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Correlators in the Heisenberg XX0 Chain as Fredholm Determinants

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Abstract

The two-point time and temperature dependent correlation functions for the XX0 one-dimensional model in constant magnetic field are represented (in the thermodynamical limit) as Fredholm determinants of linear integral operators.

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Recently essential developments have been made in the theory of quantum correlation functions showing that correlators of quantum exactly solvable models satisfy classical completely integrable differential equations [1]-[6] (this program for the example of the nonrelativistic Bose gas is now fulfilled and presented in Ref. [7]). An important preliminary step to obtain these differential equations is to represent correlation functions as the determinants of Fredholm linear integral operators. For the nonrelativistic Bose gas these representations were given in papers [8, 9] in the time independent case and in [10] in the time-dependent case.

In this paper the determinant representations of this kind are obtained for the distance, time and temperature dependent two-point correlation functions of the XX0 Heisenberg chain. In order to write differential equations and to calculate their asymptotics our plan further is to construct and solve a matrix Riemann problem, similarly to the case of the nonrelativistic Bose gas [5], [11]-[13] (see also Ref. [7]).

We would like to mention that the autocorrelator (time-dependent correlator of local spins at the same site of the lattice) was presented in paper [2] as a Fredholm determinant and this representation (different from the one obtained below in this particular case) was used in [6] to produce the differential equations for the autocorrelator.

The XX0 chain is the isotropic case of the XY model [14], being also the free fermions point for the XXZ chain. The Hamiltonian describing the nearest neighbour interaction of local 1/2 spins situated at the sites of the one-dimensional periodical lattice in transverse magnetic field, with M (even) sites, is given as

$$H(h) = -\sum_{m=1}^{M} [\sigma_x^{(m)}\sigma_x^{(m+1)} + \sigma_y^{(m)}\sigma_y^{(m+1)} + h\sigma_z^{(m)}] \,. \tag{1}$$

Pauli matrices are normalized as $(\sigma_s^{(m)})^2 = 1$ $(s = x,\ y,\ z)$. Moreover, we define $\sigma_\pm^{(m)} \equiv \frac{1}{2}[\sigma_x^{(m)} \pm i\sigma_y^{(m)}]$. Due to the similarity transformation,

$$H(h) \ \rightarrow \ H(-h) = UH(h)U^{-1}; \qquad\qquad U = \prod_{m=1}^{M} \sigma_x^{(m)} \,,$$

it is sufficient to consider only nonnegative magnetic fields, $h \geq 0$. Furthermore the choice of the minus sign at the r.h.s. of eq. (1) is just a matter of convenience due to the property

$$H(h) \rightarrow -H(-h) = VH(h)V^{-1}; \qquad V = \prod_{m=1}^{M/2} \sigma_z^{(2m)} .$$

The ferromagnetic state $\mid 0\rangle \equiv \otimes_{m=1}^{M} |\uparrow\rangle_m$ (all spins up) is an eigenstate of the Hamiltonian. All the other eigenstates can be obtained by filling this ferromagnetic state with N quasiparticles ($N = 1, 2, ..., M$) with different quasimomenta p_a, $-\pi < p_a \leq \pi$, ($a = 1, ..., N$) and energies $\varepsilon(p_a)$,

$$\varepsilon(p) \equiv \varepsilon(p,h) = -4\cos p + 2h . \tag{2}$$

Periodical boundary conditions imply:

$$\exp[iMp_a] = (-1)^{N+1}, \qquad a = 1, ..., N \tag{3}$$

for the allowed values of quasimomenta. All the momenta of the quasiparticles of a given eigenstate should be different, so that, *e.g.*, for $N = M$ one gets only one eigenstate $\mid 0'\rangle_M = \otimes_{m=1}^{M} |\downarrow\rangle_m$ which is the ferromagnetic state with all spins down.

The model in the thermodynamical limit ($M \to \infty$, h fixed) is the most interesting. For $h \geq h_c \equiv 2$, ferromagnetic state $\mid 0\rangle$ (all spins up) is the ground state of the Hamiltonian. For $0 \leq h < h_c$, the ground state $\mid \Omega\rangle$ is obtained by filling the ferromagnetic state with quasiparticles possessing all the allowed momenta inside the Fermi zone, $-k_F \leq p \leq k_F$, where

$$k_F = \arccos(h/2) ; \qquad h \leq h_c = 2 , \tag{4}$$

is the Fermi momentum. At non zero temperature $T > 0$, the density of quasiparticles in the momentum space is given as $\vartheta(p)/2\pi$, where $\vartheta(p) \equiv \vartheta(p,h,T)$ is the Fermi weight:

$$\vartheta(p) = \frac{1}{1 + \exp[\varepsilon(p)/T]} , \tag{5}$$

Temperature and time dependent correlators of local spins $\sigma_s^{(m)}(t) \equiv \exp[iHt]\, \sigma_s^{(m)} \exp[-iHt]$, $\sigma_s^{(m)} \equiv \sigma_s^{(m)}(0)$, $s = x, y, z$, are defined as usual:

$$g_{sr}^{(T)}(m,t) \equiv \langle \sigma_s^{(n_2)}(t_2)\sigma_r^{(n_1)}(t_1)\rangle_T = \frac{\mathrm{Sp}\left\{\exp[-H/T]\sigma_s^{(n_2)}(t_2)\sigma_r^{(n_1)}(t_1)\right\}}{\mathrm{Sp}\left\{\exp[-H/T]\right\}} . \tag{6}$$

Due to translation invariance the correlators depend only on the differences,

$$m \equiv n_2 - n_1 \;, \qquad t = t_2 - t_1 \;. \tag{7}$$

At zero temperature, only the ground state contributes to the traces in (6),

$$g_{sr}^{(0)}(m,t) \equiv \frac{\langle \Omega \mid \sigma_s^{(n_2)}(t_2)\sigma_r^{(n_1)}(t_1) \mid \Omega \rangle}{\langle \Omega \mid \Omega \rangle} \qquad (T=0)\;. \tag{8}$$

In Ref. [14] the time-independent correlators of XY model were calculated at $h=0$. The simple answer for the correlator of the third spin components was given; for the XX0 chain it reduces essentially to the square modulus of the Fourier transform of the Fermi weight. The result was generalized to the case of nonzero transverse magnetic field and to the time-dependent correlator [15]; in our notation, for the XX0 model the last result may be written

$$\begin{aligned} g_{zz}^{(T)}(m,t) &= \langle \sigma_z \rangle_T^2 - \frac{1}{\pi^2} \left| \int_{-\pi}^{\pi} dp\; \exp[imp + 4it\cos p]\vartheta(p) \right|^2 + \\ &+ \frac{1}{\pi^2} \left(\int_{-\pi}^{\pi} dp\; \exp[-imp - 4it\cos p]\vartheta(p) \right) \left(\int_{-\pi}^{\pi} dq\; \exp[imq + 4it\cos q] \right) \end{aligned} \tag{9}$$

(for $t=0$, the last term in the r.h.s. is equal to zero). Here

$$\begin{aligned} \langle \sigma_z \rangle_T &\equiv \langle \sigma_z^{(n)}(t) \rangle_T = 1 - \frac{1}{\pi} \int_{-\pi}^{\pi} dp\, \vartheta(p)\;, \\ \langle \sigma_z \rangle_0 &= 1 - \frac{2k_F}{\pi}\;, \end{aligned} \tag{10}$$

is the magnetization (not depending neither on n nor on t due to translation invariance). Properties of these quantities were considered in much detail [14]-[17]. Real systems for experimental comparisons were found [18].

Correlators of the other local spin components are indeed more complicated. In Ref. [14] these correlators (for the XY model at $t=0$, $h=0$) were represented as the determinants of $m \times m$ matrices (m is the distance between correlating spins). This representation was investigated in detail in [16] (see also [19]). In Ref. [20] the structure of the time-dependent correlators was investigated on the basis of an extension of the thermodynamic Wick theorem. In Ref. [2], representation of the autocorrelator ($m=0$, $t \neq 0$) in the transverse Ising chain in critical magnetic field (closely related to correlators in the XX0 chain at h=0) were given as Fredholm determinants of a linear integral operator.

In this paper the correlators (see (6), (7) for the notations)

$$g_+^{(T)}(m,t) = \langle \sigma_+^{(n_2)}(t_2)\sigma_-^{(n_1)}(t_1)\rangle_T \ , \tag{11}$$

$$g_-^{(T)}(m,t) = \langle \sigma_-^{(n_2)}(t_2)\sigma_+^{(n_1)}(t_1)\rangle_T \ , \tag{12}$$

for the XX0 model in a transverse magnetic field are given as Fredholm determinants of linear integral operators. These representations, quite different from those of paper [14], are instead similar to the representations of two-point correlators previously obtained for the one-dimensional Bose gas [8]-[10].

In order to obtain these representations we proceed as follows. The explicit form for the eigenfunctions of Hamiltonian (1) is well known, being just the simplest case of eigenfunctions of the XXZ model [21] with vanishing of two-particle scattering phases. Using this explicit form one can represent the normalized mean value of, *e.g.*, operator $\sigma_+^{(n_2)}(t_2)\sigma_-^{(n_1)}(t_1)$ on the periodical lattice with finite number M of sites (with respect to any eigenfunction with N quasiparticles over the ferromagnetic vacuum) as the determinant of a $N \times N$ matrix. Then, in the thermodynamical limit, correlator (11) is given by the Fredholm determinant of a linear integral operator.

Corresponding derivations (similar to the case of the impenetrable bosons of ref. [10]) as well as the answer for finite lattice will be given in a more detailed paper. Here only the results in the thermodynamical limit are presented. Due to space and time reflection invariance, correlators (11) and (12) possess the following property:

$$g_\pm^{(T)}(m,t) = g_\pm^{(T)}(-m,t) = \left[g_\pm^{(T)}(-m,-t)\right]^* \ , \tag{13}$$

so that all the answers are given for $m \geq 0$.

We start with correlator (11), which, at zero temperature, is represented as follows:

$$g_+^{(0)}(m,t) = \exp[-2iht]\left[G(m,t) + \frac{\partial}{\partial z}\right] \det\left[\hat{I} + \hat{V} - z\hat{R}^{(+)}\right]\Big|_{z=0} \ . \tag{14}$$

In the r.h.s. there is a Fredholm determinant. Linear operators $\hat{V}$ and $\hat{R}^{(+)}$ act on functions $f(p)$ on the interval $-k_F \leq p \leq k_F$ (k_F is the Fermi momentum (4)) as, *e.g.*,

$$\left(\hat{V}f\right)(p) = \frac{1}{2\pi}\int_{-k_F}^{k_F} dq\, V(p,q)f(q) \ . \tag{15}$$

Operator $\hat{I}$ is the identity operator (with kernel $\delta(p-q)$). The kernels of operators $\hat{V}$, $\hat{R}^{(+)}$, are

$$\begin{aligned} V(p,q) &= \frac{E_+(p)E_-(q)-E_-(p)E_+(q)}{\tan\frac{1}{2}(p-q)} - G(m,t)E_-(p)E_-(q)\ , && (16) \\ R^{(+)}(p,q) &= E_+(p)E_+(q)\ , && (17) \end{aligned}$$

where functions E_+, E_-, are given as

$$\begin{aligned} E_-(p) &\equiv E_-(m,t,p) = \exp[-\frac{i}{2}mp - 2it\cos p]\ , \\ E_+(p) &\equiv E_+(m,t,p) = E_-(p)E(m,t,p) && (18) \end{aligned}$$

Functions $G(m,t)$ and $E(m,t,p)$ are defined as follow:

$$G(m,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dq\ \exp[imq + 4it\cos q] = i^m J_m(4t)\ , \qquad (19)$$

(J_m is the Bessel function) and

$$\begin{aligned} E(m,t,p) &= \frac{1}{2\pi}\mathcal{P}\int_{-\pi}^{\pi} dq\ \frac{\exp[imq+4it\cos q]}{\tan\frac{1}{2}(q-p)} \equiv \\ &\equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} dq\ \frac{\exp[imq+4it\cos q]-\exp[imp+4it\cos p]}{\tan\frac{1}{2}(q-p)}\ ; && (20) \end{aligned}$$

here $\mathcal{P}$ means the principal value. It should be mentioned that $k_F = 0$ for $h \geq h_c \equiv 2$. In this case the ground state is the ferromagnetic state $\mid 0\rangle$ and the correlator is just the "wave packet":

$$\begin{aligned} g_+^{(0)}(m,t) &= \exp[-2iht]G(m,t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dq\ \exp[imq - it\varepsilon(q)]\ , \\ g_+^{(0)}(m,t=0) &= \delta_{m,0}\ , \qquad h \geq h_c\ . && (21) \end{aligned}$$

In the equal time case ($t = 0$) functions $G(m,0)$ and $E(m,0,p)$ can be explicitly calculated:

$$G(m,0) = \delta_{m,0}\ ; \qquad E(m,0,p) = i(1-\delta_{m,0})\exp[imp]\ ,$$

so that one obtains for the equal-time correlator at zero temperature

$$\begin{aligned} g_+^{(0)}(m,0) &= \frac{\partial}{\partial z}\det\left[\hat{I}+\hat{v}+z\hat{r}^{(+)}\right]\Big|_{z=0}\ , \qquad m>0 \\ g_+^{(0)}(0,0) &= \frac{1}{2}+\frac{1}{2}\langle\sigma_z\rangle_{T=0} = 1-\frac{k_F}{\pi}\ . && (22) \end{aligned}$$

Operators $\hat{v}$, $\hat{r}$, act on the interval $[-k_F, k_F]$, with kernels

$$\begin{aligned} v(p,q) &= -2\frac{\sin\frac{m}{2}(p-q)}{\tan\frac{1}{2}(p-q)}\,, \\ r^{(+)}(p,q) &= \exp[\frac{i}{2}m(p+q)]\,. \end{aligned} \tag{23}$$

In the case of non zero temperature ($T > 0$) the representations are similar:

$$g_+^{(T)}(m,t) = \exp[-2iht][G(m,t) + \frac{\partial}{\partial z}] \det\left[\hat{I} + \hat{V}_T - z\hat{R}_T^{(+)}\right]\Big|_{z=0}\,, \qquad m \geq 0\,, \tag{24}$$

$$\begin{aligned} g_+^{(T)}(m,0) &= \frac{\partial}{\partial z} \det\left[\hat{I} + \hat{v}_T + z\hat{r}_T^{(+)}\right]\Big|_{z=0}\,, \qquad\qquad m > 0\,, \\ g_+^{(T)}(0,0) &= \frac{1}{2} + \frac{1}{2}\langle\sigma_z\rangle_T\,. \end{aligned} \tag{25}$$

Operators $\hat{V}_T$, $\hat{v}_T$, $\hat{R}_T^{(+)}$, $\hat{r}_T^{(+)}$, act over the interval $[-\pi,\ \pi]$,

$$(\hat{V}_T f)(p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} dq\, V_T(p,q) f(q)\,, \tag{26}$$

their kernels being equal to

$$\begin{aligned} V_T(p,q) &= \sqrt{\vartheta(p)}\, V(p,q)\, \sqrt{\vartheta(q)}\,, \\ R_T^{(+)}(p,q) &= \sqrt{\vartheta(p)}\, R^{(+)}(p,q)\, \sqrt{\vartheta(q)}\,, \\ v_T(p,q) &= \sqrt{\vartheta(p)}\, v(p,q)\, \sqrt{\vartheta(q)}\,, \\ r_T^{(+)}(p,q) &= \sqrt{\vartheta(p)}\, r^{(+)}(p,q)\, \sqrt{\vartheta(q)}\,, \end{aligned} \tag{27}$$

where $\vartheta(p)$ is the Fermi weight (5) and functions $V(p,q)$, $R^{(+)}(p,q)$, $v(p,q)$, $r^{(+)}(p,q)$ are defined in eqs. (16), (17), (23).

Analogous representations are valid also for correlator (12):

$$g_-^{(0)}(m,t) = \exp[2iht]\,\frac{\partial}{\partial z}\,\det\left[\hat{I} + \hat{V} + z\hat{R}^{(-)}\right]\Big|_{z=0}\,, \tag{28}$$

$$g_-^{(T)}(m,t) = \exp[2iht]\,\frac{\partial}{\partial z}\,\det\left[(\hat{I} + \hat{V}_T + z\hat{R}_T^{(-)}\right]\Big|_{z=0}\,, \tag{29}$$

where $\hat{V}$ and $\hat{V}_T$ are the same operators as in (14), (24) and the kernels of operators $\hat{R}^{(-)}$ and $\hat{R}_T^{(-)}$ (acting over the interval $[-k_F,\ k_F]$, see (15), and $[-\pi,\ \pi]$, see (26), respectively) are

$$R^{(-)}(p,q) = E_-(p)E_-(q)\,, \tag{30}$$

$$R_T^{(-)}(p,q) = \sqrt{\vartheta(p)}\, R^{(-)}(p,q)\, \sqrt{\vartheta(q)}\,, \tag{31}$$

with functions $E_-(q)$ defined in (18). It is worth mentioning that the zero-temperature correlator (28) is equal to zero for magnetic field $h \geq h_c = 2$.

The Fredholm determinant representations of the kind of eqs. (22), (25), can be obtained also for the equal time "generating functional" $\langle \exp[\alpha Q(m)] \rangle_T$ of the third spin component correlators, where $Q(m)$ is the operator of the "number of quasiparticles" at the first m sites of the lattice:

$$Q(m) \equiv \sum_{n=1}^{m} \frac{1}{2}(1 - \sigma_z^{(n)}) . \tag{32}$$

For this expectation value one gets in the thermodynamical limit, at $T = 0$,

$$\langle \exp[\alpha Q(m)] \rangle_0 = \det \left[\hat{I} + \gamma \hat{U}(m) \right] \Big|_{\gamma = e^\alpha - 1} , \tag{33}$$

and for non zero temperature

$$\langle \exp[\alpha Q(m)] \rangle_T = \det \left[\hat{I} + \gamma \hat{U}_T(m) \right] \Big|_{\gamma = e^\alpha - 1} . \tag{34}$$

The kernels of operators $\hat{U}(m)$ and $\hat{U}_T(m)$ (acting on intervals $[-k_F,\, k_F]$ and $[-\pi,\, \pi]$ respectively) turn out to be

$$\begin{aligned} U(p, q; m) &= \frac{\sin \frac{m}{2}(p - q)}{\sin \frac{1}{2}(p - q)} , && (35) \\ U_T(p, q; m) &= \sqrt{\vartheta(p)}\, \frac{\sin \frac{m}{2}(p - q)}{\sin \frac{1}{2}(p - q)} \sqrt{\vartheta(q)} . && (36) \end{aligned}$$

It is well known [14] that the XX0 model is equivalent to the free fermion model, the free fermion fields being related to the local spin operators by means of Jordan-Wigner transform:

$$\begin{aligned} \psi(m) &= \exp[i\pi Q(m-1)]\sigma_+^{(m)} , \\ \psi^\dagger(m) &= \sigma_-^{(m)} \exp[i\pi Q(m-1)] . \end{aligned} \tag{37}$$

Due to this equivalence the linear Fredholm operators in representations (22), (35) (and (25), (36), correspondingly) should be in fact the same (for $\alpha = i\pi$ or $\gamma = -2$). Indeed, it is possible to rewrite representations (22), (25), in the equivalent form

$$g_+^{(0)}(m, 0) = \frac{\partial}{\partial z} \det \left[\hat{I} - 2\hat{U}(m-1) + z \hat{r}^{(+)} \right] , \tag{38}$$

$$g_+^{(T)}(m, 0) = \frac{\partial}{\partial z} \det \left[\hat{I} - 2\hat{U}_T(m-1) + z \hat{r}_T^{(+)} \right] , \tag{39}$$

(appearance of argument $(m-1)$ in operator $\hat{U}$ is quite natural due to (37)).

It was already mentioned that representations similar to those obtained above gave an opportunity to obtain differential equations for correlation functions in the case of impenetrable bosons (the V Painlevé transcendent in the equal time zero temperature case [1] and integrable partial differential equations for time and temperature dependent correlators [4, 5]). This allowed to construct exact asymptotics for the correlators [1, 12, 13]. Corresponding results are expected to be obtained for the XX0 chain.

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