On some representations of the six vertex model partition function

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Abstract

The partition function of the six vertex model on the finite lattice with domain wall boundary conditions is considered. Starting from Hankel determinant representation, some alternative representations for the partition function are given. It is argued that one of these representations can be rephrased in the language of the angular quantization method applied to certain fermionic model.

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The six-vertex model on a square lattice, with domain wall boundary conditions (DWBC), was introduced and solved by Korepin and Izergin [1, 2]. This model, with these very peculiar boundary conditions, (as opposed to, e.g., periodic ones [3, 4, 5]) is particularly interesting, both for its relationship with the theory of Alternating Sign Matrices [6, 7], and with some tiling problems [8], and for the fact that bulk quantities are sensitive to boundary conditions even in the thermodynamic limit [9, 10]. This last question is in turn related to the so called Arctic Circle Theorem [11].

An explicit expression for the partition function of the model on an $N \times N$ square lattice, Z_N, was given in [2, 12] in terms of the determinant of an $N \times N$ Hankel matrix. Such representation, in connection with semi-infinite Toda chain and with random matrix models, has been sucessfully used in [9, 10] to evaluate the bulk free energy of the model. Recently, the Izergin-Korepin determinant formula has been generalized to the case of boundary one-point correlation functions (polarizations) [13, 14]. However such expressions turns out to be too intricate, preventing any further, more explicit, answer, except for very particular cases. This may indicate in fact that the representation of partition function and correlators in term of Hankel determinants, even if a natural outcome from the analysis of the more general inhomogeneous model, is not the most convenient form to investigate its homogeneous version (the most interesting from the point of view of statistical mechanics). This suggests to look for other approaches.

In the present letter we derive some alternative determinant representations for the partition function starting from the Hankel determinant formula, and we argue that one of these representations can quite naturally be rephrased in the language of the angular quantization method applied to certain fermionic model. First, following the lines suggested in Ref. [15], we express the partition function as the Fredholm determinant of an integrable integral operator. Such representations has proven fruitful in the investigation of correlation functions of integrable models [16]. Second, we give one more representation for the partition function as the ordinary determinant of a $N \times N$ symmetric matrix. Its entries, in contrast to those of the original Hankel matrix, can be explicitely evaluated with a neat result. Finally, by closely investigating this last representation, we suggest how the core of the partition function can be reinterpreted as the trace of some boost-like transformation over the Fock space of N free fermions. Due to this interpretation, it turns out clear that the whole construction fits well into the language of the angular quantization approach, widely used in integrable quantum field theory. We believe this connection with the angular quantization approach may turn out useful in the quest for a systematic derivation of the correlation functions of the model.

With no loss of generality, we can restrict ourselves to the case of the disordered (or critical) regime. The procedure we follow, suggested by [15, 17], will be only sketched.

By analytic continuation our results can be extended to all regimes of the model. The disordered regime of the six-vertex model is characterized by values of the quantity $\Delta \equiv (a^2+b^2-c^2)/2ab$ in the interval $-1 < \Delta < 1$, where the Boltzmann vertex weights a, b, c, are conveniently parametrized in terms of the spectral parameter λ and the deformation parameter η as follow:

$$a = \sin(\lambda+\eta)\ , \qquad b = \sin(\lambda-\eta)\ , \qquad c = \sin(2\eta)\ , \tag{1}$$

with λ, η such that the Boltzmann weights are real and positive. The partition function Z_N of the model on a $N\times N$ square lattice with DWBC has the following expression [2, 12]:

$$Z_N = \frac{[\sin(\lambda-\eta)\sin(\lambda+\eta)]^{N^2}}{\prod_{m=0}^{N-1}(m!)^2}\,\tau_N \tag{2}$$

where τ_N is the determinant of an $N\times N$ Hankel matrix:

$$\tau_N = \det{}_N\left[\partial_\lambda^{j+k}\,\frac{\sin(2\eta)}{\sin(\lambda-\eta)\sin(\lambda+\eta)}\right]\ , \qquad j,k=0,1,\dots,N-1. \tag{3}$$

Introducing the notation

$$\phi_\pm \equiv \lambda \pm \eta\ , \tag{4}$$

and using the identities

$$\det{}_N\left[\partial_\phi^{j+k}\mathrm{e}^{-i\phi}f(\phi)\right] = \det{}_N\left[\mathrm{e}^{-i\phi}\partial_\phi^{j+k}f(\phi)\right] = \mathrm{e}^{-iN\phi}\det{}_N\left[\partial_\phi^{j+k}f(\phi)\right] \tag{5}$$

we may factorize the Hankel determinant, and write:

$$\begin{aligned}\tau_N &= \det{}_N[\mathrm{e}^{-i\phi_-}A_- - \mathrm{e}^{-i\phi_+}A_+]\\ &= \mathrm{e}^{-iN\phi_-}\ \det{}_N[A_-]\ \det{}_N[1-\zeta A_-^{-1}A_+]\end{aligned} \tag{6}$$

where $\zeta = \mathrm{e}^{i(\phi_- - \phi_+)} = \mathrm{e}^{-2i\eta}$ and‡

$$(A_\pm)_{jk} = \partial_\pm^{j+k}\left[\frac{1}{\sin\phi_\pm}\right] \qquad j,k=0,1,\dots,N-1. \tag{7}$$

Using standard techniques related to the theory of orthogonal polynomials [15, 17], both the determinant and the inverse of a matrix of the form (7) can be explicitely worked out. The whole approach is founded on the possibility of expressing the first element of matrix (7), $(A_\phi)_{00}$, as a Laplace transform; in our case we may write:

$$\frac{1}{\sin\phi} = \int_{-\infty}^{+\infty}\frac{\mathrm{e}^{\phi x}}{1+\mathrm{e}^{\pi x}}\,dx\ , \tag{8}$$

‡Whenever no ambiguity could arise, we shall use $\pm$ instead of $\phi_\pm$ in subscripts and superscripts

therefore identifying an integration measure,

$$\mu^{(\phi)}(x) = \frac{\mathrm{e}^{\phi x}}{1+\mathrm{e}^{\pi x}} \ , \tag{9}$$

which (for $0 < \phi < \pi$, indeed fulfilled in the critical regime) decays exponentially fast as $x \to \pm\infty$ and allows to define a complete set of orthogonal polynomials. In our specific case, the polynomials $\mathcal{P}_n^{(\phi)}(x)$ associated to measure (9) essentially coincide with the so called Meixner-Pollaczek polynomials [18, 19], $P_n^{(\lambda)}(x;\phi)$, where the parameter λ (not to be confused with the spectral parameter appearing in the Boltzmann weights, eq. (1)) must be set equal to the value 1/2:

$$\begin{aligned}\mathcal{P}_n^{(\phi)}(x) &= \sqrt{\sin\phi}\ P_n^{(1/2)}\left(\frac{x}{2};\phi\right) \\ &= \sqrt{\sin\phi}\ \mathrm{e}^{in\phi}\ {}_2F_1\left(\begin{matrix} -n, & 1/2+ix/2 \\ & 1 \end{matrix} \ \middle|\ 1-\mathrm{e}^{-2i\phi} \right)\end{aligned} \tag{10}$$

They satisfy the following orthogonality relation:

$$\int_{-\infty}^{+\infty} \mathcal{P}_n^{(\phi)}(x)\ \mathcal{P}_m^{(\phi)}(x)\ \mu^{(\phi)}(x)\ dx = \delta_{nm} \ , \tag{11}$$

the three-terms recurrence relation:

$$(n+1)\mathcal{P}_{n+1}^{(\phi)}(x) = [(2n+1)\cos\phi + x\sin\phi]\,\mathcal{P}_n^{(\phi)}(x) - n\mathcal{P}_{n-1}^{(\phi)}(x) \ , \tag{12}$$

and the Christoffel-Darboux identity:

$$\sum_{k=0}^{N-1} \mathcal{P}_k^{(\phi)}(x)\ \mathcal{P}_k^{(\phi)}(y) = \frac{N}{\sin\phi}\frac{\mathcal{P}_N^{(\phi)}(x)\mathcal{P}_{N-1}^{(\phi)}(y) - \mathcal{P}_{N-1}^{(\phi)}(x)\mathcal{P}_N^{(\phi)}(y)}{x-y} \ . \tag{13}$$

For future refererence, let us moreover introduce $k_j = (\sin\phi)^{j+\frac{1}{2}}/j!$, the leading coefficient of x^j in $\mathcal{P}_j^{(\phi)}(x)$.

Once the measure (9) has been identified, it is evident that the determinant of matrix A_ϕ may be rewritten as the determinant of the matrix built from the moments of the integration weight $\mu_\phi(x)$. Using standard properties of orthogonal polynomials one readily evaluates

$$\begin{aligned}\det{}_N\,[A_\phi] &= \det{}_N\left[\int_{-\infty}^{+\infty} \frac{1}{k_j k_k}\ \mathcal{P}_j^{(\phi)}(x)\mathcal{P}_k^{(\phi)}(x)\ \mu^{(\phi)}(x)\right]\ dx \\ &= \frac{1}{[\sin\phi]^{N^2}}\prod_{m=0}^{N-1}(m!)^2 \ .\end{aligned} \tag{14}$$

Moreover the inverse matrix A_ϕ^{-1} can be simply expressed in terms of derivatives of the kernel

$$K_N^{(\phi)}(x,y) = \sum_{k=0}^{N-1} \mathcal{P}_k^{(\phi)}(x)\ \mathcal{P}_k^{(\phi)}(y) \tag{15}$$

since the following relation holds:

$$K_N^{(\phi)}(x,y) = \sum_{j,k=0}^{N-1} (A_\phi^{-1})_{jk}\; x^j\; y^k \tag{16}$$

provided that the kernel $K_N^{(\phi)}(x,y)$ is built just in terms of the orthogonal polynomials relative to the measure $\mu^{(\phi)}(x)$ exactly given by the Laplace transform of $(A_\phi)_{00}$. A last ingredient of the whole construction resides in the following: let us introduce the linear integral operator V_N, whose kernel is given by

$$V_N(x,y) = K_N^{(-)}(x,y)\mu^{(+)}(y)\Big|_{\mathbf{R}}\;, \tag{17}$$

and the matrix:

$$(W)_{jk} = \int_{-\infty}^{+\infty} \mathcal{P}_j^{(-)}(x)\mathcal{P}_k^{(-)}(x)\;\mu^{(+)}(x)\;dx\;, \tag{18}$$

which is in fact semi-infinite, but will be considered in the following as truncated to its first $N \times N$ entries, $j,k = 0,1,\ldots,N-1$. The following identities

$$\mathrm{tr}_N\left[(A_-^{-1}\,A_+)^m\right] = \mathrm{tr}\left[(V_N)^m\right] = \mathrm{tr}_N\left[(W)^m\right] \tag{19}$$

are easily proven for any positive integer power m, and imply

$$\det{}_N\left[1-\zeta\,A_-^{-1}\,A_+\right] = \det\left[1-\zeta\,V_N\right] = \det{}_N\left[1-\zeta\,W\right]\;. \tag{20}$$

Using all these ingredients, the partition function of the six-vertex model with DWBC on the $N \times N$ lattice, eq. (2), may therefore equivalently be expressed in terms of a Fredholm determinant:

$$Z_N = \left[\sin\phi_+\right]^{N^2}\mathrm{e}^{-iN\phi_-}\;\det\left[1-\zeta\,V_N\right] \tag{21}$$

where the kernel of integral operator V_N may be written more explicitely as:

$$V_N(x,y) = \frac{N}{\sin\phi_-}\frac{\mathcal{P}_N^{(-)}(x)\mathcal{P}_{N-1}^{(-)}(y)-\mathcal{P}_{N-1}^{(-)}(x)\mathcal{P}_N^{(-)}(y)}{x-y}\mu^{(+)}(x)\Bigg|_{\mathbf{R}}\;; \tag{22}$$

its integrability in the sense of Ref. [16] is self-evident.

It is worth emphasizing that in the rational case, obtained by substituting $\lambda \to \epsilon\lambda$, $\eta \to \epsilon\eta$, suitably rescaling variables and vertex weights, and taking the limit $\epsilon \to 0$, the measure and polynomials become exactly Laguerre ones. The result of [15] is therefore reproduced, but we have been able here to generalize it to the whole disordered regime, still preserving the integrable structure of the linear integral operator.

Analogous expressions can be given in other regimes, namely antiferroelectric and ferroelectric ones, through analytic continuation, in terms of Meixner polynomials [18]. The main difference resides in the discrete character of the measure in these cases.

From the previous discussion it appears that a pre-eminent role is played by Meixner-Pollaczek polynomials. It is to be mentioned these polynomials had already appeared in connection with the free-fermion six vertex model [20, 21]; however in our case they arise for generic vertex weights.

Let us now give one more alternative representation for the partition function of the six-vertex models with DWBC on the $N \times N$ lattice, this time in terms of the determinant of an N dimensional matrix:

$$Z_N = [\sin \phi_+]^{N^2} \, \mathrm{e}^{-iN\phi_-} \, \det{}_N \left[1 - \zeta W \right] \tag{23}$$

with W given in eq. (18). Interestingly enough, the integral appearing in the definition of the matrix elements of W can be evaluated explicitely in terms of the Boltzmann weights of the model, eq.(1), with a result of pleasing simplicity:

$$W_{jk} = \frac{b}{a} \, \left(\frac{c}{a} \right)^{j+k} {}_2F_1 \left(\begin{array}{c} -j, \ -k \\ 1 \end{array} \middle| \left(\frac{b}{c} \right)^2 \right) \,, \tag{24}$$

where the truncated hypergeometric function is in fact Meixner polynomial [18].

Using a “quantum mechanical” picture, the connection between the two proposed representations, eq. (21) and (23) is quite natural: let us consider a one dimensional quantum particle in a confining potential, such that its energy eigenfunctions are just given by $\mathcal{P}_N^{(-)}(x) \sqrt{\mu^{(-)}(x)}$. Let us moreover introduce the operator ϱ which in the “coordinates” representation acts multiplicatively as $\mu^{(+)}(x)/\mu^{(-)}(x) = \mathrm{e}^{2i(\phi_+ - \phi_-)x}$. We have thus shown that the partition function of the six-vertex model on the $N \times N$ square lattice with DWBC is essentially given (up to trivial prefactors) by the determinant of the projection of operator $\mathbf{1} - \varrho$ onto the subspace of the N lowest energy levels of the considered quantum mechanical system. Working in the “coordinates” or “energy” representation gives rise to expressions (21) and (23) respectively. In the coordinate representation operator ϱ is diagonal, but of course the projector on the N lowest levels, $K_N^{(-)}(x, y)\mu^{(-)}(y)$, is not. Conversely, in the energy representation, the projector is manifestly diagonal, while ϱ is not, its matrix element on the j^{th}, k^{th} eigenlevel $\langle j|\varrho|k\rangle$ being given just by $(W)_{jk}$, eq. (18).

We shall end by discussing a reinterpretation of the determinant representation for the partition function Z_N, inspired by the finite size determinant formula, eq. (23). From eq. (24), and the definition of hypergeometric function, follows one more representation for matrix W, in a Gauss decomposition form, i.e., as a product of lower-triangular, diagonal and upper-triangular matrices:

$$W = \mathrm{e}^{\gamma J_+} \beta^{2J_0} \mathrm{e}^{\gamma J_-} \,, \tag{25}$$

where $\beta = b/a$ and $\gamma = c/a$, see eq. (1), and matrices J_+, J_0 and J_- have entries

$$(J_+)_{nm} = n \, \delta_{n-1,m}, \qquad (J_0)_{nm} = (n + 1/2) \, \delta_{n,m}, \qquad (J_-)_{nm} = (n+1) \, \delta_{n+1,m} \,. \tag{26}$$

It is useful to rewrite eq. (23) in the form

$$Z_N = [\sin(\lambda+\eta)]^{N^2}\, \mathrm{e}^{-iN(\lambda-\eta)}\, \det_N \left(I + \mathrm{e}^{2\eta K'_N} \right) \ , \tag{27}$$

where matrix K'_N is defined as

$$K'_N = K_N + i\xi I, \qquad K_N = \frac{1}{2\eta} \ln \left(\mathrm{e}^{\gamma J_+} \beta^{2J_0} \mathrm{e}^{\gamma J_-} \right). \tag{28}$$

with $\xi = (\pi/2\eta) - 1$. Note that, as emphasized by the notation, and in contrast to the case of matrix W, the entries of matrix K_N (and hence of K'_N) depend on N.

If we now consider more closely eq. (27), we find that the first factor can be absorbed by a suitable normalization of vertex weights, while the second factor is just a "boundary term", related to the standard symmetric form of the six-vertex model R-matrix, and would be absent if $U_q(sl_2)$-invariant R-matrix [22] were used instead. Thus, the only non trivial quantity contributing to the partition function is the last factor in (27). The important point which we would like to stress here is that such a determinant formula is typical in evaluating traces over Fock space of N canonical Fermi operators [23]:

$$\mathrm{Tr} \left(\mathrm{e}^{2\eta \mathbf{K}'_N} \right) = \det_N \left(I + \mathrm{e}^{2\eta K'_N} \right) \ . \tag{29}$$

Here $\mathbf{K}'_N$ denotes quantum operator

$$\mathbf{K}'_N = \mathbf{K}_N + i\xi \mathbf{N}_N, \tag{30}$$

where

$$\mathbf{K}_N = \sum_{n,m=0}^{N-1} c_n^\dagger (K_N)_{nm} c_m \ , \qquad \mathbf{N}_N = \sum_{n=0}^{N-1} c_n^\dagger c_n \ , \tag{31}$$

and, clearly, $[\mathbf{K}_N, \mathbf{N}_N] = 0$. Note that operator $\mathbf{K}_N$ (and hence $\mathbf{K}'_N$), though bilinear in fermions, is essentially non-local. However, as argued in the following, $\mathbf{K}_N$ has a simple interpretation as *boost operator* for a finite size system. Even though the entries of matrix K_N governing the structure of operator $\mathbf{K}_N$ are of overwhelming complexity, they simplify considerably for infinite N. Indeed in this case matrices $J_{\pm,0}$ appearing in eq. (28) are semi-infinite, and now satisfy $su(1,1)$ algebra commutation relations

$$[J_-, J_+] = 2J_0, \qquad [J_\pm, J_0] = \mp J_\pm \ . \tag{32}$$

Then, due to standard arguments [24] one has

$$\mathbf{K} = \sum_{n=0}^{\infty} (n+1)\, h_{n,n+1}, \tag{33}$$

with free-fermion Hamiltonian density

$$h_{n,n+1} = \frac{1}{\sin\phi_-} \left\{ c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n - \cos\phi_- \left(c_n^\dagger c_n + c_{n+1}^\dagger c_{n+1} \right) \right\} \ . \tag{34}$$

Formula (33) is standard expression for boost operator as it usually appears in integrable lattice models, with Hamiltonian expressed simply as a sum of $h_{n,n+1}$'s [25]. In our case the corresponding model is manifestly free-fermionic, both for finite and infinite N. This is to be interpreted as a consequence of the specific choice of boundary conditions (DWBC) applied to the six-vertex model with arbitrary vertex weigths. The important point we want to stress here is that the whole construction, especially eq. (29) with operator $\mathbf{K}_N$ regarded as a boost operator, fits well into the language of the angular quantization approach, widely used in integrable models of (1+1)-dimensional Quantum Field Theory for calculation of correlation functions [26]. We believe this treatment of the determinant representation for the partition function, as possibly arising from some angular quantization scheme, could be fruitfully applied to the derivation of the correlation functions of the model.

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