a_0}\right) \sim \sqrt{2\pi n}\, e^{-n-a_1/2a_0}\left(n+\frac{a_1}{2a_0}\right)^{n+a_1/2a_0} \quad (n \to \infty).$$

The last factor has the following asymptotic behavior:

$$n^{n+\frac{a_1}{2a_0}}\left(1+\frac{a_1}{2na_0}\right)^{n+a_1/2a_0} \sim n^{n+a_1/2a_0} e^{a_1/2a_0} \quad (n \to \infty),$$

which yields the final result

$$\Gamma\left(n+1+\frac{a_1}{2a_0}\right) \sim \sqrt{2\pi n}\, e^{-n} n^{n+a_1/2a_0} \quad (n \to \infty).$$

Similarly,

$$\Gamma\left(n+\frac{c_1}{2c_0}\right) \sim \sqrt{2\pi n}\, n^{n+\frac{c_1}{2c_0}-1} e^{-n} \quad (n \to \infty),$$

so that

$$y_{2n} \sim (-1)^n n^{\alpha}\left(\frac{c_0}{a_0}\right)^n \frac{\Gamma\left(1+\frac{a_1}{2a_0}\right)}{\Gamma\left(\frac{c_1}{2c_0}\right)} y_0 \quad (n \to \infty),$$

$$\alpha = \frac{c_1}{2c_0} - \frac{a_1}{2a_0} - 1. \qquad (46.27)$$

This implies that the y_{2n} increase (or decrease) as $n \to \infty$ like exponential functions.

Let us now return to Eq. (46.24). We introduce the *generating function* $w(z)$ of the sequence $\{y_n\}$ in the following manner:

$$w(z) = \sum_{n=0}^{\infty} y_n z^n. \qquad (46.28)$$

If the series in (46.28) is convergent in a circle $|z| < r$, then the y_n can be expressed in terms of $w(z)$ thus:

$$y_n = \frac{1}{2\pi i} \int\limits_{|z|=\rho} z^{-n-1} w(z)\, dz, \tag{46.29}$$

where $0 < \rho < r$.

Let us derive a differential equation for the generating function. We multiply the nth equation in (46.24) by z^n and add all the products. Since

$$\sum_{n=0}^{\infty} (c_0 n + c_1) y_n z^n = c_0 z w'(z) + c_1 w(z),$$

$$\sum_{n=0}^{\infty} [b_0(n+1) + b_1] y_{n+1} z^n = b_0 w'(z) + b_1 z^{-1} (w(z) - y_0),$$

$$\sum_{n=0}^{\infty} [a_0(n+2) + a_1] y_{n+2} z^n$$
$$= a_0 z^{-1} (w'(z) - y_1) + a_1 z^{-2} (w(z) - y_0 - y_1 z),$$

we can find a first order inhomogeneous ordinary differential equation for $w(z)$:

$$z(c_0 z^2 + b_0 z + a_0) w' + (c_1 z^2 + b_1 z + a_1) w$$
$$= f(z) \equiv (a_1 + b_1 z) y_0 + (a_0 + a_1) z y_1. \tag{46.30}$$

Suppose z_1 and z_2 are the roots of the equation

$$c_0 z^2 + b_0 z + a_0 = 0. \tag{46.31}$$

We will assume that these roots are distinct and nonzero. Then

$$\frac{c_1 z^2 + b_1 z + a_1}{z(c_0 z^2 + b_0 z + a_0)} = -\frac{r-1}{z} - \frac{p-1}{z-z_1} - \frac{q-1}{z-z_2}, \quad r = 1 - \frac{a_1}{a_0}. \tag{46.32}$$

The homogeneous equation (46.30) (for $y_0 = 0$ and $y_1 = 0$) has

$$w_0(z) = z^{r-1} \left(1 - \frac{z}{z_1}\right)^{p-1} \left(1 - \frac{z}{z_2}\right)^{q-1} \tag{46.33}$$

as solution, and for this reason every solution of the inhomogeneous equation (46.30) has the form

$$w(z) = c w_0(z) + \int\limits_{z0}^{z} \frac{w_0(z)}{w_0(t)} \frac{f(t)}{t(c_0 t^2 + b_0 t + a_0)}\, dt. \tag{46.34}$$

Here c is a constant that we must select in a way such that the function $w(z)$ be regular at point $z = 0$, in accordance with (46.28). We select the branches of $(1 - z/z_1)^{p-1}$ and $(1 - z/z_2)^{q-1}$ in a way such that for $z = 0$ they be equal to unity.

We will restrict our discussion to the case where a_1/a_0 is a real positive number, i.e.

$$a_1/a_0 > 0, \tag{46.35}$$

so that $r < 1$. In (46.34) we put $z_0 = 0$ and note that

$$\frac{f(t)}{w_0(t)\, t\,(c_0t^2 + b_0t + a_0)} = t^{-r} g(t),$$

where $g(t)$ is a function that is regular at point $t = 0$, $g(0) = a_1a_0^{-1}y_0$. Whence,

$$\int_0^z t^{-r} g(t)\, dt = z^{-r+1} h(z),$$

where $h(z)$ is regular at point $z = 0$, $h(0) = y_0$. This can be verified by expanding $g(t)$ in a Taylor series in powers of t and integrating the series termwise. If in (46.34) we put $c = 0$, we finally arrive at

$$w(z) = \int_0^z \frac{w_0(t)}{w_0(z)} \frac{(a_1 + b_1t)\, y_0 + (a_0 + a_1)\, ty_1}{t\,(c_0t^2 + b_0t + a_0)}\, dt, \tag{46.36}$$

where $w_0(z)$ is given by (46.33). By what we have just proved, $w(z)$ is regular at point $z = 0$.

Thus, the process of finding the general term y_n of the sequence $\{y_n\}$ resulted in two integrals, (46.29) and (46.36). The integral (46.36) can be expressed only in terms of the hypergeometric function. Obviously, if we are looking only for a limited number of the first term in $\{y_n\}$, it is simpler to calculate them from the recurrence relations directly. The formulas (46.29) and (46.36) are of interest primarily because they enable us to establish the asymptotic behavior of y_n as $n \to \infty$.

Only the points z_1 and z_2 can be the finite singular points of $w(z)$, which means that the series (46.28) is convergent in the circle $|z| < R$, where $R = \min(|z_1|, |z_2|)$. Hence

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|y_n|} \leqslant 1/R,$$

in view of the Cauchy-Hadamard test. But if p and q are not integers, then $w(z)$ will have singularities at z_1 and z_2, and in this case

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|y_n|} = 1/R. \tag{46.37}$$

In particular, we always have the estimate

$$|y_n| \leqslant C_\varepsilon \left(\frac{1}{R} - \varepsilon\right)^n, \quad n = 0, 1, 2, \ldots,$$

for any ε such that $0 < \varepsilon < R^{-1}$ and with a constant C_ε, so that the solution of Eq. (46.24) cannot grow faster than the terms of a geometric sequence.

Chapter VIII

Operational Calculus

An important application of the theory of functions of a complex variable is the method of integrating linear differential equations based on Laplace's integral transformation (the operational method). To each function of a real variable the Laplace transformation assigns a function of a complex variable, the result function. The importance of the method lies in the fact that operations on the result functions prove to be much simpler than operations on the initial functions, or object functions. For instance, a linear ordinary differential equation for the object function is replaced by an algebraic equation for the result function. Solving the equation for the result function and restoring the object function from the result function, we arrive at the sought-for solution of the given differential equation.

47 Basic Properties of the Laplace Transformation

47.1 The Laplace transform. The object and result functions Suppose a function $f(t)$ of the real variable t is defined on the semiaxis $t \geqslant 0$. Its Laplace transform is defined as the following function of the complex variable p:

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt. \tag{47.1}$$

We will consider complex valued functions $f(t)$ defined on the entire real axis t and satisfying the following conditions:

(1) In each finite interval on the real t axis the function $f(t)$ is continuous everywhere except, perhaps, at a finite number of jump discontinuities.

(2) $f(t) = 0$ for $t < 0$.

(3) There are constants C and α such that

$$|f(t)| \leqslant Ce^{\alpha t} \tag{47.2}$$

for all nonnegative values of t.

A function $f(t)$ that satisfies conditions (1)-(3) is said to be an *object function*, and its Laplace transform, i.e. the function $F(p)$, a *result function*. We will denote the relationship between an object function and its result function in either of two ways

$$f(t) \risingdotseq F(p) \quad \text{or} \quad F(p) \risingdotseq f(t).$$

Note that functions that describe physical processes usually satisfy conditions (1)-(3).

Example 1. Let us take the Heaviside unit function

$$\theta(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geqslant 0. \end{cases}$$

The function $F(p) = \int\limits_0^\infty e^{-pt} dt$ is defined in the half-plane $\operatorname{Re} p > 0$, and $F(p)) = -e^{-pt}/p\,|_0^\infty = 1/\text{p}$. Hence,

$$\theta(t) \risingdotseq \frac{1}{p}. \ \square \tag{47.3}$$

Note that if we have a function $g(t)$ that satisfies conditions (1) and (3) but does not satisfy condition (2), for

$$f(t) = \theta(t)\,g(t) = \begin{cases} g(t), & t \geqslant 0, \\ 0, & t < 0 \end{cases}$$

condition (2) is satisfied and, hence, $f(t)$, is an object function. For instance, the functions $\theta(t)\,t$, $\theta(t)\,e^t$, and $\theta(t)\cos t$ are object functions.

In what follows we will always drop the factor $\theta(t)$ and simply assume that all such functions are zero for $t < 0$. For instance, instead of writing $\theta(t)$, $\theta(t)\,t^2$, or $\theta(t)\sin t$ we will always write 1, t^2, or $\sin t$. Then formula (47.3) becomes

$$1 \risingdotseq \frac{1}{p}. \tag{47.4}$$

Example 2. Let us find the result function for $e^{\lambda t}$, where λ is a complex valued constant. The integral

$$F(p) = \int\limits_0^\infty e^{-(p-\lambda)t}\,dt$$

has a finite value in the domain $\operatorname{Re} p > \operatorname{Re} \lambda$, and $F(p) = \frac{1}{p-\lambda}$. Hence,

$$e^{\lambda t} \risingdotseq \frac{1}{p-\lambda}. \ \square \tag{47.5}$$

Let us show that a Laplace transform is regular in a certain half-plane. In Sec. 16 we found (see Theorem 3) that if a function $f(t)$

is continuous on the semiaxis $t \geqslant 0$ and satisfies condition (47.2), then the function $F(p)=\int\limits_0^\infty e^{-pt} f(t)\,dt$ is regular in the half-plane $\operatorname{Re} p > \alpha$. This proposition remains true for the case where $f(t)$ has a finite number of jump discontinuities. We call the greatest lower bound of the values of α for which condition (2) for $f(t)$ is still met the *exponent of growth of function* $f(t)$. Then we have the following.

Theorem *For each object function $f(t)$ the result function $F(p)$ is regular in the half-plane* $\operatorname{Re} p > \alpha_0$, *where α_0 is the exponent of growth of $f(t)$.*

Corollary *If $f(t)$ is an object function, then*

$$\lim_{\operatorname{Re} p \to +\infty} F(p)=0. \tag{47.6}$$

Indeed, if α_0 is the exponent of growth of $f(t)$ and $s = \operatorname{Re} p$, then $|f(t)| \leqslant c_1 e^{(\alpha_0+\varepsilon)t}$, where $c_1 > 0$, $\varepsilon > 0$, and

$$|F(p)| \leqslant c_1 \int\limits_0^\infty e^{(\alpha_0+\varepsilon-s)t}\,dt=\frac{c_1}{s-(\alpha_0+\varepsilon)},$$

from which (47.6) follows.

47.2 Properties of the Laplace transformation

(1) *Linearity.* If $f(t) \risingdotseq F(p)$ and $g(t) \risingdotseq G(p)$, then

$$\lambda f(t)+\mu g(t) \risingdotseq \lambda F(p)+\mu G(p)$$

for every complex valued λ and μ. This property follows from the definition of the Laplace transform and the linearity of integrals.

Example 3. Let us find the result functions for the trigonometric and hyperbolic functions $\sin \omega t$, $\cos \omega t$, $\sinh \omega t$, and $\cosh \omega t$. From the fact that $\sin \omega t=\frac{e^{i\omega t}-e^{-i\omega t}}{2i}$ and formula (47.5) we find that $\sin \omega t \risingdotseq \frac{1}{2i}\left(\frac{1}{p-i\omega}-\frac{1}{p+i\omega}\right)=\frac{\omega}{p^2+\omega}$, so that

$$\sin \omega t \risingdotseq \frac{\omega}{p^2+\omega^2}. \tag{47.7}$$

From the fact that $\cos \omega t=\frac{e^{i\omega t}+e^{-i\omega t}}{2i}$ we conclude that

$$\cos \omega t \risingdotseq \frac{p}{p^2+\omega^2} \tag{47.8}$$

Similarly, if we use the formulas $\sinh \omega t = \frac{e^{\omega t} - e^{-\omega t}}{2}$ and $\cosh \omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}$, we find that

$$\sinh \omega t \doteqdot \frac{\omega}{p^2 - \omega^2}, \quad \cosh \omega t \doteqdot \frac{p}{p^2 - \omega^2}. \ \square$$

(2) *The similarity theorem.* If $f(t) \doteqdot F(p)$, then for every positive α we have

$$f(\alpha t) \doteqdot \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right).$$

Indeed, if we put $\alpha t = \tau$, we find that

$$\int_0^\infty f(\alpha t) e^{-pt} \, dt = \frac{1}{\alpha} \int_0^\infty f(\tau) e^{-\frac{p}{\alpha}\tau} \, d\tau = \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right).$$

(3) *Differentiating the object function.* If $f(t), f'(t), \ldots, f^{(n)}(t)$ are object functions and $f(t) \doteqdot F(p)$, then

$$f^{(n)}(t) \doteqdot p^n F(p) - p^{n-1} f(0) - p^{u-2} f'(0) - \ldots - p f^{(n-2)}(0) - f^{(n-1)}(0), \quad (47.9)$$

where $f^{(k)}(0) = \lim\limits_{t \to +0} f^{(k)}(t)$, $k = 0, 1, \ldots, n-1$.

Indeed, if we integrate by parts, we obtain

$$\int_0^\infty f'(t) e^{-pt} \, dt = [f(t) e^{-pt}]_0^\infty + p \int_0^\infty f(t) e^{-pt} \, dt.$$

If $\operatorname{Re} p > \alpha_0$, with α_0 the exponent of growth of $f(t)$, the upper limit in the first term on the right-hand side yields zero and therefore

$$f'(t) \doteqdot pF(p) - f(0). \quad (47.10)$$

The validity of (47.9) for an arbitrary value of n is established by induction.

Formula (47.9) simplifies if $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$. In this case $f^{(n)}(t) \doteqdot p^{(n)} F(p)$ and, in particular, $f'(t) \doteqdot pF(p)$, i.e. differentiation of the object function is equivalent to multiplying $F(p)$ by p. This is one of the more important properties of the Laplace transformation.

(4) *Differentiating the result function.* If $F(p) \doteqdot f(t)$, then

$$F^{(n)}(p) \doteqdot (-t)^n f(t). \quad (47.11)$$

Indeed, since $F(p)$ is regular in the half-plane $\operatorname{Re} p > \alpha_0$, where α_0 is the exponent of growth of $f(t)$, then by Theorem 2 of Sec. 16,

$$F'(p)=\int_0^{\infty}\frac{\partial}{\partial p}\left(e^{-pt}f(t)\right)dt=\int_0^{\infty}e^{-pt}\left(-tf(t)\right)dt.$$

Hence, $F'(p) \risingdotseq (-1)f(t)$. The general formula (47.11) can be proved by induction.

Example 4. Let us find the result functions for t^n, $t^n e^{\lambda t}$, $t^n \sin \omega t$ and $t^n \cos \omega t$.

From (47.4), (47.5), (47.7), and (47.8) it follows

$$t^n \risingdotseq \frac{n!}{p^{n+1}}, \quad t^n e^{\lambda t} \risingdotseq \frac{n!}{(p-\lambda)^{n+1}}, \tag{47.12}$$

$$t\sin\omega t \risingdotseq \frac{2p\omega}{(p^2+\omega^2)^2}, \quad t\cos\omega t \risingdotseq \frac{p^2-\omega^2}{(p^2+\omega^2)^2}.$$

Putting $\lambda = i\omega$ in (47.12), we find that

$$t^n e^{i\omega t} \risingdotseq \frac{n!}{(p-i\omega)^{n+1}}=\frac{n!\,(p+i\omega)^{n+1}}{(p^2+\omega^2)^{n+1}},$$

from which we obtain

$$t^n\cos\omega t \risingdotseq n!\,\frac{\operatorname{Re}(p+i\omega)^{n+1}}{(p^2+\omega^2)^{n+1}},$$

$$t^n\sin\omega t \risingdotseq n!\,\frac{\operatorname{Im}(p+i\omega)^{n+1}}{(p^2+\omega^2)^{n+1}}$$

(we assume p to be real). □

(5) *Integrating the object function.* If $f(t) \risingdotseq F(p)$, then

$$\int_0^t f(\tau)\,d\tau \risingdotseq \frac{F(p)}{p}. \tag{47.13}$$

Indeed, if $f(t)$ is an object function, we can easily verify that $g(t) = \int_0^t f(\tau)\,d\tau$ is an object function, too, with $g'(t) = f(t)$ and $g(0) = 0$. If $g(t) \risingdotseq G(p)$, then from (47.10) we find that

$$f(t) = g'(t) \risingdotseq pG(p), \text{ i.e. } F(p) = pG(p),$$

from which (47.13) follows.

(6) *Integrating the result function.* If $f(t) \risingdotseq F(p)$ and $f(t)/t$ is an object function, then

$$\frac{f(t)}{t} \risingdotseq \int_p^{\infty} F(\zeta)\,d\zeta. \tag{47.14}$$

Indeed, suppose $f(t)/t \risingdotseq \Phi(p)$. Differentiating $\Phi(p)$, which is regular in the half-plane $\operatorname{Re} p > \alpha$ we obtain

$$\Phi'(p) = -\int_0^\infty e^{-pt} f(t)\,dt = -F(p),$$

whence

$$\Phi(p) - \Phi(\infty) = \int_p^\infty F(\zeta)\,d\zeta, \tag{47.15}$$

where the path of integration from p to ∞ lies in the half-plane $\operatorname{Re} p > \alpha$. Since $\Phi(\infty) = 0$ in view of (47.6), from (47.15) follows (47.14).

Example 5. Let us find the result function for the sine-integral function

$$\operatorname{si} t = \int_0^t \frac{\sin\tau}{\tau}\,dt.$$

Using (47.7) and (47.14), we obtain

$$\frac{\sin t}{t} \risingdotseq \int_p^\infty \frac{d\zeta}{1+\zeta^2} = \frac{\pi}{2} - \arctan p = \operatorname{arccot} p,$$

from which, in view of (47.13), we obtain

$$\operatorname{si} t \risingdotseq \frac{\operatorname{arccot} p}{p}. \ \square$$

(7) *The result function of a translation.* If $f(t) \risingdotseq F(p)$ and $f(t) = 0$ for $t < \tau$, with $\tau > 0$, then

$$f(t-\tau) \risingdotseq e^{-p\tau} F(p). \tag{47.16}$$

Indeed, if we put $t - \tau = \xi$, we find that

$$f(t-\tau) \risingdotseq \int_0^\infty f(t-\tau)\,e^{-pt}\,dt = \int_\tau^\infty f(\xi)\,e^{-p(\tau+\xi)}\,d\xi$$

$$= e^{-p\tau}\int_0^\infty f(\xi)\,e^{-p\xi}\,d\xi = e^{-p\tau}F(p),$$

from which formula (47.16) follows.

Example 6. Let us find the result function for the step function

$$f(t)=\begin{cases} 0, & t<0 \\ (n+1)h, & n\tau<t<(n+1)\tau, \quad n=0,\ 1,\ 2,\ \ldots, \end{cases}$$

with $\tau>0$ and $h=\text{const}$.

Note that $f(t)=h[\theta(t)+\theta(t-\tau)+\ldots+\theta(t-k\tau)+\ldots]$, where $\theta(t)$ is the Heaviside unit function, and by (47.16) we have

$$f(t)=h\left[\frac{1}{p}+\frac{1}{p}e^{-p\tau}+\ldots+\frac{1}{p}e^{-kp\tau}+\ldots\right].$$

Let $\operatorname{Re} p$ be positive. Then $|e^{-p\tau}|<1$ and the series $\sum\limits_{k=0}^{\infty} e^{-kp\tau}$ is convergent and its sum is equal to $1/(1-e^{-p\tau})$. Hence, $f(t) \risingdotseq \dfrac{h}{p(1-e^{-p\tau})}$, or

$$f(t) \risingdotseq \frac{h}{2p}\left(1+\coth\frac{p\tau}{2}\right). \ \square$$

Example 7. Let us find the result function for a function $f(t)$ that is periodic for $t>0$ with a period of $T>0$.

Consider the function

$$\varphi(t)=\begin{cases} f(t), & 0\leqslant t\leqslant T, \\ 0, & t<0,\ t>T. \end{cases}$$

Then

$$f(t)=\varphi(t)+f(t-T). \tag{47.17}$$

If $f(t) \risingdotseq F(p)$ and $\varphi(t) \risingdotseq \Phi(p)$, then from (47.17) and (47.16) we obtain

$$F(p)=\Phi(p)+e^{-Tp}F(p),$$

whence

$$F(p)=\frac{\int\limits_0^T e^{-pt}f(t)\,dt}{1-e^{-Tp}}. \tag{47.18}$$

We use formula (47.18) to find the result function for the periodic function $f(t)=|\sin t|$ with a period $T=\pi$. We have

$$\int\limits_0^{\pi} e^{-pt}\sin t\,dt=\frac{e^{-pt}}{p^2+1}(-p\sin t-\cos t)\Big|_0^{\pi}=\frac{1+e^{-\pi p}}{1+p^2}.$$

Hence,

$$|\sin t| \doteqdot \frac{1+e^{-\pi p}}{(1-e^{-\pi p})(1+p^2)} = \frac{\coth(\pi p/2)}{1+p^2}. \quad \square$$

(8) *Translation of the result function.* If $f(t) \doteqdot F(p)$, then

$$e^{\lambda t} f(t) \doteqdot F(p-\lambda)$$

for any complex valued λ.

Indeed,

$$e^{\lambda t} f(t) \doteqdot \int_0^\infty f(t) e^{-(p-\lambda)t}\, dt = F(p-\lambda).$$

Example 8. Let us find the result functions for $e^{\lambda t}\cos\omega t$ and $e^{\lambda t}\sin\omega t$.

Since $\cos\omega t \doteqdot \frac{p}{p^2+\omega^2}$ and $\sin\omega t \doteqdot \frac{\omega}{p^2+\omega^2}$, the property of translation of the result function yields

$$e^{\lambda t}\cos\omega t \doteqdot \frac{p-\lambda}{(p-\lambda)^2+\omega^2}, \quad e^{\lambda t}\sin\omega t \doteqdot \frac{\omega}{(p-\lambda)^2+\omega^2}. \quad \square$$

(9) *The result function of a convolution.* The *convolution* $f * g$ of two functions f and g is the function

$$(f*g)(t) = \int_0^t f(\xi) g(t-\xi)\, d\xi.$$

Let us prove that the convolution of two object functions is equivalent to multiplying the respective result functions, i.e. if $f(t) \doteqdot F(p)$ and $g(t) \doteqdot G(p)$, then

$$(f*g)(t) \doteqdot F(p)G(p). \tag{47.19}$$

Let us first demonstrate that $\varphi(t) = (f*g)(t)$ is an object function. Indeed, the function $\varphi(t)$ satisfies conditions (1) and (2), since both $f(t)$ and $g(t)$ are object functions. We introduce the notation $\gamma = \max(\alpha, \beta)$, where α and β are the exponents of growth of functions $f(t)$ and $g(t)$. Then

$$|f(t)| \leqslant Ce^{(\gamma+\varepsilon)t}, \quad |g(t)| \leqslant Ce^{(\gamma+\varepsilon)t},$$

where $C > 0$ and $\varepsilon > 0$, whence

$$|\varphi(t)| \leqslant C^2 \int_0^t e^{\gamma(\xi+\varepsilon)+\gamma(t+\varepsilon-\xi)}\, d\xi = C^2 t e^{\gamma(t+2\varepsilon)}.$$

By fixing a positive ρ, we find a positive C_1 such that $C^2 t \leqslant C_1 e^{\delta t}$ for $t \geqslant 0$. Hence, $|\varphi(t)| \leqslant C_1 e^{(\gamma+2\varepsilon+\delta)t}$, i.e. $\varphi(t)$ is an object function.

Note that the exponent of growth of $\varphi(t)$ cannot exceed the greatest of the two exponents of growth of $f(t)$ and $g(t)$, since ε and δ can be taken as small as desired.

Let us find the result function $\Phi(p)$ for $\varphi(t)$. By definition of a result function,

$$\Phi(p)=\int_0^\infty e^{-pt}\left(\int_0^t f(\xi)\,g(t-\xi)\,d\xi\right)dt.$$

Since the double integral is absolutely convergent for $\operatorname{Re} p>\gamma$, we can change the order of integration and obtain

$$\Phi(p)=\int_0^\infty f(\xi)\,d\xi\int_\xi^\infty e^{-pt}g(t-\xi)\,dt.$$

Putting $t-\xi=\tau$ in the inner integral, we obtain

$$\Phi(p)=\int_0^\infty e^{-p\xi}f(\xi)\,d\xi\int_0^\infty e^{-p\tau}g(\tau)\,d\tau=F(p)\,G(p).$$

We have thus proved the validity of (47.19). Below we give a table of object and result functions that are most commonly used in practical calculations.

Object Function	Result Function	Object Function	Result Function
1	$\frac{1}{p}$	$e^{\lambda t}\cos\omega t$	$\frac{p-\lambda}{(p-\lambda)^2+\omega^2}$
t^n	$\frac{n!}{p^{n+1}}$	$e^{\lambda t}\sin\omega t$	$\frac{\omega}{(p-\lambda)^2+\omega^2}$
$e^{\lambda t}$	$\frac{1}{p-\lambda}$	$t\sin\omega t$	$\frac{2\omega p}{(p^2+\omega^2)^2}$
$t^n e^{\lambda t}$	$\frac{n!}{(p-\lambda)^{n+1}}$	$t\cos\omega t$	$\frac{p^2-\omega^2}{(p^2+\omega^2)^2}$
$\cos\omega t$	$\frac{p}{p^2+\omega^2}$	$\cosh\omega t$	$\frac{p}{p^2-\omega^2}$
$\sin\omega t$	$\frac{\omega}{p^2+\omega^2}$	$\sinh\omega t$	$\frac{\omega}{p^2-\omega^2}$

48 Reconstructing Object Function from Result Function

48.1 The inverse Laplace transform

Theorem 1 *Let $f(t)$ be an object function and $F(p)$ its result function. If $f(t)$ is continuous at point t and has finite one-sided derivatives* *at this point, then*

$$f(t)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{pt}F(p)\,dp. \qquad (48.1)$$

The integral in (48.1) is taken along any straight line $\operatorname{Re} p = b > \alpha_0$, with α_0 the exponent of growth of $f(t)$, and is understood in the principal-value sense.

Proof. Consider the function $g(t) = e^{-bt} f(t)$, with $b > \alpha_0$. The function $F(b+iu)$ is the Fourier transform of $g(t)$, since

$$F(b+iu)=\int_{0}^{\infty} f(t)\,e^{-(b+iu)t}\,dt=\int_{-\infty}^{\infty} g(t)\,e^{-iut}\,dt.$$

By the hypothesis of Theorem 1, $g(t)$ is absolutely integrable over the entire length of the straight line $\operatorname{Re} p = b$, is continuous at point t, and has finite one-sided derivatives at this point. By Fourier's inversion theorem (see Kudryavtsev [1]),

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(b+iu)\,e^{iut}\,du,$$

where the integral is understood in the principal-value sense. Hence,

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(b+iu)\,e^{(b+iu)t}\,du=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} F(p)\,e^{pt}\,dp.$$

We have thus proved the validity of (48.1). It is known as the *inverse Laplace transform*, or *Mellin's formula.*

Corollary *The object function $f(t)$ is determined uniquely by its result function $F(p)$ at all points where $f(t)$ is differentiable.*

48.2 The existence conditions for an object function

Theorem 2 *Suppose $F(p)$ is regular in the half-plane* $\operatorname{Re} p > \alpha$ *and satisfies the conditions*

(1) *The integral* $\int_{-\infty}^{\infty} |F(a+i\sigma)|\,d\sigma$ *has a finite value for every* $a > \alpha$.

(2) $M(R) = \max_{p\in\Gamma_R} |F(p)| \to 0$ *as* $R \to \infty$, *where* Γ_R *is the arc* $|p| = R$, $\operatorname{Re} p \geqslant a > \alpha$.

Then $F(p)$ is the result function corresponding to

$$f(t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{pt}F(p)\,dp, \qquad (48.2)$$

where $a > \alpha$, and the integral is understood in the principal-value sense.

Proof. Let us first demonstrate that the value of (48.2) is independent of the choice of a ($a > \alpha$). Indeed, the integral of $e^{pt}F(p)$ taken along the boundary of the rectangle whose vertices are at points $a \pm ib$ and $a_1 \pm ib$ ($a > \alpha$, $a_1 > \alpha$, $b > 0$) is zero, according to Cauchy's integral theorem. The integrals taken along the horizontal sides of the rectangle tend to zero as $b \to \infty$, by condition (2). Hence,

$$\lim_{b\to\infty} \int_{a-ib}^{a+ib} e^{pt}F(p)\,dp = \lim_{b\to\infty} \int_{a_1-ib}^{a_1+ib} e^{pt}F(p)\,dp,$$

i.e. (48.2) does not depend on a and is a function of only one variable, t.

Let us prove that $f(t)$ given by (48.2) is the object function corresponding to $F(p)$, i.e. satisfies conditions (1)-(3) in Sec. 47.

By condition (1) this integral has a finite value and

$$|f(t)| \leqslant \frac{e^{at}}{2\pi} \int_{-\infty}^{\infty} |F(a+i\sigma)|\,d\sigma = Ce^{at}. \tag{48.3}$$

From (48.3) follows the uniform convergence of (48.2) in t on any finite segment $[0, T]$ and the continuity of $f(t)$ for $t \geqslant 0$.

Let us show that $f(t) = 0$ for $t < 0$. Consider the closed contour γ_R consisting of the segment $[a - iR, a + iR]$ and the arc Γ_R: $|p| = R$, $\operatorname{Re} p \geqslant a$. By Cauchy's integral theorem, the integral of $e^{pt}F(p)$ taken along γ_R is zero, while the same integral but taken along Γ_R tends to zero as $R \to \infty$ ($t < 0$), by Jordan's lemma (Sec. 29). Whence,

$$f(t) = \lim_{R\to\infty} \int_{a-iR}^{a+iR} e^{pt}F(p)\,dp = \int_{a-i\infty}^{a+i\infty} e^{pt}F(p)\,dp = 0, \quad t < 0.$$

Let us show that every p_0 ($\operatorname{Re} p_0 > a$) the result function for $f(t)$ is $F(p_0)$. We have

$$\int_0^{\infty} e^{-p_0 t} f(t)\,dt = \frac{1}{2\pi i} \int_0^{\infty} e^{-p_0 t} \int_{a-i\infty}^{a+i\infty} e^{pt}F(p)\,dp\,dt. \tag{48.4}$$

Since

$$|F(p)\,e^{(p-p_0)t}| = |F(a+i\sigma)|\,e^{-(\operatorname{Re} p_0 - a)t},$$

with $\int_{-\infty}^{\infty} |F(+ai\sigma)|\, d\sigma$ and $\int_{0}^{\infty} e^{-(\mathrm{Re}\, p_0 - a)t}\, dt$ being convergent and the inner integral in (48.4) being uniformly convergent, we can change the order of integration, i.e.

$$\int e^{-p_0 t} f(t)\, dt = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(p)\, dp \int_{0}^{\infty} e^{-(p-p_0)t}\, dt$$

$$= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{F(p)}{p_0 - p}\, dp. \qquad (48.5)$$

Let us select $R > 0$ in a way such that point $p = p_0$ is inside contour γ_R. By the residue theorem,

$$\frac{1}{2\pi i} \int_{\gamma_R} \frac{F(p)}{p - p_0}\, dp = F(p_0).$$

Note that

$$\left| \frac{1}{2\pi i} \int_{\Gamma_R} \frac{F(p)}{p - p_0}\, dp \right| \leqslant \frac{1}{2\pi} \frac{M(R)\, 2\pi R}{R - |p_0|} \to 0 \quad (R \to \infty).$$

Whence, if we go over to the limit as $R \to \infty$ in

$$F(p_0) = \frac{1}{2\pi i} \int_{a-iR}^{a+iR} \frac{F(p)}{p_0 - p}\, dp + \frac{1}{2\pi i} \int_{\Gamma_R} \frac{F(p)}{p - p_0}\, dp$$

and use (48.5), we find that

$$\int_{0}^{\infty} e^{-p_0 t} f(t)\, dt = F(p_0).$$

Since p_0 is an arbitrary point in the half-plane $\mathrm{Re}\, p > a$, we conclude that $f(t) \risingdotseq F(p)$. Note that (48.2) coincides with the inversion formula (48.1).

Example 1. Let us find, via the inversion formula, the object function corresponding to

$$F(p) = \frac{1}{p} e^{-\alpha \sqrt{p}}, \quad \alpha > 0.$$

Suppose D is the complex p plane with a cut along the negative part of the real axis. The function $F(p)$, where $\sqrt{p}$ is the branch of the root that assumes positive values for $\operatorname{Im} p = 0$, $\operatorname{Re} p > 0$, satisfies for $\operatorname{Re} p > 0$ the hypothesis of Theorem 2.

Let us consider the contour Γ_R consisting of the arc C_R: $|p| = R$, $\operatorname{Re} p \leqslant a$ $(a > 0)$, the chord l_R: $\operatorname{Re} p = a$, $-\sqrt{R^2 - a^2} \leqslant \operatorname{Im} p < \sqrt{R^2 - a^2}$, the circle C_ρ: $|p| = \rho < R$, and the segments lying on the banks of the cut γ: $\operatorname{Im} p = 0$, $-R \leqslant \operatorname{Re} p \leqslant -\rho$. By Cauchy's integral theorem,

$$\int_{\Gamma_R} F(p)\, e^{pt}\, dp = 0. \tag{48.6}$$

Let $p = re^{i\varphi}$. Then on the upper bank of the cut $\varphi = \pi$, $p = -r$, and $\sqrt{p} = i\sqrt{r}$, while on the lower bank $\varphi = -\pi$, $p = -r$, and $\sqrt{p} = -i\sqrt{r}$. Then (48.6) yields

$$0 = \frac{1}{2\pi i}\int_{C_R} e^{pt} F(p)\, dp + \frac{1}{2\pi i}\int_{a-i\sqrt{R^2-a^2}}^{a+i\sqrt{R^2-a^2}} e^{pt} F(p)\, dp$$
$$+ \frac{1}{2\pi i}\int_{\rho}^{R} e^{-rt}\, \frac{e^{ia\sqrt{r}} - e^{-ia\sqrt{r}}}{r}\, dr - \frac{1}{2\pi i}\int_{C_\rho} e^{pt} F(p)\, dp. \tag{48.7}$$

Since $\int_{C_R} e^{pt} F(p)\, dp \to 0$ as $R \to \infty$, $t > 0$ (Jordan's lemma), $\frac{1}{2\pi i}\int_{a-i\sqrt{R^2-a^2}}^{a+i\sqrt{R^2-a^2}} e^{pt} F(p)\, dp \to f(t)$ as $R \to \infty$ (the inversion formula), and $\frac{1}{2\pi i}\int_{C_\rho} e^{pt} F(p)\, dp = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\alpha\sqrt{\rho}\, e^{i\frac{\varphi}{2}}}\, d\varphi \to 1$ as $\rho \to 0$, we can go over to the limit in (48.7) as $\rho \to 0$ and $R \to \infty$ and obtain

$$f(t) = -\frac{1}{\pi}\int_0^\infty e^{-rt}\, \frac{\sin \alpha\sqrt{r}}{r}\, dr + 1. \tag{48.8}$$

Putting $\sqrt{r} = x$ in (48.8), we find that

$$f(t) = -\frac{2}{\pi}\int_0^\infty e^{-tx^2}\, \frac{\sin \alpha x}{x}\, dx + 1. \tag{48.9}$$

To evaluate the integral in (48.9), we employ the well-known result (see Example 17 in Sec. 29)

$$\int_0^\infty e^{-tx^2}\cos\alpha x\,dx=\frac{1}{2}\sqrt{\frac{\pi}{t}}\,e^{-\frac{\alpha^2}{4t}}. \tag{48.10}$$

We introduce the notation

$$I(\alpha)=\int_0^\infty e^{-tx^2}\frac{\sin\alpha x}{x}\,dx.$$

Then (48.10) yields

$$I'(\alpha)=\frac{1}{2}\sqrt{\frac{\pi}{t}}\,e^{-\alpha^2/4t},$$

whence

$$I(\alpha)=\sqrt{\pi}\int_0^\alpha \frac{1}{2\sqrt{t}}\,e^{-\left(\frac{\xi}{2\sqrt{t}}\right)^2}d\xi=\sqrt{\pi}\int_0^{\alpha/2\sqrt{t}} e^{-\tau^2}\,d\tau,$$

so that $I(0)=0$. This enables us to write (48.9) in the form

$$f(t)=1-\frac{2}{\sqrt{\pi}}\int_0^{\alpha/2\sqrt{t}} e^{-\tau^2}\,dt=1-\operatorname{erf}\left(\frac{\alpha}{2\sqrt{t}}\right),$$

with

$$\operatorname{erf}(x)=\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\tau^2}\,d\tau$$

the error integral. Putting $1-\operatorname{erf}(x)=\operatorname{Erf}(x)$, we obtain the final result:

$$\frac{1}{p}\,e^{-\alpha\sqrt{p}}\doteq\operatorname{Erf}\left(\frac{\alpha}{2\sqrt{t}}\right). \ \square \tag{48.11}$$

48.3 Expansion theorems Knowing the result function $F(p)$, it is easy to find the object function $f(t)$ if $F(p)$ is regular at the point at infinity. In this case we can expand $F(p)$ in a Laurent series about point $p=\infty$:

$$F(p)=\sum_{n=0}^\infty \frac{C_n}{p^n}.$$

Note that $C_0=0$ because $F(p)\to 0$ as $\operatorname{Re} p\to\infty$ (see formula (47.6)).

Theorem 3 *Suppose $F(p)$ is regular at point $p = \infty$, $F(\infty) = 0$, and let its Laurent expansion about point $p = \infty$ be*

$$F(p) = \sum_{n=1}^{\infty} \frac{C_n}{p^n}. \tag{48.12}$$

Then the object function $f(t)$ corresponding to $F(p)$ is given by the formula

$$f(t) = \sum_{n=0}^{\infty} \frac{C_{n+1}}{n!} t^n. \tag{48.13}$$

Proof. Let us select R_0 so large that in the set $|p| \geqslant R_0$ there are no singular points of $F(p)$. Since $p = \infty$ is a zero of $F(p)$, there exists a positive M and an $R_1 \geqslant R_0$ such that

$$|F(p)| \leqslant M/R \quad \text{for } |p| = R \geqslant R_1.$$

Cauchy's inequalities (Sec. 17) imply that

$$|C_n| \leqslant MR^{n-1}. \tag{48.14}$$

From (48.13) and (48.14) we obtain

$$\sum_{n=0} \left| \frac{C_{n+1}}{n!} t^n \right| \leqslant M \sum_{n=0}^{\infty} \frac{R^n |t|^n}{n!} = Me^{R|t|}. \tag{48.15}$$

In view of (48.15), the series (48.13) converges in the entire plane, and the sum of this series, $f(t)$, is an entire function. It is called an *entire function of the exponential type.*

Let us multiply the series (48.13) by e^{-pt} and integrate the product term-by-term with respect to t form 0 to ∞. Using the fact that $t^n \risingdotseq n!/p^{n+1}$ and the linearity of the Laplace transformation, we obtain

$$\sum_{n=0}^{\infty} \frac{C_{n+1}}{n!} t^n \risingdotseq \sum_{n=0}^{\infty} \frac{C_{n+1}}{p^{n+1}} = F(p).$$

Theorem 3 is known as the *first expansion theorem.*

Remark. The theorem that is the converse of Theorem 3 is also valid, namely, if $f(t) = \theta(t) g(t)$ is an object function, with $g(t)$ an entire function of the exponential type, then its result function $F(p)$ is regular at the point at infinity.

Example 2. Let us find the object function $f(t)$ for

$$F(p) = \frac{1}{p^{n+1}} e^{-1/p} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, p^{n+k+1}},$$

where n is a positive integer.

By Theorem 3, $f(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{n+k}}{k!\,(n+k)!}$. Using the following formula for the Bessel function:

$$t^{n/2} J_n(2\sqrt{t}) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{n+k}}{k!\,(n+k)!},$$

we find that

$$t^{n/2} J_n(2\sqrt{t}) \risingdotseq \frac{1}{p^{n+1}} e^{-1/p}. \tag{48.16}$$

In particular

$$J_0(2\sqrt{t}) \risingdotseq \frac{1}{p} e^{-1/p}. \square \tag{48.17}$$

Theorem 4 *Let a meromorphic function $F(p)$ be regular in the half-plane* Re $p > \alpha$ *and satisfy the following conditions:*

(1) *There is a system of circles*

$$C_n\colon\ |p| = R_n,\quad R_1 < R_2 < \ldots,\ R_n \to \infty\ (n \to \infty)$$

such that $\max_{p \in C_n} |F(p)| \to 0$ as $n \to \infty$.

(2) *The integral* $\int_{-\infty}^{\infty} |F(a+\sigma)|\,d\sigma$ *has a finite value for every* $a > \alpha$.

Then $F(p)$ is a result function whose object function is given by the formula

$$f(t) = \sum_{(p_k)} \operatorname*{Res}_{p=p_k} [F(p)\,e^{pt}], \tag{48.18}$$

where the sum is taken over all the poles of $F(p)$.

Proof. The function $F(p)$ satisfies the hypothesis of Theorem 2 and, in view of this theorem, $F(p)$ is the result function of

$$f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{pt} F(p)\,dp. \tag{48.19}$$

Suppose Γ_n is the arc of C_n that lies to the left of the straight line Re $p = a$, the points $a \pm ib_n$ are the points at which C_n intersects this straight line, and γ_n is the closed contour consisting of the segment $[a - ib_n,\ a + ib_n]$ and the arc Γ_n.

Since the integral in (48.19) is taken in the principal-value sense and $b_n \to \infty$ as $n \to 0$, we conclude that

$$f(t) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{a-ib_n}^{a+ib_n} e^{pt} F(p)\,dp. \tag{48.20}$$

For $t > 0$, by Jordan's lemma,

$$\lim_{n\to\infty} \int_{\Gamma_n} e^{pt} F(p)\, dp = 0.$$

For this reason we can write formula (48.20) thus:

$$f(t) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\gamma_n} e^{pt} F(p)\, dp. \tag{48.21}$$

Applying the residue theorem to the integral in (48.21), we arrive at (48.18). This theorem is known as the *second expansion theorem.*

Corollary *If $F(p) = A_n(p)/B_m(p)$, where A_n and B_m are polynomials of degrees n and m, respectively, without common zeros, and if $n < m$, then*

$$f(t) = \sum_{k=1}^{l} \frac{1}{(m_k-1)!} \frac{d^{m_k-1}}{dp^{m_k-1}} \{F(p)\, e^{pt} (p-p_k)^{m_k}\}\big|_{p=p_k}, \tag{48.22}$$

where $p_1, p_2, \ldots, p_l$ are the distinct zeros of $B_m(p)$, and m_k is the order of the zero p_k.

We can derive (48.22) from (48.18) if we use the rule by which the residue at a pole of order m_k can be calculated (see Sec. 28). In particular, if all the poles of $F(p)$ are simple, formula (48.22) takes the form

$$\frac{A_n(p)}{B_m(p)} \risingdotseq \sum_{k=1}^{m} \frac{A_n(p_k)}{B'_m(p_k)} e^{p_k t}. \tag{48.22'}$$

Example 3. Let us find the object function $f(t)$ corresponding to the result function

$$F(p) = \frac{p+8}{p^2+p-2}.$$

Since the function $F(p)$ has simple poles at $p_1 = 1$ and $p_2 = -2$, formula (48.22′) yields

$$f(t) = \left(\frac{p+8}{2p+1} e^{pt}\right)_{p=1} + \left(\frac{p+8}{2p+1} e^{pt}\right)_{p=-2}.$$

Hence, $f(t) = 3e^t - 2e^{-2t}$. We arrive at the same result if we write the partial-fraction expansion for $F(p)$, or $F(p) = \frac{3}{p-1} - \frac{2}{p+2}$, and use the formula $\frac{1}{p-\lambda} \risingdotseq e^{\lambda t}$. □

Example 4. Let us find the object $f(t)$ function corresponding to the result function

$$F(p) = \frac{1+2p^2}{(1+p^2)^2}.$$

The function $F(p)$ has second order poles at $p_1 = i$ and $p_2 = -i$. If we apply (48.18) and the rule for calculating the residue at a second order pole (see Sec. 28), we find that

$$f(t)=\left[\frac{1+2p^2}{(p+i)^2}e^{pt}\right]'_{p=i}+\left[\frac{1+2p^2}{(p-i)^2}e^{pt}\right]'_{p=-i}.$$

This yields $f(t) = \frac{1}{2}t\cos t + \frac{3}{2}\sin t$. The same result follows from the fact that

$$\frac{1+2p^2}{(1+p^2)^2}=\frac{1}{2}\,\frac{p^2-1}{(p^2+1)^2}+\frac{3}{2}\,\frac{1}{p^2+1}$$

and the formulas $\frac{p^2-1}{(1+p^2)^2} \risingdotseq t\cos t$ and $\frac{1}{1+p^2} \risingdotseq \sin t$. □

Example 5. Let us find the object function corresponding to the result function

$$F(p)=\frac{1}{(p^2+1)^3}.$$

The result function $F(p)$ has third order poles at $p_1 = i$ and $p_2 = -i$. If we apply (48.18) and the rule for calculating the residue at a third order pole, we obtain

$$f(t)=\frac{1}{2}\left[\frac{e^{pt}}{(p+i)^3}\right]''_{p=i}+\frac{1}{2}\left[\frac{e^{pt}}{(p-i)^3}\right]''_{p=-i}.$$

from which we find that

$$f(t)=\frac{3}{8}\sin t-\frac{3}{8}t\cos t-\frac{1}{8}t^2\sin t. \quad \square$$

48.4 The result functions for some elementary and special functions

(a) *The power function.* Let us take the function $f(t) = t^\beta$. If $-1 < \beta < 0$, then $f(0) = \infty$, and hence $f(t)$ does not satisfy the conditions (1)-(3) that an object function must satisfy (Sec. 47). But for $-1 < \beta < 0$ and $\operatorname{Re} p > 0$ the integral

$$F(p)=\int_0^\infty t^\beta e^{-pt}\,dt \tag{48.23}$$

has a finite value and represents a function that is regular in the domain $\operatorname{Re} p > 0$. We wish to evaluate this integral.

Let p be real and positive. Putting $pt = \tau$ in (48.23), we obtain

$$F(p)=\int_0^\infty t^\beta e^{-pt}\,dt=\frac{1}{p^{\beta+1}}\int_0^\infty e^{-\tau}\tau^\beta\,d\tau$$

Hence,

$$F(p)=\frac{\Gamma(\beta+1)}{p^{\beta+1}}, \quad p>0 \tag{48.24}$$

If we continue the analytic function $F(p)$ from the semiaxis $(0, +\infty)$ into the half-plane $\operatorname{Re} p > 0$, we find that (48.24) is valid for $\operatorname{Re} p > 0$, so that

$$t^{\beta} \doteqdot \frac{\Gamma(\beta+1)}{p^{\beta+1}} \quad (\beta > -1, \ \operatorname{Re} p > 0). \tag{48.25}$$

Note the important case of $\beta = -1/2$ in formula (48.25). Since $\Gamma(1/2) = \sqrt{\pi}$, we have

$$\frac{1}{\sqrt{\pi t}} \doteqdot \frac{1}{\sqrt{p}}. \tag{48.26}$$

(b) *Impulse functions.* Consider the function

$$\delta_n(t)=\begin{cases} \frac{1}{h}, & 0<t<h, \\ 0, & t<0, \ t>h. \end{cases} \tag{48.27}$$

For small h's this function can be interpreted as a force of constant magnitude $1/h$ acting within a small time interval $0 < t < h$ and having an impulse equal to unity:

$$\int_{-\infty}^{\infty} \delta_h(t)\,dt = \int_0^h \frac{dt}{h} = 1. \tag{48.28}$$

Let us introduce an idealized function that will be the limit for the family of functions $\delta_h(t)$ as $h \to 0$. We denote this function by $\delta(t)$ and call it the *unit impulse function of order one* (it is more commonly known as the *delta function of Dirac*).

In view of (48.27) and (48.28), this function must satisfy the following conditions:

$$\delta(t)=\begin{cases} \infty, & t=0, \\ 0, & t\neq 0, \end{cases} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1,$$

which no ordinary function can satisfy simultaneously. Nevertheless, the delta function is used as a criptic notation for a limiting physical process in which an infinitely large quantity (e.g. force) acts over an infinitely small time interval with a total effect (the impulse) equal to unity.

Note that

$$\delta_h(t) = \frac{1}{h}[\theta(t) - \theta(t-h)], \tag{48.29}$$

where $\theta(t)$ is the Heaviside unit function. We will assume that the result function for the delta function is the limit of the result function for $\delta_h(t)$ as $h \to 0$. Since $\delta_h(t) \doteqdot (1 - e^{-ph})/ph$, we can write

$$\delta(t) \doteqdot \lim_{h \to 0} \frac{1 - e^{-ph}}{ph},$$

or

$$\delta(t) \doteqdot 1. \tag{48.30}$$

We also note that in view of (48.29) the delta function can formally be considered as the derivative of $\theta(t)$, i.e.

$$\delta(t) = \theta'(t). \tag{48.31}$$

For this reason we can arrive at (48.30) by using the formula $\theta(t) \doteqdot 1/p$ and the rule of differentiating object functions.

Using (48.30) and the rule for finding the result function of a translation, we obtain

$$\delta(1 - \tau) \doteqdot e^{-p\tau}, \quad \tau > 0.$$

Reasoning along similar lines, we can introduce the impulse functions $\delta^{(n)}(t)$ for $n \geqslant 2$ and arrive at

$$\delta^{(n)}(t) \doteqdot p^n. \tag{48.32}$$

The substantiation of (48.32) can be achieved on the basis of the theory of so-called generalized functions (e.g. see Vladimirov [1]).

(c) *Bessel functions.* The function

$$F(p) = \frac{1}{\sqrt{p^2+1}} = \frac{1}{p}\left(1 + \frac{1}{p^2}\right)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{2^{2k}(k!)^2 p^{2k+1}}$$

is regular at infinity and $F(\infty) = 0$. By the first expansion theorem,

$$F(p) \doteqdot \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{2^{2k}(k!)^2} = J_0(t).$$

We have therefore found that

$$J_0(t) \doteqdot \frac{1}{\sqrt{p^2+1}}. \tag{48.33}$$

Substituting $i\beta t$ for t in (48.33) and employing the similarity theorem, we obtain $J_0(i\beta t) \risingdotseq \frac{1}{\sqrt{p^2-\beta^2}}$ from which by the rule for finding the translation of a result function we have

$$e^{-\alpha t} J_0(i\beta t) \risingdotseq \frac{1}{\sqrt{(p+\alpha)^2-\beta^2}}. \tag{48.34}$$

We can also write this formula in the following form:

$$e^{-\alpha t} I_0(\beta t) \risingdotseq \frac{1}{\sqrt{(p+\alpha)^2-\beta^2}},$$

where $I_0(t)$ is the modified Bessel function.

Using formula (48.33) and the induction method, we can find that

$$J_n(t) \risingdotseq \frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}. \tag{48.35}$$

For $n = 0$ this formula coincides with (48.33). Since

$$J_1(t) = -J_0'(t), \quad J_0(0) = 1, \tag{48.36}$$

we can find, using the rule for differentiating object functions and (48.36) and (48.33), that

$$J_1(t) \risingdotseq (\sqrt{p^2+1} - p)/\sqrt{p^2+1},$$

i.e. (48.35) is valid for $n = 1$. Let it be valid for all positive integers less than n ($n \geqslant 2$). Using the formula $J_n(t) = J_{n-2}(t) - 2J'_{n-1}(t)$ and the fact that $J_{n-1}(0) = 0$, we find that

$$J_n(t) \risingdotseq \frac{(\sqrt{p^2+1}-p)^{n-2}}{\sqrt{p^2+1}} - 2p\,\frac{(\sqrt{p^2+1}-p)^{n-1}}{\sqrt{p^2+1}}.$$

The final result is

$$J_n(t) \risingdotseq \frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}},$$

(d) *Functions related to the error integral.* Let us consider the functions

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-\tau^2}\, d\tau,$$

$$\operatorname{Erf}(t) = 1 - \operatorname{erf}(t), \quad f(t) = e^t \operatorname{erf}(\sqrt{t}).$$

There are object functions, with

$$f'(t) = f(t) + \frac{1}{\sqrt{\pi t}}. \tag{48.37}$$

Suppose $f(t) \risingdotseq F(p)$. Then, combining (48.37), the formula $1/\sqrt{\pi t} \risingdotseq 1/\sqrt{p}$, and the rule for differentiating object functions, we find that $pF(p) = F(p) + 1/\sqrt{p}$, whence $F(p) = \frac{1}{(p-1)\,p^{1/2}}$, i.e.

$$e^t \operatorname{erf}(\sqrt{t}) \risingdotseq \frac{1}{(p-1)\sqrt{p}}. \tag{48.38}$$

From (48.38) and the rule for calculating the translation of result functions we find that

$$\operatorname{erf}(\sqrt{t}) \risingdotseq \frac{1}{p\sqrt{p+1}}. \tag{48.39}$$

Consider the function

$$g(t) = e^t \operatorname{Erf}(\sqrt{t}) = e^t - e^t \operatorname{erf}(\sqrt{t}). \tag{48.40}$$

Combining (48.40) with (48.39), we obtain

$$g(t) \risingdotseq \frac{1}{p-1} - \frac{1}{(p-1)\sqrt{p}} = \frac{1}{p+\sqrt{p}}.$$

Hence, $e^t \operatorname{Erf}(\sqrt{t}) \risingdotseq \frac{1}{p+\sqrt{p}}$, which yields

$$\operatorname{Erf}(\sqrt{t}) \risingdotseq \frac{1}{p+1+\sqrt{p+1}}. \tag{48.41}$$

Further, from

$$\frac{\sqrt{p+\alpha}}{p} = \frac{1}{\sqrt{p+\alpha}} + \frac{\alpha}{p\sqrt{p+\alpha}}$$

and the fact that $\frac{1}{\sqrt{p}} \risingdotseq \frac{1}{\sqrt{\pi t}}$ it follows that

$$\frac{\sqrt{p+\alpha}}{p} \risingdotseq e^{-\alpha t}\frac{1}{\sqrt{\pi t}} + \alpha\int_0^t e^{-\alpha\tau}\frac{d\tau}{\sqrt{\pi\tau}},$$

or

$$\frac{\sqrt{p+\alpha}}{p} \risingdotseq \frac{1}{\sqrt{\pi t}}\, e^{-\alpha t} + \sqrt{\alpha}\operatorname{erf}(\sqrt{\alpha t}). \quad \alpha > 0. \tag{48.42}$$

49 Solving Linear Differential Equations via the Laplace Transformation

49.1 Ordinary differential equations Let us take an nth order linear differential equation with constant coefficients

$$Lx = x^{(n)}(t) + a_1 x^{(n-1)}(t) + \dots + a_{n-1}x'(t) + a_n x(t) = f(t). \quad (49.1)$$

The Cauchy problem is formulated as follows: to find the solution of Eq. (49.1) that obeys the condition

$$x(0) = x_0,\ x'(0) = x_1,\ \dots,\ x^{(n-1)}(0) = x_{n-1}, \quad (49.2)$$

where $x_0, x_1, \dots, x_{n-1}$ are given constants. Assuming that $f(t)$ is an object function, we seek the solution $x(t)$ of the Cauchy problem (49.1), (49.2) such that $x(t) = 0$ for $t < 0$. Let $x(t) \risingdotseq X(p)$ and $f(t) \risingdotseq F(p)$. By the rule of differentiating an object function and the property of linearity, we go over to result functions in Eq. (49.1) and, in view of the boundary condition (49.2), we obtain

$$\begin{aligned} p^n X(p) - p^{n-1}x_0 - \dots - px_{n-2} - x_{n-1} \\ + a_1(p^{n-1}X(p) - p^{n-2}x_0 - \dots - px_{n-3} - x_{n-2}) \\ + \dots + a_{n-1}(pX(p) - x_0) + a_n X(p) = F(p), \end{aligned}$$

or

$$A(p)X(p) - B(p) = F(p),$$

where

$$A(p) = p^n + a_1 p^{n-1} + \dots + a_{n-1}p + a_n$$

is the characteristic polynomial of the equation $Lx = 0$, and

$$\begin{aligned} B(p) = x_0(p^{n-1} + a_1 p^{n-2} + \dots + a_{n-1}) \\ + x_1(p^{n-2} + a_1 p^{n-3} + \dots + a_{n-2}) \\ + \dots + x_{n-2}(p + a_1) + x_{n-1}. \end{aligned}$$

We have $X(p) = [B(p) + F(p)]/A(p)$. To find the sought-for solution $x(t)$ of the Cauchy problem (49.1), (49.2), we need only reconstruct the object function $x(t)$ from the result function $X(p)$. This can be done by the inversion formula. In practical applications of the operational method, however, the more customary approach is to use the table of object and result functions. For instance, if $f(t)$ is a quasi-polynomial (i.e. a linear combination of terms of the type $t^r e^{\lambda t}$), then $X(p)$ proves to be a rational function. To find the object function in this case it often convenient to represent $X(p)$ as a partial-fraction expansion.

To substantiate the possibility of applying the operational method to the Cauchy problem (49.1), (49.2) it is sufficient to verify that $x(t), x'(t), \ldots, x^{(n)}(t)$ are object functions. Let us represent $x(t)$ as the sum $\tilde{x}(t) + x_0(t)$, where $\tilde{x}(t)$ is the solution of the homogeneous equation

$$Lx = 0 \tag{49.3}$$

with given initial conditions (49.2), and $x_0(t)$ is the solution of Eq. (49.1) with zero Cauchy data. Note that $\tilde{x}(t)$ is a linear combination of functions of the $t^r e^{\lambda t}$ type (r is a nonnegative integer), which implies that all the derivatives of $\tilde{x}(t)$ are object functions. We can write $x_0(t)$ in the form of the convolution

$$x_0(t) = \int_0^t f(\xi)\, z(t-\xi)\, d\xi, \tag{49.4}$$

where $z(t)$ is the solution of Eq. (49.3) satisfying the conditions

$$z(0) = z'(0) = \ldots = z^{(n-2)}(0) = 0, \quad z^{(n-1)}(0) = 1. \tag{49.5}$$

From (49.4) and (49.5) it follows that

$$x_0^{(k)}(t) = \int_0^t f(\xi)\, z^{(k)}(t-\xi)\, d\xi + f(t)\, z^{(k)}(0), \quad k = 1, 2, \ldots n. \tag{49.6}$$

Since the result function for $z(t)$ is $Z(p) = 1/A(p)$, which is regular at infinity, $Z(\infty) = 0$, we conclude that $z(t)$ is an entire function of the exponential type (see Theorem 3 of Sec. 48), and, in view of (49.4) and (49.6), that the functions $x_0(t), x_0'(t), \ldots, x_0^{(n)}(t)$ are object functions. Hence, $x(t), x'(t), \ldots, x^{(n)}(t)$ are object functions, too.

The function $z(t)$ is often called the *function of a unit point source* for the equation $Lz = 0$, while the function $\varepsilon(t) = \theta(t)\, z(t)$ is known as the *fundamental solution* of operator L, i.e. the solution of the equation

$$L\varepsilon = \delta(t),$$

where $\delta(t)$ is the delta function of Dirac.

Remark. If the initial conditions (49.2) are specified at time $t = t_0$ rather than at time $t = 0$, then by substituting τ for $t - t_0$ we can replace the Cauchy problem (49.1), (49.2) with

$$Ly(\tau) = f(\tau + t_0)$$

with the initial conditions at $\tau = 0$.

Example 1. Solve the Cauchy problem for

$$x''(t) - 3x'(t) + 2x(t) = 6e^{-t}$$

with the initial conditions $x(0) = 2$ and $x'(0) = 0$. Let $x(t) \risingdotseq X(p)$. Then

$$x'(t) \risingdotseq pX(p) - x(0) = pX(p) - 2,$$
$$x''(t) \risingdotseq p^2X(p) - px(0) - x'(0) = p^2X(p) - 2p.$$

Going over to result functions in the equations, we obtain

$$p^2X(p) - 2p - 3(pX(p) - 2) + 2X(p) = \frac{6}{p+1},$$
$$X(p) = \frac{2p}{p^2-1},$$

which leads to the sought-for solution $x(t) = 2\cosh t$. □

Example 2. Let us find the solution of the equation

$$x'''(t) + x'(t) = \cos t$$

with the initial conditions $x(0) = 0$, $x'(0) = -2$, and $x''(0) = 0$. Let $x(t) \risingdotseq X(p)$. Then $x'(t) \risingdotseq pX(p)$ and $x'''(t) \risingdotseq p^3X(p) + 2p$.

Going over to result functions in the equation, we find that $(p^3 + p)X(p) + 2p = \frac{p}{1+p^2}$, whence $X(p) = -\frac{1+2p^2}{(1+p^2)^2}$. Thus (see Example 4 in Sec. 48),

$$X(p) = -\frac{1}{2}\cdot\frac{p^2-1}{(p^2+1)^2} - \frac{3}{2}\cdot\frac{1}{p^2+1},$$
$$x(t) = -\frac{1}{2}t\cos t - \frac{3}{2}\sin t. \ \square$$

Example 3. Let us solve the equation

$$x^{\mathrm{IV}}(t) + 2x''(t) + x(t) = \sin t$$

with zero initial conditions. If $x(t) \risingdotseq X(p)$, then

$$(p^4 + 2p^2 + 1)X(p) = \frac{1}{p^2+1},$$

which leads to $X(p) = \frac{1}{(1+p^2)^3}$. Whence (see Example 5 in Sec. 48),

$$x(t) = \frac{3}{8}\sin t - \frac{3}{8}t\cos t - \frac{1}{8}t^2\sin t. \ \square$$

The operational method can also be applied by a similar technique to the solution of systems of linear differential equations with constant coefficients.

Example 4. Let us solve the Cauchy problem for the system

$$\begin{cases} x'(t)+y'(t)+x(t)+y(t)=t, \\ x''(t)-y'(t)+2x(t)=3(e^{-t}+1) \end{cases}$$

with the initial conditions $x(0) = y(0) = 0$ and $x'(0) = -1$. Let $x(t) \fallingdotseq X(p)$ and $y(t) \fallingdotseq Y(p)$. Then

$$x'(t) \fallingdotseq pX(p), \quad x''(t) \fallingdotseq p^2X(p) + 1, \quad y'(t) \fallingdotseq pY(p).$$

Going over to result functions in the system of equations, we find that

$$\begin{cases} pX(p)+pY(p)+X(p)+Y(p)=\frac{1}{p^2}, \\ p^2X(p)+1-pY(p)+2X(p)=3\left(\frac{1}{p+1}-\frac{1}{p}\right). \end{cases}$$

Solving this system, we find that

$$X(p)=\frac{1}{p+1}-\frac{1}{p}, \quad Y(p)=\frac{1}{p^2}.$$

whence $x(t) = e^{-t} - 1$ and $y(t) = t$. □

49.2 Volterra integral equations Let us consider a Volterra integral equation of the second kind with a kernel K that depends on the difference of the independent variables, i.e. an equation of the type

$$\varphi(t)=f(t)+\int_0^t K(t-\xi)\,\varphi(\xi)\,d\xi, \tag{49.7}$$

where $K(t)$ and $f(t)$ are given functions and $\varphi(t)$ is the unknown function.

Let $\varphi(t) \fallingdotseq \Phi(p)$, $f(t) \fallingdotseq F(p)$, and $K(t) \fallingdotseq G(p)$. Going over to result functions in Eq. (49.7) and using the rule for finding the result function of a convolution, we obtain $\Phi(p) = F(p) + G(p)F(p)$, whence

$$\Phi(p)=\frac{F(p)}{1-G(p)}.$$

The object function corresponding to $\Phi(p)$ is the sought-for solution of Eq. (49.7).

Example 5. Let us solve the integral equation

$$\varphi(t)=\sin t+\int_0^t (t-\xi)\,\varphi(\xi)\,d\xi.$$

Going over to result functions in the equation, we obtain

$$\Phi(p) = \frac{1}{p^2+1} + \frac{1}{p^2}\Phi(p),$$

whence

$$\Phi(p) = \frac{p^2}{(p^2-1)(p^2+1)} = \frac{1}{2}\left(\frac{1}{p^2+1} + \frac{1}{p^2-1}\right),$$

$$\varphi(t) = \frac{1}{2}(\sin t + \sinh t). \quad \square$$

Let us study a Volterra integral equation of the first kind with a kernel K that depends only on the difference of the independent variables, i.e. an equation of the type

$$\int_0^t K(t-\xi)\,\varphi(\xi)\,d\xi = f(t), \tag{49.8}$$

where $f(t)$ is given and $\varphi(t)$ the unknown function. We put $f(t) \risingdotseq F(p)$, $K(t) \risingdotseq G(p)$, and $\varphi(t) \risingdotseq \Phi(p)$. Then Eq. (49.8) yields $\Phi(p) = F(p)/G(p)$. The object function corresponding to $\Phi(p)$ is the sought-for solution of Eq. (49.8).

Example 6. Let us solve the integral equation

$$\int_0^t e^{t-\xi}\varphi(\xi)\,d\xi = t.$$

Going over to result functions, we find that $\frac{1}{p-1}\Phi(p) = \frac{1}{p^2}$, whence $\Phi(p) = \frac{1}{p} - \frac{1}{p^2}$, and $\varphi(t) = 1 - t$. $\square$

49.3 Equations with partial derivatives Let us study the problem of vibrations of a string $0 < x < l$ with fixed ends, assuming that the initial velocities of the points of the string and the initial deviations of these points from equilibrium are given. This problem (see Vladimirov [2]) is formulated as follows: to find the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \tag{49.9}$$

for zero boundary conditions

$$u\,|_{x=0} = u\,|_{x=l} = 0 \tag{49.10}$$

and given initial conditions

$$u\,|_{t=0} = u_0(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x). \tag{49.11}$$

Let us assume that the functions $u_0(x)$ and $u_1(x)$ are sufficiently smooth and obey the additional conditions at the end points of the segment $[0, l]$, which ensure that there is a smooth solution of the problem (49.9)-(49.11).

Suppose $U(p, x)$ is the result function for $u(x, t)$. Then, assuming p to be a parameter, we obtain

$$\frac{\partial^2 u}{\partial x^2} \risingdotseq \int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-pt}\, dt = \frac{d^2 U}{dx^2}.$$

Using the rule of differentiating object functions and the initial conditions (49.11), we find that

$$\frac{\partial^2 u}{\partial t^2} \risingdotseq pU(x, p) - pu_0(x) - u_1(x).$$

Going over to result functions in Eq. (49.9), we obtain

$$a^2 \frac{d^2 U}{dx^2} - p^2 U + pu_0(x) + u_1(x) = 0, \qquad (49.12)$$

while the boundary conditions (49.10) yield

$$U\,|_{x=0} = U\,|_{x=l} = 0. \qquad (49.13)$$

Solving the problem (49.12), (49.13), we arrive at the result function $U(x, p)$ and then at the object function $u(x, t)$.

Example 7. Let us solve the problem (49.9)-(49.11) for $a = 1$, $l = 1$, $u_0(x) = \sin \pi x$, and $u_1(x) = 0$. In this case Eq. (49.12) takes the form

$$\frac{d^2 U}{dx^2} - p^2 U = -p \sin \pi x.$$

Solving this equation for $U\,|_{x=0} = U\,|_{x=1} = 0$, we obtain $U(x, p) = \frac{p \sin \pi x}{p^2 + \pi^2}$, which yields $u(x, t) = \cos \pi t \sin \pi x$. □

Example 8. Solve the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0,$$

for zero initial conditions $u\,|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$ and the following boundary conditions:

$$u\,|_{x=0} = 0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1} = \sin \omega t.$$

Let $u(x, t) \risingdotseq U(x, p)$. Then $\frac{\partial^2 u}{\partial x^2} \risingdotseq p^2 \frac{\partial^2 U}{\partial x^2}$. Going over to result functions and taking into account the initial conditions, we obtain $\frac{d^2U}{dx^2} = p^2U$, from which we find the general solution $U(x, p) = C_1 \cosh px + C_2 \sinh px$. Since $\sin \omega t \risingdotseq \omega/(p^2+\omega^2)$, the boundary conditions yield

$$U\,|_{x=0} = 0, \quad \left.\frac{dU}{dx}\right|_{x=1} = \frac{\omega}{p^2+\omega^2}.$$

Hence

$$U(x, p) = \frac{\omega \sin xp}{p(p^2+\omega^2)\cosh p}.$$

The function $U(x, p)$ has poles at $\pm i\omega$ and $\pm i\omega_k$, where $\omega_k = (k - 1/2)\pi$, $k = 1, 2, \ldots$. By the second expansion theorem,

$$u(x, t) = \operatorname*{Res}_{p=i\omega} G(p) + \operatorname*{Res}_{p=-i\omega} G(p) + \sum_{k=1}^{\infty} \left[\operatorname*{Res}_{p=p_k} G(p) + \operatorname*{Res}_{p=-p_k} G(p)\right],$$

with $p_k = i\omega_k$ and $G(p) = U(x, p)\, e^{pt}$.

We will assume that $|\omega| \neq \omega_k$ $(k = 1, 2, \ldots)$. Then all the residues of $G(p)$ are simple, with

$$\operatorname*{Res}_{p=i\omega} G(p) = \left(\frac{\omega e^{pt} \sin xp}{p \cosh p}\right)_{p=i\omega} \frac{1}{(p^2+\omega^2)'_{p=i\omega}} = -\frac{i \sin \omega x e^{i\omega t}}{2\omega \cos \omega},$$

$$\operatorname*{Res}_{p=p_k} G(p) = \left(\frac{\omega e^{pt} \sin xp}{p(p^2+\omega^2)\sinh p}\right)_{p=p_k} = (-1)^{k-1} \frac{i\omega e^{i\omega_k t} \sin \omega_k x}{(\omega_k^2-\omega^2)\,\omega_k}.$$

Bearing in mind that

$$\operatorname*{Res}_{p=i\omega} G(p) + \operatorname*{Res}_{p=-i\omega} G(p) = 2 \operatorname{Re} \operatorname*{Res}_{p=i\omega} G(p) = \frac{\sin \omega x \sin \omega t}{\omega \cos \omega},$$

$$\operatorname*{Res}_{p=p_k} G(p) + \operatorname*{Res}_{p=-p_k} G(p) = 2 \operatorname{Re} \operatorname*{Res}_{p=p_k} G(p)$$

$$= 2(-1)^k \frac{\omega \sin \omega_k x \sin \omega_k t}{\omega_k (\omega_k^2 - \omega^2)},$$

we finally obtain

$$u(x, t) = \frac{\sin \omega x \sin \omega t}{\omega \cos \omega} + 2\omega \sum_{k=1}^{\infty} (-1)^k \frac{\sin \omega_k x \sin \omega_k t}{\omega_k (\omega_k^2 - \omega^2)}. \quad \square$$

Let us discuss the application of the operational method to the problem of the heat conduction equation. We wish to find the temperature distribution in a semi-infinite rod $0 < x < \infty$, assuming that the rod's initial temperature was zero and the left end is in a heat bath whose temperature is a known function of time.

The problem consists in finding the solution $u(x, t)$ of equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0, \tag{49.14}$$

that is bounded for $x \geqslant 0$ and obeys the following initial and boundary conditions:

$$u|_{t=0} = 0, \quad u|_{x=0} = f(t) \tag{49.15}$$

Let $f(t)$ be an object function and $u(x, t) \risingdotseq U(x, p)$. Then bearing in mind (49.15), we have

$$\frac{\partial u}{\partial t} \risingdotseq pU, \quad \frac{\partial^2 u}{\partial x^2} \risingdotseq \frac{d^2 U}{dx^2}.$$

Going over to result functions in Eq. (49.14), we arrive at the boundary value problem

$$\frac{d^2 U}{dx^2} - \frac{p^2}{a^2} U = 0, \tag{49.16}$$

$$U|_{x=0} = F(p), \quad |U(x, p)| < \infty, \tag{49.17}$$

where $F(p) \risingdotseq f(t)$. Solving the problem (49.16), (49.17), we obtain

$$U(x, p) = F(p) e^{-\frac{\sqrt{p}}{a} x}$$

The sought-for solution $u(x, t)$ can be found from the result function $U(x, p)$ via the inversion formula, but it is more convenient to represent $U(x, p)$ as

$$U(x, p) = pF(p) \frac{1}{p} e^{-\frac{x}{a}\sqrt{p}},$$

use the rules for calculating the result functions of derivatives and convolutions, and a formula we obtained in Sec. 48.2,

$$\frac{1}{p} e^{-\frac{x}{a}\sqrt{p}} \risingdotseq 1 - \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) = \frac{2}{\sqrt{2}} \int\limits_{\frac{x}{2a\sqrt{t}}}^{\infty} e^{-\xi^2} d\xi = G(x, t).$$

We have

$$U(p, x) \risingdotseq u(x, t) = \int\limits_0^t f(\tau) \frac{\partial}{\partial t} G(x, t-\tau) d\tau,$$

where

$$\frac{\partial}{\partial t} G(x, t-\tau) = \frac{x}{2a\sqrt{\pi}} (t-\tau)^{-3/2} e^{-\frac{x^2}{4a^2(t-\tau)}}.$$

Hence,

$$u(x, t) = \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{f(\tau)}{(t-\tau)^{3/2}} e^{-\frac{x^2}{4a^2(t-\tau)}} d\tau,$$

50 String Vibrations from Instantaneous Shock

50.1 A semi-infinite string Small natural vibrations of a homogeneous string are described by the wave equation

$$u_{tt} = a^2 u_{xx}. \tag{50.1}$$

Here $u(t, x)$ is the deviation of the string from equilibrium at point x and at time t, and a is a positive constant.

Suppose the string is semi-infinite ($0 < x < \infty$), its end $x = 0$ is free, and at the initial moment ($t = 0$) the string was at rest, i.e. the Cauchy data are

$$u|_{t=0} \equiv 0, \quad u_t|_{t=0} \equiv 0. \tag{50.2}$$

At time $T > 0$ the end $x = 0$ is subjected to an instantaneous shock, so that the following boundary condition holds:

$$u_x(t, 0) = V\delta(t - T), \tag{50.3}$$

where δ is the delta function of Dirac, and V is a nonzero constant.

Let us solve the mixed problem (50.1)-(50.3) for $0 < x < \infty$ and $0 < t < \infty$. We introduce the Laplace transform

$$v(p, x) = \int_0^\infty u(t, x) e^{-pt} dt. \tag{50.4}$$

Then for v we have the ordinary differential equation

$$v''_{xx} - \frac{p^2}{a^2} v = 0 \quad (0 < x < \infty) \tag{50.5}$$

and the boundary condition

$$v'_x|_{x=0} = Ve^{-pT}. \tag{50.6}$$

We will impose the following boundary condition at infinity:

$$v(p, x) \to 0 \quad (\operatorname{Re} p \to +\infty). \tag{50.7}$$

Then $v(p, x) = -\frac{aV}{p} e^{-px/a-pT}$, and the inversion formula yields

$$u(t, x) = -\frac{aV}{2\pi i} \int_{c-\infty i}^{c+i\infty} \frac{1}{p} e^{p\left(t-T-\frac{x}{a}\right)} dp,$$

with c positive. This yields (see Sec. 48)

$$u(t, x) \equiv 0 \quad \left(t < T + \frac{x}{a}\right), \quad u(t, x) = -aV \quad \left(t > T + \frac{x}{a}\right)$$

or

$$u(t, x) = -aV\theta\left(t - T - \frac{x}{a}\right), \tag{50.8}$$

where θ is the Heaviside unit function.

Thus, the impact on the left end of the string results in a plane wave (a rectangle step of height $a \mid V \mid$) that travels along the string with a speed a.

String vibrations in the presence of friction are described by the equation

$$u_{tt} = a^2 u_{xx} + \alpha u_t, \tag{50.9}$$

where a and α are positive constants. The initial and boundary conditions are again taken in the form (50.2), (50.3). We arrive at the following equation for the Laplace transform $v(p, x)$ of the function u:

$$v''_{xx} - \frac{p^2 + \alpha p}{a^2} v = 0 \tag{50.10}$$

and the boundary conditions (50.6), (50.7). This yields

$$v = -\frac{aV}{\sqrt{p^2+\alpha p}} e^{-\frac{1}{a}\sqrt{p^2+\alpha p}\, x - pT},$$

and from the inversion formula we obtain

$$u(t, x) = -\frac{aV}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{p(t-T)-\frac{x}{a}\sqrt{p^2+\alpha p}}}{\sqrt{p^2+\alpha p}} dp. \tag{50.11}$$

The function $\sqrt{p^2 + \alpha p}$ has two branch points: $p = 0$ and $p = -\alpha$. Its regular branch in the right half-plane is selected in such a way that the root is positive for positive p's, so that $v(p, x)$ satisfies condition (50.7).

As $p \to \infty$, the exponent in the exponential function in (50.11) becomes

$$p\left(t - T - \frac{x}{a}\right) - \frac{\alpha x}{2a} + o(1).$$

The integrand in (50.11) has no poles in the right half-plane Re $p > 0$, and by Jordan's lemma $u(t, x) \equiv 0$ for $t - T - x/a < 0$.

For $t - T - x/a > 0$ the integral (50.11) cannot be expressed in terms of elementary functions, and we will use a well-known formula of operational calculus (see Lavrent'ev and Shabat [1]),

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{-\tau\sqrt{p^2-b^2}+tp}}{\sqrt{p^2-b^2}} = I_0\left(b\sqrt{t^2-\tau^2}\right)\theta(t-\tau),$$

where I_0 is a modified Bessel function. We can reduce the integral in (50.11) to this form if we substitute $p = \tilde{p} - \alpha/2$, so that $b = \alpha/2$ and $\tau = x/a$. The final result is

$$u(t, x) = -aVe^{-\frac{\alpha}{2}(t-T)} I_0\left(\frac{\alpha}{2}\sqrt{(t-T)^2 - \frac{x^2}{a^2}}\right)\theta\left(t - T - \frac{x}{a}\right). \tag{50.12}$$

For $\alpha = 0$ we arrive at (50.8). The solution given by (50.12) describes a wave, too. Its wavefront $x = a(t - T)$, moves to the right with a speed a.

We fix a point $x > 0$ and study the behavior of $u(t, x)$ as $t \to +\infty$. Using the asymptotic formula

$$I_0(x) \sim \frac{e^{-x}}{\sqrt{2\pi x}} \quad (x \to +\infty),$$

we obtain from (50.12) the following: $u(t, x) \sim -\frac{aV}{\sqrt{\alpha\pi t}} \ (t \to +\infty)$, so that the vibrations at a fixed point fade out, in contrast to (50.8). This is the result of friction.

50.2 A finite string vibrating without friction Suppose the string is finite ($0 < x < l$), its left end $x = 0$ is free, and the right end $x = l$ is fixed, so that

$$u(0, l) = 0. \tag{50.13}$$

Just as in Sec. 50.1, we will fix the zero Cauchy data and the condition that at $t = T > 0$ the left end is subjected to a sudden shock. Then $u(t, x)$ satisfies Eq. (50.1), the Cauchy data (50.2), and the boundary condition (50.3) and (50.13). Going over to the Laplace transform (50.4), we arrive at Eq. (50.5) for v and at the boundary conditions

$$v'_x|_{x=0} = Ve^{-pT}, \quad v|_{x=l} = 0. \tag{50.14}$$

This yields

$$v(p, x) = \frac{aVe^{-pT}}{p\cosh(pl/a)} \sinh\left[\frac{p}{a}(x - l)\right], \tag{50.15}$$

and using the inversion formula, we obtain

$$u(t, x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{pt}\,v(p, x)\,dp,$$

where c is positive. We evaluate this integral using the theory of residues. The singular points of the integrand coincide with the zeros of the function $\cosh(pl/a)$ (point $p=0$ is not a singular point for this function). Hence, the integrand has poles at $p=p_n$, where

$$p_n=\frac{ia\pi}{l}\left(n+\frac{1}{2}\right),\quad n=0, \pm 1, \ldots$$

All of these poles are simple and lie on the imaginary axis, with

$$\operatorname*{Res}_{p=p_n}(e^{pt}v(p, x))=\frac{aVe^{p_n(t-T)}\sinh\left(\frac{p_n}{a}(x-l)\right)}{p_n\left(\cosh\frac{pl}{a}\right)'\Big|_{p=p_n}}$$

$$=\frac{2iaV(-1)^{n+1}}{\pi(2n+1)}e^{i\frac{\pi a}{l}\left(n+\frac{1}{2}\right)(t-T)}\varphi_n(x), \qquad (50.16)$$

where

$$\varphi_n(x)=\sin\left[\frac{\pi}{l}\left(n+\frac{1}{2}\right)(x-l)\right]. \qquad (50.17)$$

Let us show that

$$u(t, x)=-\sum_{n=-\infty}^{\infty}\operatorname*{Res}_{p=p_n}(e^{pt}v(p, x)) \qquad (50.18)$$

for $t>T+l/a$. Since a is the speed with which the shock travels along the string, during this time the perturbation is able to reach the right end of the string and be reflected from it. But if $t<T+l/a$, we can easily see that the solution has the form (50.8).

Consider the integral

$$J_N=\frac{1}{2\pi i}\int_{\Gamma_N} e^{pt}v(p, x)\,dp, \qquad (50.19)$$

where Γ_N is a rectangle with its vertices at $c\pm iy_N$ and $\pm iy_N, -y_N$, with $y_N=\pi aN/l$. The integral J_N is equal to the sum of residues at the poles of the integrand that lie inside Γ_N. The integral taken along the segment $[c-iy_N, c+iy_N]$ tends to $u(t, x)$ as $N\to\infty$. What remains to be proved is that the integral taken along the contour $\tilde{\Gamma}_N$ consisting of the other three segments tends to zero as $N \to \infty$. We have

$$\cosh\frac{pl}{a} = \frac{1}{2}e^{-pl/a}\varphi(p), \quad \varphi(p) = 1 + e^{2pl/a}.$$

Let us show that $|\varphi(p)| \geqslant A > 0$ for $p \in \tilde{\Gamma}_N$, where the constant A is independent of N. On $[-y_N + iy_N, c + iy_N]$ we have

$$|\varphi(p)| = 1 + e^{\frac{2l}{a}y}, \quad -y_N \leqslant y \leqslant C,$$

so that $|\varphi(p)| \geqslant 1$. The same estimate holds on $[-y_N - iy_N, c - iy_N]$, while on the remaining segment the function $e^{2lp/a}$ decreases exponentially as $N \to \infty$. For this reason the integrand in (50.15) is equal to

$$\frac{aV}{2\varphi(p)\,p}\left[e^{p\left(t-T+\frac{l}{a}+\frac{x-l}{a}\right)} - e^{p\left(t-T+\frac{l}{a}-\frac{x-l}{a}\right)}\right].$$

The first factor is of the order of $O(1/p)$ on the $\tilde{\Gamma}_N$ as $N \to \infty$, while the exponents in the first and second terms in the brackets are strictly positive, since $t > T + l/a$, $0 \leqslant x \leqslant l$. By Jordan's lemma, the integral taken along $\tilde{\Gamma}_N$ tends to zero as $N \to \infty$.

From (50.8) and (50.6) we find that

$$u(t, x) = -\frac{2iaV}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n e^{\frac{i\pi a}{l}\left(n+\frac{1}{2}\right)(t-T)}\frac{\varphi_n(x)}{2n+1}.$$

Combining the terms with numbers n and $-n-1$ ($n = 0, 1, 2, \ldots$) into pairs and allowing for the fact that $\varphi_{-n-1}(x) = -\varphi_n(x)$, we arrive at the final result

$$u(t, x) = \frac{4Va}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\sin(\omega_n(t-T))\sin\left(\frac{\omega_n}{a}(x-l)\right), \tag{50.20}$$

where

$$\omega_n = \frac{\pi a}{l}\left(n + \frac{1}{2}\right). \tag{50.21}$$

Each term in this sum corresponds to a natural vibration mode of the string with end $x = 0$ free and end $x = l$ fixed. The natural frequencies of these vibrations are equal to ω_n.

Equation (50.20) shows that a shock excites all natural vibration modes of the string. The amplitudes of the various vibration modes decrease like $1/n$ as the frequency grows, but the energies E_n of all natural vibrations are approximately the same. Indeed, let ρ be the density of the material of the string, Q is the strain, and $a^2 = Q/\rho$. Then the energy E_n of the nth natural vibration mode is

$$E_n = \frac{1}{2} \int_0^l \left[\rho \left(\frac{\partial u_n}{\partial t} \right)^2 + Q \left(\frac{\partial u_n}{\partial x} \right)^2 \right] dx$$

$$= \frac{2a^2V^2}{l} [\rho a^2 \cos^2 (\omega_n (t - T) + T \sin^2 (\omega_n (t - T))]. \quad (50.22)$$

Let us study the response of the string to a series of N periodic shocks. Suppose all shocks have the same strength and act on the string at times T, $2T$, . . ., NT. This means that the boundary condition (50.3) is replaced by

$$\frac{\partial u_N}{\partial x} (t, 0) = V \sum_{m=1}^{N} \delta (x - mT).$$

Obviously, for $t > NT + l/a$ we have

$$u_N (t, x) = \sum_{m=1}^{N} u_1 (t - (m-1) T, x),$$

where $u_1 (t, x)$ is the solution of the (50.20) type. We have (see Example 6 in Sec. 1)

$$\sum_{m=1}^{N} \sin (\omega (t - mT)) = \frac{\sin \left(\omega_n \frac{NT}{2} \right) \sin \left(\omega_n \left(t - \frac{N+1}{2} \right) T \right)}{\sin \left(\frac{\omega_n T}{2} \right)}.$$

The final result is

$$u_N (t, x) = \frac{4aV}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \sin \left(\omega_n \frac{NT}{2} \right)}{(2n+1) \sin \left(\frac{\omega_n T}{2} \right)}$$

$$\times \sin \left(\omega_n \left(t - \frac{N+1}{2} \right) T \right) \sin \left(\frac{\omega_n}{a} (x - l) \right) \quad (50.23)$$

The amplitude A_n of the nth vibration mode is

$$A_n = \frac{4aV (-1)^n \sin (\omega_n NT/2)}{\pi (2n+1) \sin (\omega_n T/2)} \quad (50.24)$$

Let us study the case when resonance sets in, i.e. the period between the shocks coincides with one of the periods of the natural vibrations, precisely, $T = 2\pi/\omega_n$. Then from (50.24) we obtain

$$A_n = \frac{4aVN(-1)^n}{\pi(2n+1)}, \tag{50.25}$$

which means that $|A_n|$ assumes the greatest possible value and increases without limit as $N \to \infty$.

The most interesting case is when T coincides with the period of the principal vibration mode, i.e.

$$T = 2\pi/\omega_0 = 4l/a. \tag{50.26}$$

In this case we have

$$u(t, x) = \frac{4aVN}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \sin(\omega_n T) \sin\left(\frac{\omega_n}{a}(x-l)\right), \tag{50.27}$$

so that

$$u_N(t, x) = Nu_1(t, x). \tag{50.28}$$

Thus, the vibration $u_1(t, x)$ is amplified N times after the $(N-1)$st shock.

50.3 A finite string vibrating with friction In this case the function $u(t, x)$ satisfies Eq. (50.9), while the Cauchy data and the boundary conditions are the same as in Sec. 50.2. Going over to the Laplace transform, we arrive at the equation (50.10) and the boundary condition (50.14) for the function $v(p, x)$. Solving this problem, we obtain

$$v(p, x) = \frac{aVe^{-pT} \sinh\left(\frac{x-l}{a}\sqrt{p^2+\alpha p}\right)}{\sqrt{p^2+\alpha p}\cosh\left(\frac{l}{a}\sqrt{p^2+\alpha p}\right)}. \tag{50.29}$$

Note that $v(p, x)$ is a single-valued function of p, since $\sinh\sqrt{z}/\sqrt{z}$ and $\cosh\sqrt{z}$ are single-valued functions of z (see Examples 18 and 19 in Sec. 22). The singular points of $v(p, x)$ coincide with the roots of the equation $\cosh\left(\frac{l}{a}\sqrt{p^2+\alpha p}\right) = 0$, which are

$$p_n^{\pm} = -\frac{\alpha}{2} \pm \frac{i\sqrt{D_n}}{2}, \quad D_n = \frac{\pi^2 a^2}{l^2}(2n+1)^2 - \alpha^2. \tag{50.30}$$

All the singularities are simple poles that lie on the straight line $\operatorname{Re} p = -\alpha/2$ except, perhaps, a finite number, since D_n is positive for large n's. If α is greater than $\pi a/l\ (2n + 1)$ for a certain n, the roots $p_n^{\pm}$ are real and negative.

As in the previous case, the integral in $u\ (t,\ x)$ is equal to the sum of residues over all the poles of the integrand. In evaluating the residues we must bear in mind that the choice of the value of $\sqrt{p^2 + \alpha p}$ is irrelevant, the only requirement is that this value be the same in all expressions containing this root. Evaluating the integral

$$u(t,\ x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{pt} V(p,\ x)\, dp,$$

we obtain

$$u(t,\ x) = \frac{2ia^2V}{l} \sum_{n=-\infty}^{\infty} \left(e^{p_n^-(t-T)} - e^{p_n^+(t-T)}\right) \frac{(1-)^n \varphi_n(x)}{\sqrt{D_n}},$$

where $\varphi_n(x)$ is the same as in (50.17). Let us transform this expression. We have

$$e^{p_n^-(t-T)} - e^{p_n^+(t-T)} = -2ie^{-\frac{\alpha}{2}(t-T)} \sin\left(\frac{\sqrt{D_n}}{2}(t-T)\right).$$

Joining the terms with numbers n and $-n-1$ into pairs, we find that

$$u(t,\ x) = \frac{4a^2V}{l} e^{-\frac{\alpha}{2}(t-T)} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{D_n}} \sin\left(\frac{\sqrt{D_n}}{2}(t-T)\right) \times \sin\left(\frac{\pi}{l}\left(n+\frac{1}{2}\right)(x-l)\right). \quad (50.31)$$

For $\alpha = 0$ this expression coincides with (50.20). From (50.31) it follows that friction changes the natural frequencies of string vibrations. In the case at hand,

$$\omega_n = \sqrt{\frac{\pi^2 a^2}{l^2}\left(n+\frac{1}{2}\right)^2 - \frac{\alpha^2}{4}}. \quad (50.32)$$

Let us assume, for the sake of definiteness, that α is less than $\pi a/l$. Then $\sqrt{D_n}$ is positive for all values of n. The solution is $u(t, x) = O(e^{-\alpha t/2})$ at $t \to +\infty$, i.e. exponentially decreases, which is due to the presence of friction. Let us see what is the response of the string to a series of N identical shocks acting on the string at $T, 2T, \ldots, NT$. Summing these vibrations, we find that at $t > NT + l/a$,

$$u_N(t, x) = \frac{4a^2V}{l} e^{-\frac{\alpha}{2}t} \times \sum_{n=0}^{\infty} \frac{(-1)^n \varphi_n(x)}{\sqrt{D_n}\left[1 - 2e^{-\alpha T/2} \cos\left(\frac{\sqrt{D_n}\,T}{2}\right) + e^{-\alpha T}\right]}$$

$$\times \left[\sin\left(\frac{\sqrt{D_n}}{2} t\right) - e^{\frac{\alpha}{2}T} \sin\left(\frac{\sqrt{D_n}}{2}(t+T)\right)\right.$$

$$- e^{-\frac{\alpha}{2}(N+1)T}\left[\sin \frac{\sqrt{D_n}}{2}(t-(N+1)T)\right)$$

$$\left. + e^{\frac{\alpha}{2}(N+1)T} \sin\left(\frac{\sqrt{D_n}}{2}(t - NT)\right)\right]. \quad (50.33)$$

Suppose there are very many shocks, i.e. $N \to \infty$. Then $e^{-\alpha NT/2}$ is exponentially small, we can approximately replace (50.33) by

$$u_N(t, x) \approx \frac{4a^2V}{l} e^{-\frac{\alpha}{2}(\tau - T)} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{D_n}} \varphi_n(x) \times \frac{e^{\frac{\alpha}{2}T} \sin\left(\frac{\sqrt{D_n}}{2}\tau\right) - \sin\left(\frac{\sqrt{D_n}}{2}(\tau - T)\right)}{1 - 2e^{-\frac{\alpha}{2}T} \cos\left(\frac{D_n}{2}T\right) + e^{-\alpha T}}, \quad (50.34)$$

where we have put

$$t = NT + \tau, \quad \tau > l/a. \quad (50.35$$

Let us consider the case where the period between each shock coincides with the period of the principal natural vibration mode, i. e. $T = 2\pi/\omega_0 = 4\pi/\sqrt{D_0}$. Then (50.34) takes the form

$$u_N(t, x) \approx \frac{4a^2V}{l} e^{-\frac{\alpha}{2}(\tau-T)} \left\{ -\frac{\varphi_0(x) \sin\left(-\frac{\sqrt{D_0}}{2}\tau\right) e^{\frac{\alpha}{2}T}}{\sqrt{D_0}\left(1-e^{-\frac{\alpha}{2}T}\right)} \right.$$

$$\left. + \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{D_n}} \frac{\varphi_n(x)\left[e^{\frac{\alpha}{2}T} \sin\left(\frac{\sqrt{D_n}}{2}T\right) - \sin\left(\frac{\sqrt{D_n}}{2}\right)(\tau-T)\right)\right]}{\left[1-2e^{-\alpha T/2}\cos\left(\frac{\sqrt{D_n}\,T}{2}\right)+e^{-\alpha T}\right]} \right\}. \tag{50.36}$$

In this case the resonance manifests itself much weaker (cf. (50.18)) because in the presence of friction the natural frequencies ω_n are not integral multiples of ω_0.

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