Canonical Observables
versus the Algebra of Invariant Charges
for the Open Nambu String

F. Colomo
Dipartimento di Fisica
Università di Firenze
Largo E. Fermi 2, Arcetri
50125 Firenze, Italy

and

L. Lusanna
Istituto Nazionale di Fisica Nucleare
Sezione di Firenze
Largo E. Fermi 2, Arcetri
50125 Firenze, Italy

Abstract

The relationship between the canonical observables and the Pohlmeyer-Rehren infinite-dimensional tensor algebra of invariant charges is analysed for the open Nambu string.

In two recent papers[1,2] the classical Nambu string has been studied by means of the many-time approach. The original pair of first-class constraints, which are only in weak involution, have been locally replaced by two sets of strictly abelian (*i.e.* in involution under Poisson brackets) constraints by multiplying them by suitable functions suggested by the light-cone coordinates. In this way it is possible to describe nearly all the constraint manifold using two overlapping charts, one with $P^+ \neq 0$, the other with $P^- \neq 0$. The remaining part of such a manifold contains only longitudinal modes. In each chart the Hamilton-Dirac equations of motion are replaced by many-time functional Hamilton equations with the abelian constraints as Hamiltonians. These equations have been solved in an arbitrary gauge. Moreover a complete set of canonical observables, *à la* Dirac, has ben found for each chart; this set reduce to the DDF oscillators[3] in the orthonormal gauge. This allows the construction of a canonical transformation, in each chart, from the original symplectic basis $x^\mu(\sigma)$, $P^\mu(\sigma)$, to a new one which is made of: 1) the abelian constraints and the conjugated gauge variables; 2) all the previous transverse canonical observables; 3) the three independent components (in four space-time dimension) of the total momentum and the three conjugated variables for the center of mass (which are Dirac observables too).

Therefore the set of all canonical observables plays the game of the $2d-1$ independent constants of motion of a completely integrable d-dimensional system.

The draw-back of this construction is the loss of manifest covariance. In a series of papers[4–8] Pohlmeyer and Rehren have introduced a set of invariant tensor charges for the closed Nambu string, which are Dirac observables, *i.e.* have zero Poisson brackets with the original constraints. Their form for the open string is

$$\mathbf{Z}^{\mu_1,\ldots,\mu_n}_{[n]\pm} = \mathbf{R}^{\mu_1,\ldots,\mu_n}_{[n]\pm} + (\text{cyclic permutations}), \tag{1}$$

with

$$\mathbf{R}^{\mu_1,\ldots,\mu_n}_{[n]\pm} = \int_{-\pi}^{\pi} d\sigma_1 A^{\mu_1}_{\pm}(\sigma_1) \int_{-\pi}^{\sigma_1} d\sigma_2 A^{\mu_2}_{\pm}(\sigma_2) \quad \ldots \quad \int_{-\pi}^{\sigma_{n-1}} d\sigma_n A^{\mu_n}_{\pm}(\sigma_n), \tag{2}$$

where $A^{\mu}_{\pm}(\sigma)$ is given by

$$A^{\mu}_{\pm}(\sigma) = P^{\mu}(\sigma) \pm N x'^{\mu}(\sigma). \tag{3}$$

The original constraints are $\chi_{\pm}(\sigma) = A^2_{\pm}(\sigma) \approx 0$ and the canonical Poisson brackets are

$$\{x^{\mu}(\sigma), P^{\nu}(\sigma')\} = -\eta^{\mu\nu}\Delta_{+}(\sigma,\sigma');$$

(for every notation see references [1,2]).

On the other hand, in the chart with $P^{+} \neq 0$, where the abelian constraints are $\tilde{\chi}_{\pm}(\sigma) = \chi_{\pm}(\sigma)/2A^{+}_{\pm}(\sigma)$ (here $A^{+}_{\pm}(\sigma)$ is the light-cone component $\frac{1}{\sqrt{2}}[A^{0}_{\pm}(\sigma) + A^{3}_{\pm}(\sigma)]$ of $A^{\mu}_{\pm}(\sigma)$), the canonical observables of reference [2] are

$$A^{\mu}_{n} = \frac{1}{\sqrt{4\pi N}} \int_{-\pi}^{\pi} d\sigma A^{\mu}_{\pm}(\sigma) \exp\left[\pm \frac{i\pi n}{P^{+}} B^{+}_{\pm}(\sigma)\right], \qquad \begin{array}{l} n = \pm 1, \pm 2, \ldots, \\ \mu = +, 1, 2, -; \end{array} \tag{4}$$

where $B^{\mu}_{\pm}(\sigma)$ is the following special primitive of $A^{\mu}_{\pm}(\sigma)$:

$$B^{\mu}_{\pm}(\sigma) = \int_{0}^{\sigma} d\sigma' P^{\mu}(\sigma') \pm N x^{\mu}(\sigma). \tag{5}$$

For the A^{μ}_{n}'s we have the following properties:

$$A^{+}_{n} = 0 \qquad \text{for } n \neq 0, \tag{6}$$

$$A^{\mu}_{0} = \frac{P^{\mu}}{\sqrt{\pi N}}. \tag{7}$$

Moreover,

$$A^{-}_{n} = \frac{\sqrt{\pi N}}{P^{+}} (\tilde{L}_n + \tilde{U}^{-}_{n}), \tag{8}$$

with

$$
\begin{aligned}
\tilde{L}_n(\tau) =& \frac{P^+}{2\pi N}\int_{-\pi}^{\pi} d\sigma \tilde{\chi}_\pm(\sigma)\exp\left[\pm\frac{i\pi n}{P^+}B^+_\pm(\sigma)\right] = \frac{1}{2}\sum_{m=-\infty}^{+\infty} A^\mu_m\cdot A_{n-m,\mu} = \\
=& \frac{P^+}{\pi^2}\sum_{m=-\infty}^{+\infty} L_m(\tau)\int_{-\pi}^{\pi} d\sigma\frac{e^{\mp im\sigma}}{2A^+_\pm(\sigma)}\exp\left[\pm\frac{i\pi n}{P^+}B^+_\pm(\sigma)\right]\approx 0, \qquad (9)\\
\tilde{U}^-_n(\tau) =& \frac{P^+}{2\pi N}\int_{-\pi}^{\pi} d\sigma\frac{\vec{A}^2_\pm(\sigma)}{2A^+_\pm(\sigma)}\exp\left[\pm\frac{i\pi n}{P^+}B^+_\pm(\sigma)\right] = \frac{1}{2}\sum_{m=-\infty}^{+\infty}\vec{A}_m\cdot\vec{A}_{n-m}. \qquad (9')
\end{aligned}
$$

Here the L_m's are the standard Virasoro generators. $\frac{P^+}{\sqrt{\pi N}}A^-_n$, $\tilde{L}_n$, $\tilde{U}^-_n$, all satisfy the Virasoro algebra.

The canonical observables we were speaking about are the transverse oscillators $\vec{A}_n$, $n\neq 0$, which satisfy the following algebra:

$$
\begin{aligned}
&\left\{A^a_n, A^b_m\right\} = -in\delta^{ab}\delta_{n,-m}, \qquad a,b=1,2; \qquad (10)\\
&\left\{A^a_n, \tilde{\chi}_\pm(\sigma)\right\} = \left\{A^a_n, \chi_\pm(\sigma)\right\} = 0. \qquad (11)
\end{aligned}
$$

The A^-_n's allows to rebuild the generalized Virasoro generators $\tilde{L}_n$, and have the following algebra with the constraints:

$$
\left\{A^-_n, \tilde{\chi}_\pm(\sigma)\right\}\approx 0, \qquad \left\{A^-_n, \chi_\pm(\sigma)\right\} = 0. \qquad (12)
$$

The new basis is spanned by

$$
\begin{aligned}
&1) \qquad Y^-(\sigma) = \frac{1}{2N}\left[\tilde{\chi}_-(\sigma)-\tilde{\chi}_+(\sigma)\right], \qquad \mathcal{P}^+(\sigma) = \textstyle\int_0^\sigma d\sigma' P^+(\sigma') - \frac{\sigma}{\pi}P^+;\\
&\qquad\quad x^+(\sigma), \qquad \Pi^-(\sigma) = \frac{1}{2}\left[\tilde{\chi}_-(\sigma)+\tilde{\chi}_+(\sigma)\right];\\
&2) \qquad A^a_n;
\end{aligned}
$$

and, for the center of mass :

$$
3) \qquad P^+, \vec{P}; \qquad Z^-, \vec{Z}; \quad (Z^+=0),
$$

where

$$
\begin{aligned}
Z^a =& X^a - \frac{1}{2P^+}\int_{-\pi}^{\pi} d\sigma\left[x^+(\sigma)P^a(\sigma) - y^a(\sigma)\mathcal{P}^+(\sigma)\right] = -\frac{J^{+a}}{P^+}\\
Z^- =& X^- \mp \frac{1}{4NP^+}\int_{-\pi}^{\pi} d\sigma\left(\mathcal{P}^+(\sigma)\pm Nx^+(\sigma)\right)\frac{\vec{A}^2_\pm(\sigma)}{A^+_\pm(\sigma)} = \\
=& -\frac{J^{+-}}{P^+} \pm \frac{\pi N X^+}{2{P^+}^2}\sum_{m=-\infty}^{\infty} A^\mu_m\cdot A_{-m,\mu} + \\
&+ \frac{i}{4\pi P^+}\sum_{n\neq 0}\sum_{m=-\infty}^{\infty}\frac{A^\mu_m\cdot A_{n-m,\mu}}{n}\int_{-\pi}^{\pi} d\sigma \exp\left[\mp\frac{i\pi n}{P^+}B^+_\pm(\sigma)\right].
\end{aligned} \qquad (13)
$$

P^+, $\vec{P}$, Z^-, $\vec{Z}$, $\vec{A}_n$ are those constant of motion which describe the independent Cauchy data of the string. $J^{\mu\nu}$, with $\mu,\nu = +,1,2,-$, are the generators of the Lorentz group:

$$J^{\mu\nu} = \frac{1}{2}\int_{-\pi}^{\pi} d\sigma \left[x^\mu(\sigma)P^\nu(\sigma) - x^\nu(\sigma)P^\mu(\sigma)\right].$$

We have also the following relations between the abelian constraints and the Virasoro generators:

$$\begin{aligned}
Y^-(\sigma) =& -\frac{\pi}{2{P^+}^2}\sum_{n,m=-\infty}^{\infty} A_m^\mu \cdot A_{n-m,\mu}\left(A_+^+(\sigma)\exp\left[-\frac{in\pi}{P^+}B_+^+(\sigma)\right] - A_-^+(\sigma)\exp\left[\frac{in\pi}{P^+}B_-^+(\sigma)\right]\right),\\
\Pi^-(\sigma) =& \frac{1}{\pi}\Xi^-_{tot} + \frac{\partial}{\partial\sigma}\Xi^-_{rel}(\sigma),\\
\Xi^-_{tot} =& \frac{1}{2}\int_{-\pi}^{\pi} d\sigma \Pi^-(\sigma) = \frac{\pi N}{2P^+}\sum_{n=-\infty}^{\infty} A_n^\mu \cdot A_{-n,\mu},\\
\Xi^-_{rel}(\sigma) =& \int_0^\sigma d\bar\sigma \Pi^-(\bar\sigma) - \frac{\sigma}{\pi}\Xi^-_{tot} = \frac{\pi N}{2{P^+}^2}\sum_{n=-\infty}^{\infty} A_n^\mu \cdot A_{-n,\mu}\mathcal{P}^+(\sigma)+\\
&+\frac{iN}{4P^+}\sum_{n\neq 0}\sum_{m=-\infty}^{\infty}\frac{A_m^\mu \cdot A_{n-m,\mu}}{n}\left(\exp\left[-\frac{in\pi}{P^+}B_+^+(\sigma)\right] - \exp\left[\frac{in\pi}{P^+}B_-^+(\sigma)\right]\right),
\end{aligned} \tag{14}$$

and the following expression for the Lorentz generators which do not appear in equations (13):

$$\begin{aligned}
J^{12} =& \epsilon^{ab}\left(Z^aP^b + \frac{i}{2}\sum_{n\neq 0}\frac{A_n^aA_{-n}^b}{n}\right),\\
J^{-a} =& Z^-P^a - Z^aP^- - \frac{i}{2\sqrt{\pi n}}\sum_{n\neq 0}\frac{A_n^aA_{-n}^-}{n} - \frac{\pi N}{2{P^+}^2}X^+P^a\sum_{n=-\infty}^{\infty} A_n^\mu \cdot A_{-n,\mu} -\\
&-\frac{iP^a}{4\pi P^+}\sum_{n\neq 0}\sum_{m=-\infty}^{\infty}\frac{A_m^\mu \cdot A_{n-m,\mu}}{n}\int_{-\pi}^{\pi} d\sigma \exp\left[-\frac{in\pi}{P^+}B_+^+(\sigma)\right].
\end{aligned} \tag{15}$$

From equations (14) the mass spectrum is given by

$$P^- = \frac{1}{2P^+}\left(\vec{P}^2 + 2\pi N\sum_{n=1}^{\infty}\vec{A}_n \cdot \vec{A}_{-n}\right) + \Xi^-_{tot}. \tag{16}$$

In terms of the new basis we have

$$\begin{aligned}
\vec{A}_\pm(\sigma) =& A_\pm^+(\sigma)\left(\frac{\vec{P}}{P^+} + \frac{\sqrt{N\pi}}{P^+}\sum_{n\neq 0}\vec{A}_n\exp\left[\mp\frac{i\pi n}{P^+}B_\pm^+(\sigma)\right]\right);\\
A_\pm^+(\sigma) =& P^+(\sigma) \pm N{x'}^+(\sigma);\\
A_\pm^-(\sigma) =& A_\pm^+(\sigma)\left(\frac{N\pi}{{P^+}^2}\sum_{n=-\infty}^{\infty}(\tilde{L}_n + \frac{1}{2}\sum_{m=-\infty}^{\infty}\vec{A}_m \cdot \vec{A}_{n-m})\exp\left[\mp\frac{i\pi n}{P^+}B_\pm^+(\sigma)\right]\right).
\end{aligned} \tag{17}$$

We see from equations (13), (15) that the new canonical basis $Y^-(\sigma)$, $\mathcal{P}^+(\sigma)$, $x^+(\sigma)$, $\Pi^-(\sigma)$, A^a_n, P^+, Z^-, $\vec{P}$, $\vec{Z}$, adapted to the abelian constraints $\tilde{\chi}_\pm(\sigma)$, is not adapted to the generators of the Lorentz group, which is a symmetry group of the system. This is due to the fact that the abelian constraints are only weakly Lorentz invariant, so that the Lorentz generators are only weak observables and not strong ones. Therefore it is not possible to rebuild $J^{\mu\nu}$ using only the strong observables A^a_n, P^+, Z^-, $\vec{P}$, $\vec{Z}$, plus the Virasoro constraints (either $\tilde{L}_n = \sum_{m=-\infty}^{\infty} A^\mu_m \cdot A_{n-m,\mu}$ or A^-_n or else $Y^-(\sigma)$, $\Pi^-(\sigma)$). From equations (13),(15) we see that J^{+-} and J^{-a} depends on the gauge variables X^+ and $\int_{-\pi}^{\pi} d\sigma \exp\left[\mp \frac{i\pi n}{P^+} B^+_\pm(\sigma)\right]$, which are conjugated to the constraints Ξ^-_{tot} and

$$\begin{aligned}\hat{L}_n =& \frac{i}{8\pi N n} \int_{-\pi}^{\pi} d\sigma \chi_\pm(\sigma) \exp\left[\mp \frac{i\pi n}{P^+} B^+_\pm(\sigma)\right] = \\ =& \frac{i}{4n{P^+}^2} \sum_{m=-\infty}^{\infty} \tilde{L}_m \int_{-\pi}^{\pi} d\sigma {A^+_\pm}^2(\sigma) \exp\left[\mp \frac{i\pi}{P^+}(n+m) B^+_\pm(\sigma)\right],\end{aligned}$$

where the following representation of the original constraints has been used

$$\chi_\pm(\sigma) = 2A^+_\pm(\sigma)\tilde{\chi}_\pm(\sigma) = \frac{2\pi n}{{P^+}^2} {A^+_\pm}^2(\sigma) \sum_{n=-\infty}^{\infty} \tilde{L}_n \exp\left[\mp \frac{i\pi n}{P^+} B^+_\pm(\sigma)\right]. \tag{18}$$

The previous discussion shows that the use of light-cone coordinates for the abelianization of the constraints is not natural from the point of view of the Poincaré group. A different abelianization, in which the abelian constraints are Poincaré scalars, is now under investigation: in this last case all the Poincarè generators turn out to be strong observables.

Another consequence of the previous discussion is that the set of generators $J^{\mu\nu}$, A^μ_n (where A^μ_0 contains P^μ and A^-_n the generalized Virasoro generators) does not close even if we consider their enveloping algebra: indeed, beside the Lorentz algebra we get

$$\begin{aligned}\left\{A^a_n, A^b_m\right\} =& -in\delta^{ab}\delta_{n+m}, \\ \left\{A^a_n, A^-_m\right\} =& -i\frac{\sqrt{\pi N}}{P^+} n A^a_{n+m}, \\ \left\{A^-_n, A^-_m\right\} =& i\frac{\sqrt{\pi N}}{P^+}(m-n) A^-_{n+m}, \\ \{P^\mu, A^\nu_m\} =& \frac{i\pi N n}{P^+} \eta^{\mu +} A^\nu_n, \\ \{J^{\mu\nu}, A^\alpha_n\} =& -\eta^{\mu\alpha} A^\nu_n + \eta^{\nu\alpha} A^\mu_n + \frac{i\pi N n}{P^+}(\eta^{\nu +} Z^\mu - \eta^{\mu +} Z^\nu) A^\alpha_n + \\ & + \frac{n\sqrt{\pi N}}{P^+} \sum_{n=-\infty}^{\infty} \sum_{m\neq 0} \frac{A^\alpha_{n-m}}{m} (\eta^{\mu +} A^\nu_m - \eta^{\nu +} A^\mu_m).\end{aligned} \tag{19}$$

The last line of equations (19), together with equations (13), shows that the gauge variables X^+, $\int_{-\pi}^{\pi} d\sigma \exp[-\frac{i\pi n}{P^+} B^+_+(\sigma)]$ appears in the algebra. Moreover the first line of equation (19) shows the central charge associated to the symplectic algebra:to avoid it one must consider only A^μ_n with $n \geq 0$, but in this way the Virasoro generators $\tilde{L}_n$ with $n < 0$ are lost. However this is just the pattern followed in the canonical quantization.

Coming back to the invariant tensor charges $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$, it turns out that they can be expressed in terms of the Dirac observables P^μ, $\vec{A}_n$, and of the Virasoro generators $\tilde{L}_n \approx 0$.

In particular the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s with only transverse and/or "+" indices turn out to be functions only of P^+, $\vec{P}$, $\vec{A}_n$. For the sake of simplicity from now on we shall restrict ourselves to the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]+}$'s. For an arbitrary $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]+}$ we have:

$$\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]+} = \left(\frac{N}{\pi}\right)^{\frac{n}{2}} \sum_{k_1\dots k_n=-\infty}^{+\infty} A^{\mu_1}_{k_1} \quad \dots \quad A^{\mu_n}_{k_n} \cdot \bar{\mathbf{C}}^{[n]}_{k_1,\dots,k_n}, \tag{20}$$

where

$$\bar{\mathbf{C}}^{[n]}_{k_1,\dots,k_n} = \mathbf{C}^{[n]}_{k_1,\dots,k_n} + (\text{cyclic permutations}), \tag{21}$$

and $\mathbf{C}^{[n]}_{k_1,\dots,k_n}$ are defined through the following recurrence formulas:

$$\mathbf{C}^{[n]}_{k_1,\dots,k_n} = \frac{1}{-ik_n}\mathbf{C}^{[n-1]}_{k_1,\dots,k_{n-1}+k_n} + \frac{1}{ik_n}\exp\left[-\frac{i\pi k_n}{P^+}B^+_+(-\pi)\right]\mathbf{C}^{[n-1]}_{k_1,\dots,k_{n-1}}. \tag{22}$$

Let us outline that the gauge dependent phases $\exp[-\frac{i\pi k_n}{P^+}B^+_+(-\pi)]$ disappear when we perform the sum over the cyclic permutations of the indices in $\mathbf{C}^{[n]}_{k_1,\dots,k_n}$ in order to obtain the $\bar{\mathbf{C}}^{[n]}_{k_1,\dots,k_n}$, and thus do not appear either in equation (20). When the index k_n takes the value $k_n = 0$, the recurrence relation (22) is to be read:

$$\mathbf{C}^{[n]}_{k_1,\dots,k_n} = -\frac{\pi}{P^+}B^+_+(-\pi)\cdot\mathbf{C}^{[n-1]}_{k_1,\dots,k_{n-1}}. \tag{23}$$

In particular, we have $\mathbf{C}^{[1]}_k = 2\pi\cdot\delta_{k,0}$.

Viceversa, equations (2) and (4) imply:

$$A^\mu_k = \frac{1}{\sqrt{4\pi N}}\exp\left[-\frac{ik\pi}{P^+}B^+_+(-\pi)\right]\sum_{n=0}^{+\infty}\left(\frac{ik\pi}{P^+}\right)^n \mathbf{R}^{\mu,+,\dots,+}_{[n+1]+}, \tag{24}$$

$$\begin{aligned}\tilde{L}_k =& \frac{P^+}{\sqrt{\pi N}}A^-_k - \tilde{U}^-_k = \\ =& \frac{P^+}{2\pi N}\exp\left[-\frac{ik\pi}{P^+}B^+_+(-\pi)\right]\sum_{n=0}^{+\infty}\left(\frac{ik\pi}{P^+}\right)^n\mathbf{R}^{-,+,\dots,+}_{[n+1]+} - \\ & -\frac{1}{2}\sum_{l=-\infty}^{\infty}\frac{1}{4\pi N}\exp\left[-\frac{ik\pi}{P^+}B^+_+(-\pi)\right]\sum_{n,m=0}^{+\infty}\left(\frac{il\pi}{P^+}\right)^n\left(\frac{i(k-l)\pi}{P^+}\right)^m\sum_{a=1,2}\mathbf{R}^{a,+,\dots,+}_{[n+1]+}\mathbf{R}^{a,+,\dots,+}_{[m+1]+}.\end{aligned} \tag{25}$$

Pohlmeyer and Rehren lay stress on the fact that for the d-dimensional string, instead of the Poincaré algebra $SO(1,d-1) \uplus P^d$ (where P^d is the Lie algebra of translation), one should consider the dynamical Poisson algebra $g = SO(1,d-1) \uplus [P^d \oplus h_+ \oplus h_-]$. Here the information about the center of mass coordinates Z^-, $\vec{Z}$ is contained in the Lorentz algebra $SO(1,d-1)$, while $h_\pm$ are the linear span of the invariant tensor charges (1) and of all the derived charges which may be extracted from them: indeed all these charges close over themselves under: 1) tensor product followed by cyclic symmetrization; 2) Poisson brackets (see [5] for the structure constants, which depend on P^μ). Therefore the lack of closure of equations (19) for the set of observables can be cured in a way which does not depend on the abelianization chosen for the constraints, just by going to this algebra g.

The idea of Pohlmeyer and Rehren is to try to quantize the constraints $\chi_\pm(\sigma)$ and to interpret the resulting loop equations as an infinite collection of representations conditions for the infinite-dimensional algebras $h_\pm$ (see [6] for an approach to the problem through the WKB approximation). Therefore one would like to quantize only the algebra g and not a canonical basis, in the spirit of Isham's ideas[9]. A similar situation is present in the loop representation[10] of general relativity studied with the Ashtekar approach[11]. For the closed string the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s are also topological invariant of loops on a given stationary minimal surface of the Nambu action [7]. Therefore these infinite-dimensional algebras contain a description of the global topological properties of the system. Indeed reference [7] shows, at least for the euclidean version of the closed Nambu string complexified to a Riemann surface, that there is a set of $\mathbf{Z}$'s for each homology class of loops on the surface (in particular all the $\mathbf{Z}$'s vanish for the zero homology class). The knowledge of the sets of $\mathbf{Z}$'s for all the homology class allows to rebuild the Riemann surface (patching various charts) modulo the motion of the center of mass. In the Minkowsky case (with signature $(+,-)$ for the metric in (τ,σ)) not all these loops are at constant τ: equations (1),(2) give the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s for a loop at τ =constant; otherwise the integrand has to be changed and the Hamilton equations must be used in evaluating the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s.

In this respect let us remark that our canonical transformation and its consequences for the invariant charges are not restricted to the solutions of the equations of motion.

On the other side, the canonical observables $\vec{A}_n$ describing the localized independent degrees of freedom are not contained in the set of the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s (just as the graviton degrees of freedom are not contained in the loops invariants of the loop representation of general relativity). This is due to the fact that their functional form depends of the chosen abelianization of the original constraints, *i.e.* of the local (in phase-space) identification of the gauge variables. In order to define a localized measurement in a reparametrization invariant theory, in presence of other local invariances, the previous procedure may not be bypassed. So the problem arises of how to recover a set of canonical variables (and the associated choice of the gauge variables) from the infinite-dimensional algebra, without performing the abelianization of the constraints. While this problem is still unsolved for the loop representation of general relativity, Pohlmeyer and Rehren have given an answer for the case of the closed Nambu string [7]. First of all one has to make a choice of the gauge variables, which is hidden in their choice of isothermal coordinates (this is equivalent to the orthonormal gauge) on a Riemann surface. This allows the identification of a set of Fourier coefficients which become our $\vec{A}_n$ in an arbitrary gauge. Then one notices that all the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s with $n \geq 3$ indices, ($n-k$ of which are "+", and k are transverse one (i_1, i_2, ..., i_k)), depend only of $A^{i_1}_{n_1}$, $A^{i_2}_{n_2}$, ..., $A^{i_k}_{n_k}$, with $n_1+n_2+...+n_k=0$. These monomials can be extracted with a limiting procedure for $n \longrightarrow \infty$ from such $\mathbf{Z}$'s. If we set $A^i_k = (A^i_{-k})^* = |A^i_k|e^{i\varphi^i_k}$ ($\varphi^i_{-k} = -\varphi^i_k$), the equations with $k=2$ and $i_1 = i_2$ allows the determination of all the $|A^i_k|$'s. Then all the other equations with $k \geq 2$ can now be solved in the unknown variables $\varphi^{i_1}_{n_1} + \varphi^{i_2}_{n_2} + ... + \varphi^{i_k}_{n_k=-(n_1+n_2+...+n_{k-1})}$. Therefore one can evaluate all the phases as soon as one of them (*e.g.* φ^1_1) is given. As the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s with all transverse indices depend on the same monomials, this phase is not determined by the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s. On the other hand equation (25) shows that this phase is an invariant non-local charge which has to be given together with the choice of the gauge variables. All this procedure may be performed again for the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s with "$-$" indices instead of the transverse ones to evaluate all the A^-_n's (still with an arbitrary over-all phase).

Instead of the limiting procedure one can also extract the previous momomials in the following way (see reference [8]): since we have (for $i,j = 1,2,-$)

$$\mathbf{Z}^{ij+\dots+}_{[n]+} = (\sqrt{4\pi N})^n (2P^+)^{n-2} \left[\frac{A^i_0 A^j_0}{(n-1)!} - \sum_{k\neq 0} A^i_k A^j_{-k} \sum_{l=2}^{n-1} \frac{1}{(l-1)!} \left(\frac{1}{-2i\pi k} \right)^{n-l} \right], \quad (26)$$

let's define, according to Pohlmeyer and Rehren [5], the reduced $\mathbf{Z}$'s:

$$_{[l]}\mathbf{Z}^{ij}_{[n]} = \sum_{k\neq 0} A^i_k A^j_{-k} \left(\frac{1}{k} \right)^{n-l}, \qquad l = 2,3,...,n. \quad (27)$$

We may now define in the disc $\zeta < 1$, the function:

$$F^{ij}(\zeta) = -\sum_{n=l+1}^{+\infty} \zeta^{n-l-1} \quad {}_{[l]}\mathbf{Z}^{ij}_{[n]} = \sum_{k\neq 0} A^i_k A^j_{-k} \frac{1}{\zeta - k}. \quad (28)$$

Now, extending $F^{ij}(\zeta)$ to the whole complex plane we get a meromorphic function; by taking its residue in $\zeta = m$, for $i = j$, we eventually obtain:

$$\begin{aligned} \frac{1}{2i\pi} \cdot Res_{(\zeta=m)}\ F^{ii}(\zeta) =& A^i_m A^i_{-m} = \|A^i_m\|^2 = \\ =& \left\| \exp\left[-\frac{i\pi m}{P^+} B^+_+(-\pi) \right] \cdot \sum_{n=0}^{+\infty} \left(\frac{i\pi m}{P^+} \right)^n \mathbf{R}^{i,+,\dots,+}_{[n+1]+} \right\|^2 ; \end{aligned} \quad (29)$$

showing again the impossibility to reconstruct the over-all phase from the $\mathbf{Z}^{\mu_1,\dots\mu_n}_{[n]\pm}$'s.

In a similar way equation (26) with $i = j = -$ determine the moduli of A^-_k and their relative phases. These phases can be expressed in terms of one of them (for instance φ^-_1). Then equation (26) with $i = 1,2$, $j = -$, allow to express this phase in terms of the undetermined phase of the $\vec{A}_k$ (φ^1_1). Then, by using equation (8) we can express the $\tilde{L}_n$'s in terms of the $\mathbf{Z}$'s and φ_1^1. Therefore the original constraints, which are determined by the generalized Virasoro generators $\tilde{L}_n$ and by the choice of the gauge variables (*i.e.* the $B_\pm^+(\sigma)$), as shown in equation (18), may be expressed in terms of the $\mathbf{Z}$'s, the phase φ_1^1 and the gauge variables $B_\pm^+(\sigma)$.

In reference [8] on the Casimirs elements of the algebra of the invariant charges, the orthonormal gauge Virasoro constraints L_n ($\tilde{L}_n$ in an arbitrary gauge) are replaced with an infinite set of reparametrization invariant and abelain constraints $\mathcal{S}_a^\pm - 1 \approx 0$. The latter are infinite superpositions of the L_n (or $\tilde{L}_n$) which have vanishing Poisson brackets among themselves and carry just as much information as the Virasoro generators. Therefore the constraints can be rebuild from the invariant charges once the choice of the gauge variables as been made and the previously quoted undetermined phase is given. With our canonical transformation an equivalent set of abelian constraints can be obtained by replacing the variables $Y^-(\sigma) \approx 0$, $\Pi^-(\sigma) \approx 0$ with their Fourier coefficients; their conjugate variables are the Fourier coefficients of $\mathcal{P}^+(\sigma)$, $x^+(\sigma)$. Their quantization, as known, can be done without any anomaly appearing, but the Lorentz algebra is realized only in the critical dimension.

The conclusion is that till now there is no well defined general procedure for extracting all the localized degrees of freedom from the infinite-dimensional algebras describing the global topological properties of the system.

Viceversa, once one has succeeded in finding the canonical observables through the abelianization procedure and the many-time approach, the algebra can be decomposed over this set of observables.

The Pohlmeyer-Rehren algebra of invariant charges seems to be the counterpart for the string of the dynamical $SU(3)$ algebra for the armonic oscillators. The $SU(3)$ generators are build as bilinears in the oscillators a^i, a^{*i}, which are not Cauchy data. However, instead of the oscillators one can use the symplectic basis of the action variables $I^i = a^{*i}a^i$ and the Cauchy data φ^i of the angle variables $\phi^i = \arccos \frac{a^i + a^{*i}}{2\sqrt{I^i}}$: the constants of the motion I^i, φ^i, with $\{I^i, \varphi^j\} = \delta^{ij}$, are the counterpart of the string observables $\vec{A}_n$, $\vec{A}_n^* = \vec{A}_{-n}$. The I^i's form the Cartan subalgebra of the $U(3)$ algebra obtained from $SU(3)$ adding to it the Hamiltonian (*i.e.* $\sum_i I^i$); however the Poisson brackets of the φ^i's with the $U(3)$ generators close on the enveloping algebra of the a^i's. In order to understand better the similarities with the Nambu string one should study these problems for the relativistic harmonic oscillators using the approach of reference[12]Let us stress that in the Chern-Simons-Witten[13] topological quantum field theory one has no canonical observables (there is no localized physical degree of freedom): only global topological invariants connected to the link invariants are present. This suggests that in the loop representation of general relativity there is a superposition of this situation and of localized degrees of freedom (the graviton), as in the string.

As a final remark let us check that the non-local invariant charges A_n^μ (and therefore the $\mathbf{Z}$'s, too) are generators of Noether transformations under which the Nambu action is quasi-invariant, so that they all are Lagrangian constants of the motion.

The Nambu Lagangian is

$$\mathcal{L}(\sigma,\tau) = -N\sqrt{-h(\sigma,\tau)} \equiv -P_\mu(\sigma)\dot{x}^\mu(\sigma), \tag{30}$$

with the definitions

$$P_\mu(\sigma) = -\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu(\sigma)}, \qquad\qquad \Pi_\mu(\sigma) = -\frac{\partial\mathcal{L}}{\partial x'^\mu(\sigma)},$$

$$-h = (\dot{x}\cdot x')^2 - \dot{x}^2 x'^2. \tag{31}$$

The second equality in equation (30) follows from the fact that $\mathcal{L}$ is homogeneous of first degree in the velocities. Since for a Noether transformations $\delta x^\mu(\sigma)$ generated by a phase space constant of motion $\bar{G}$ we have

$$\begin{aligned} \delta x(\sigma) &= \left\{x(\sigma), \bar{G}\right\}|_{P=-\frac{\partial\mathcal{L}}{\partial\dot{x}}}, \\ \delta\frac{\partial\mathcal{L}}{\partial\dot{x}(\sigma)} &= -\,\delta P(\sigma)|_{P=-\frac{\partial\mathcal{L}}{\partial\dot{x}}} = -\left\{P(\sigma), \bar{G}\right\}|_{P=-\frac{\partial\mathcal{L}}{\partial\dot{x}}}, \end{aligned} \tag{32}$$

we may write the variation of the Lagrangian in the following form

$$\begin{aligned} \delta\mathcal{L}(\sigma) =& \delta x^\mu(\sigma)L_\mu(\sigma) - \partial_\tau[P_\mu(\sigma)\delta x^\mu(\sigma)] - \partial_\sigma[\Pi_\mu(\sigma)\delta x^\mu(\sigma)] \equiv \\ \equiv& -\delta[P_\mu(\sigma)\dot{x}^\mu(\sigma)] = -\dot{x}^\mu(\sigma)\delta P_\mu(\sigma) + \dot{P}_\mu(\sigma)\delta x^\mu(\sigma) - \partial_\tau[P\mu(\sigma)\delta x^\mu(\sigma)], \end{aligned} \tag{33}$$

where $L_\mu(\sigma) = \dot{P}_\mu(\sigma) + \Pi'_\mu(\sigma) \doteq 0$ are the Euler-Lagrange equations of motion ($\doteq$ means "equal when evaluated on the extremals of the variational principle").

For $\bar{G} = A^\mu_n$ one gets

$$\begin{aligned} \delta^\nu_n x^\mu(\sigma) =& \{x^\mu(\sigma), A^\nu_n\} = -\eta^{\mu\nu}\frac{1}{\sqrt{4\pi N}}\left(\exp\left[\frac{in\pi}{P^+}B^+_+(\sigma)\right] + \exp\left[-\frac{in\pi}{P^+}B^+_-(\sigma)\right]\right) - \\ &- \frac{i\pi n}{P^+}\eta^{\mu+}\Big[\frac{1}{\sqrt{4\pi N}}\left(\int_\pi^\sigma d\sigma' - \int_{-\pi}^{-\sigma} d\sigma'\right)A^\nu_+(\sigma')\exp\left[\frac{in\pi}{P^+}B^+_+(\sigma')\right]\Big] + \\ &+ \frac{i\pi n}{{P^+}^2}\frac{1}{\sqrt{4\pi N}}\eta^{\mu+}\int_{-\pi}^{\pi} d\sigma' B^+_+(\sigma')A^\nu_+(\sigma')\exp\left[\frac{in\pi}{P^+}B^+_+(\sigma')\right]; \end{aligned} \tag{34$'$}$$

$$\begin{aligned} \delta^\nu_n P^\mu(\sigma) =& \{P^\mu(\sigma), A^\nu_n\} = \\ =& -\frac{in\sqrt{\pi N}}{2P^+}\eta^{\mu\nu}\left(A^+_+(\sigma)\exp\left[\frac{in\pi}{P^+}B^+_+(\sigma)\right] + A^+_-(\sigma)\exp\left[-\frac{in\pi}{P^+}B^+_-(\sigma)\right]\right) + \\ &+ \frac{in\sqrt{\pi N}}{2P^+}\eta^{\mu+}\left(A^\nu_+(\sigma)\exp\left[\frac{in\pi}{P^+}B^+_+(\sigma)\right] + A^\nu_-(\sigma)\exp\left[-\frac{in\pi}{P^+}B^+_-(\sigma)\right]\right) \end{aligned} \tag{34$''$}$$

To get this result we need

$$\left\{x^\mu(\sigma), B^+_\pm(\tilde\sigma)\right\} = \int_0^{\tilde\sigma} d\sigma' \left\{x^\mu(\sigma), P^+(\sigma')\right\} = -\,\eta^{\mu+}\left[\theta(\tilde\sigma-\sigma) + \theta(\tilde\sigma+\sigma) - 1\right],$$
$$\text{for } \sigma,\ \tilde\sigma \in (-\pi,\pi).$$

Then we get

$$
\begin{aligned}
\delta^\nu_n \mathcal{L}(\sigma,\tau) = & -\frac{\partial}{\partial\tau}\left[P_\mu(\sigma)\delta^\nu_n x^\mu(\sigma) + J^\nu_{\tau,n}(\sigma,\tau)\right] - \frac{\partial}{\partial\sigma}\left[\Pi_\mu(\sigma)\delta^\nu_n x^\mu(\sigma) + J^\nu_{\sigma,n}(\sigma,\tau)\right] + \\
& + \dot{P}^+ \frac{in}{P^+}\left\{A^\nu_n + \frac{1}{P^+\sqrt{4\pi N}}\int_{-\pi}^{\pi} d\tilde{\sigma} B^+_+(\tilde{\sigma}) A^\nu_+(\tilde{\sigma}) \exp\left[\frac{in\pi}{P^+}B^+_+(\tilde{\sigma})\right] - \right. \\
& \left. -\frac{\pi}{P^+}\frac{1}{\sqrt{4\pi N}}\left(B^+_+(\sigma)A^\nu_+(\sigma)\exp\left[\frac{in\pi}{P^+}B^+_+(\sigma)\right] - B^+_-(\sigma)A^\nu_-(\sigma)\exp\left[-\frac{in\pi}{P^+}B^+_-(\sigma)\right]\right)\right\},
\end{aligned}
\tag{35}
$$

where $\dot{P}^+ = \int_0^\pi d\sigma L^+(\sigma) \doteq 0$ from the conservation of the total momentum and

$$
\begin{aligned}
J^\nu_{\tau,n}(\sigma,\tau) = & \frac{1}{\sqrt{4\pi N}}\left(A^\nu_+(\sigma)\exp\left[\frac{in\pi}{P^+}B^+_+(\sigma)\right] + A^\nu_-(\sigma)\exp\left[-\frac{in\pi}{P^+}B^+_-(\sigma)\right]\right), \\
J^\nu_{\sigma,n}(\sigma,\tau) = & -\Pi_\mu(\sigma)\delta^\nu_n x^\mu(\sigma) - \frac{N\dot{x}^\nu(\sigma)}{\sqrt{4\pi N}}\left(\exp\left[\frac{in\pi}{P^+}B^+_+(\sigma)\right] - \exp\left[-\frac{in\pi}{P^+}B^+_-(\sigma)\right]\right) - \\
& -\frac{i\pi n}{P^+}\dot{\mathcal{P}}^+(\sigma)\left(A^\nu_n + \frac{1}{\sqrt{4\pi N}}\left(\int_{-\pi}^{\pi} d\bar{\sigma} B^+_+(\bar{\sigma}) - \int_\sigma^\pi d\bar{\sigma} - \int_{-\sigma}^{\pi} d\bar{\sigma}\right) A^\nu_+(\bar{\sigma})\exp\left[\frac{in\pi}{P^+}B^+_+(\bar{\sigma})\right]\right),
\end{aligned}
\tag{36}
$$

Equations (33) and (35) imply

$$
\frac{\partial}{\partial\tau}J^\nu_{\tau,n}(\sigma,\tau) + \frac{\partial}{\partial\sigma}J^\nu_{\sigma,n}(\sigma,\tau) \doteq 0.
$$

Moreover, since $J^\nu_{\sigma,n}(\pi,\tau) - J^\nu_{\sigma,n}(-\pi,\tau) = 0$ due to the boundary conditions [1], we have

$$
\dot{A}^\nu_n = \int_{-\pi}^{\pi} d\sigma \frac{\partial}{\partial\tau} J^\nu_{\tau,n} \doteq 0, \tag{37}
$$

as expected.

References

[1] F.Colomo, G.Longhi and L.Lusanna, *"Classical Solutions of the Many-Time Functional Equations of Motion of the Nambu String";* to appear in Int. Journal Mod. Phys. **A** .

[2] F.Colomo, G.Longhi and L.Lusanna, *"Classical Observables for the Nambu String from the multitemporal approach"";* to appear in Int. Journal Mod. Phys. **A** .

[3] E.Del Giudice,P.Di Vecchia and S.Fubini, Ann.Phys.(NY) **70**, 378 (1972).

[4] K.Pohlmeyer, *Phys. Letters* **B119** (1982) 100;
K.Pohlmeyer and K.-H.Rehren, *Commun. Math. Phys.* **114** (1988) 55.

[5] K.Pohlmeyer and K.-H.Rehren, *Commun. Math. Phys.* **105** (1986) 593.

[6] K.Pohlmeyer and K.-H.Rehren, *Commun. Math. Phys.* **105** (1986) 629.

[7] K.Pohlmeyer and K.-H.Rehren, *Commun. Math. Phys.* **114** (1988) 177.

[8] K.Pohlmeyer and K.-H.Rehren, *Commun. Math. Phys.* **114** (1988) 351.

[9] C.J.Isham, in *Relativity, Groups and Topology II*, pg.1059, Les Houches XL 1983, eds. B.S.De Witt and R.Stora, Elsevier, Amsterdam 1984.
[10] C.Rovelli and L.Smolin, *Phys. Rev. Lett.* **61** (1988) 1155;
C.Rovelli and L.Smolin, *Nucl. Phys.* **B** in press (1989);
J.Jacobson and L.Smolin, *Nucl. Phys.* **B299** (1988) 295.
[11] A.Ashtekar, *New Perspectives in Canonical Gravity*, Bibliopolis, Napoli 1986.
[12] G.Longhi and L.Lusanna, *Phys. Rev.***34** (1986) 3707.
[13] E.Witten, *Comm. Math. Phys.* **121** (1989) 351.