Frustrated Magnetic Systems and the q-Oscillator

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Abstract

We analyse the Spherical Model with frustration induced by an external gauge field. In infinite dimensions, this model has been recently mapped onto a problem of q-deformed oscillators, q parametrizing the amount of frustration. We find a complete analytic solution of the Model by using a convenient representation of the q-oscillator algebra, the q-Hermite polynomials. The low-temperature phase does not exhibit a glassy behaviour. The effect of frustration is only quantitative, with respect to the usual unfrustrated Spherical Model. A glassy low-temperature phase is anyway expected for the more complicated XY Model, whose study is in progress.

In two recent papers [1, 2], Parisi *et al.* have investigated the mean field theory of some spin models with frustration, but in the absence of any quenched disorder. Their aim was to show that those systems behave in the low temperature phase as corresponding spin-glass models with quenched disorder. This correspondence is made by matching the frustrated model and the quenched one in the high temperature phase. The point is that the model with quenched disorder can be solved in the low temperature phase by standard techniques such as the replica trick approach [3]. They considered the frustrated models in the limit of large dimensionality D of the lattice, where the saddle-point approximation becomes exact. In their analysis of these models, they showed that the high temperature expansion can be nicely rewritten by using the q-oscillator algebra [1, 2]. Here q measures the frustration per plaquette and varies continuously between the fully-frustrated case ($q = -1$, fermionic algebra) and the ferromagnetic case ($q = 1$, bosonic algebra). Similar q-deformed algebraic relations have also appeared in the Hofstadter problem of quantum

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particles hopping on a two-dimensional lattice in a magnetic field [4]. This problem is closely related to the frustrated spin models, which can be considered as simplified models of hopping in the large D and classical limits. This relation provides another motivation for our analysis. Finally, we note that the frustrated XY Model can also describe Josephson junctions arrays in a magnetic field.

In this talk we want to show how a careful investigation of the interesting connection found in [1, 2] between the coefficients of the high temperature expansion and the q-oscillator algebra allows to solve exactly the infinite dimensional, partially frustrated Spherical Model, both in the high and low temperature phases. We hope that this solution can be extended to the frustrated XY Model, which is the more interesting, because it should exhibit a glassy low-temperature phase [2].

The model we consider is described by the Hamiltonian

$$H_0 = -c(D) \sum_{\langle jk \rangle} \bar{\phi}_j U_{jk} \phi_k \ + \ \text{h.c.} \tag{1}$$

Here $c(D)$ is a normalization constant, which is chosen to obtain a sensible $D \to \infty$ limit for the high temperature expansion. results from the high temperature expansion. The spins ϕ_j are complex valued variables ($\phi_j \in \mathbf{C}$) defined on the sites (labelled by j) of a D-dimensional hypercubic lattice, and there are nearest-neighbour interactions U_{ij}. Three different models can be obtained by constraining the field as follows:

$$\beta H_G = \beta H_0 + \sum_{j=1}^{N} |\phi_j|^2 , \tag{2}$$

$$\beta H_S = \beta H_0 + \mu \left(\sum_{j=1}^{N} |\phi_j|^2 - N \right) , \tag{3}$$

$$\beta H_{XY} = \beta H_0 + \sum_{j=1}^{N} \mu_j \left(|\phi_j|^2 - 1 \right) . \tag{4}$$

The first is the Gaussian Model, which only exists in the high-temperature phase, where the mass term dominates the kinetic term (1). The second is the Spherical Model [5], which uses the Lagrange multiplier μ to enforce the condition $\sum |\phi_j|^2 = N$, N being the total number of sites. This Lagrange multiplier is promoted to a field μ_j in the XY Model (4), whose variables are constrained to live on the unit circle $|\phi_j| = 1, \ \forall j$.

The coupling U_{jk} are non zero only for nearest-neighbour sites. They are complex numbers of modulus one and satisfy the relation $U_{jk} = \bar{U}_{kj}$; in other words, the couplings U are the link variables of a $U(1)$ lattice gauge field. We are interested in the case where the couplings U are selected in such a way to give a constant magnetic field, forming an angle of 45^0 with all the axes of the lattice: the flux of the magnetic field through each elementary plaquette is $\pm B$ independently of the plane to which the plaquette belongs.

We are therefore restricting the choice of the U's in such a way that their product around an elementary plaquette is $e^{\pm iB}$. In order to avoid the choice of a preferred direction in the lattice, these signs are chosen randomly[2] [1].

If all the couplings are set equal to 1 ($B = 0$), we recover the usual ferromagnetic model; in this case the normalization constant $c(D)$ has to be chosen equal to $\frac{1}{2D}$. The maximal amount of frustration is obtained when the product of the couplings around each elementary plaquette is equal to -1; this will be referred to as the fully frustrated case, and corresponds to $B = \pi$. Of course all the intermediate, partially frustrated cases, with $0 < B < \pi$, are also interesting. It appears that as soon as some (even infinitesimally) small amount of frustration is induced in the system, the correct value for the normalization constant $c(D)$ is $\frac{1}{\sqrt{2D}}$. At least in the $D \to \infty$ limit, slightly frustrated systems cannot be studied perturbatively starting from the ferromagnetic case.

Finally, there is another completely different choice of the U's which is very well studied for Hamiltonian (1): $U_{jk} = e^{ir_{jk}}$, where r_{jk} are random numbers uniformely distributed on the interval $[-\pi, \pi]$. In this way a particular spin-glass model is obtained, the so-called gauge glass[3]. This model contains quenched disorder, and is therefore very diferent from the previous ones. Nevertheless its high temperature expansion turns out to be diagrammatically equivalent to the one of the deterministic frustrated case with $B = \pi/2$.

In order to study the previous models, the main difficulty consists in finding the spectral properties of the lattice Laplacian, defined as:

$$(\Delta f)_j = \sum_k \; U_{jk} \; f_k \quad , \tag{5}$$

in the presence of a magnetic field. In references [1, 2], the authors have performed the high temperature expansion for the Gaussian Model (see eq. (2)), and discovered an interesting connection between the coefficients of the expansion (essentially the number of closed loops with given length and given area on an infinite dimensional lattice) and the q-oscillator algebra. We shall exploit this connection to extract the full spectral properties of the infinite dimensional lattice Laplacian in the presence of a magnetic field.

The high temperature expansion for the free energy of the Gaussian Model (eq. (2)) is given as a sum over all closed loops:

$$\beta F = \sum_{\{\mathcal{C}\}} \; W(\mathcal{C}) \frac{[\beta c(D)]^{L(\mathcal{C})}}{L(\mathcal{C})} \tag{6}$$

[2] This is a small amount of randomness, of order $D(D-1)/2$, to be compared to the $\sim L^D$ randomness present in systems with quenched disorder.

[3] In this case $c(D)$ should again be chosen as $1/\sqrt{2D}$.

where $W(\mathcal{C})$ is the Wilson loop, *i.e.* the path ordered product of couplings U along the closed loop $\mathcal{C}$; $L(\mathcal{C})$ is the length of loop $\mathcal{C}$. We can also write

$$\beta F \;=\; \sum_{n \text{ even}} \langle W(\mathcal{C})\rangle_n \frac{[\beta c(D)]^n}{n} \;\mathcal{N}(n) \;=\; \sum_{n \text{ even}} \frac{\beta^n}{n}\; \mathrm{Tr}[\Delta^n] \tag{7}$$

where the Wilson loop has now been averaged over all closed circuits of length $L(\mathcal{C}) = n$ (n even), whose number is $\mathcal{N}(n)$. Parisi *et al.* [1, 2] managed to compute $\langle W(\mathcal{C})\rangle_n$ and $\mathcal{N}(n)$ under the following assumptions: *i)* $D \to \infty$, and *ii)* the magnetic field is such that its flux through each elementary plaquette of the lattice is $\pm B$. In particular they showed that for any given n,

$$\begin{aligned}
\mathrm{Tr}[\Delta^n] \;&=\; \frac{1}{\sqrt{2D}}\langle W(\mathcal{C})\rangle_n \;\mathcal{N}(n) \\
&=\; 1\cdot(\#\text{ of loops with area } = 0) + \\
&\quad\; +q\cdot(\#\text{ of loops with area } = 1) + \\
&\quad\; +q^2\cdot(\#\text{ of loops with area } = 2) + ...
\end{aligned} \tag{8}$$

where $q = \cos B$ and the previous sum stops at $q^{\frac{n(n-2)}{8}}$. Their final result is that the previous counting can be rewritten as the following ground state expectation value:

$$\mathrm{Tr}[\Delta^n] = \langle 0|(a_q^\dagger + a_q)^n|0\rangle \;, \tag{9}$$

where a_q, $a_q^\dagger$ are the q-oscillator operators, with algebra

$$a_q a_q^\dagger - q a_q^\dagger a_q = 1 \;. \tag{10}$$

This algebra cen be represented on an Hilbert space spanned by the vectors: $|m\rangle$, $m = 0,\ 1,\ ...,$ as follows [6]:

$$\begin{aligned}
a_q^\dagger|m\rangle \;&=\; \sqrt{[m+1]_q}|m+1\rangle \\
a_q|m\rangle \;&=\; \sqrt{[m]_q}|m-1\rangle \\
a_q|0\rangle \;&=\; 0
\end{aligned} \tag{11}$$

where

$$[m]_q = \frac{1-q^m}{1-q} \tag{12}$$

In general, if we are interested in calculating $\mathrm{Tr}[f(\Delta)]$, we can therefore equivalently consider the quantity

$$\langle 0|f(X_q)|0\rangle \;, \qquad\qquad X_q \equiv a_q^\dagger + a_q \;. \tag{13}$$

In coordinates representation, this amounts to integrate the function $f(x)$ over the square modulus of the ground state eigenfunction of the q-deformed harmonic oscillator, *i.e.* over the orthogonalizing measure for the q-Hermite Polynomials. The latter is given as [7, 8, 9]:

$$
\begin{aligned}
\nu_q(x) &= \frac{\sqrt{1-q}}{2\pi}\ \frac{1}{q^{1/8}}\ \Theta_1(\theta, q) \\
&= \frac{\sqrt{1-q}}{\pi} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \ \sin([2n+1)\theta] \ ,
\end{aligned} \tag{14}
$$

where $\Theta_1(\theta, q)$ is the first Jacobi Theta Function; θ is varying over the interval $[0, \pi]$, and is related to x by:

$$
x = \frac{2}{\sqrt{1-q}} \cos\theta \ . \tag{15}
$$

In other words, the lattice Laplacian, under the assumptions: *i)* $D \to \infty$, and *ii)* the magnetic field is such that its flux through each elementary plaquette of the lattice is $\pm B$, has the following spectral properties: its eigenvalues x span the interval $[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]$, and the eigenvalue density is given by ν_q as defined in (14).

Note that the previous measure is singular as $q \to 1$; nevertheless, if the limit is carefully performed, the usual gaussian weight is recovered [7]. On the other hand, for $q = 0$, the previous measure reduces to the standard Wigner semicircle distribution of the Gaussian Hermitean Ensemble of Random Matrices[4].

We have now complete knowledge of the spectrum of the Laplacian, and are therefore ready to study the models defined in eqs. (2), (3) and (4). Let us start with the Gaussian Model, the simplest one. Besides beeing an interesting warm up, this will allow us to check the high temperature expansion, and therefore test the validity of our procedure. The Gaussian Model Hamiltonian was given in (2). The partition function is simply given as:

$$
\begin{aligned}
Z &= \int \mathcal{D}\phi \ \exp[\beta \sum \bar{\phi}\Delta\phi - \sum |\phi|^2] \\
&= 1/\det[1 - \beta\Delta] \ ,
\end{aligned} \tag{16}
$$

from which we readily get the free energy: $\beta F = \mathrm{Tr}[\ln(1 - \beta\Delta)]$ and the internal energy $U(\beta)$:

$$
1 - \beta U(\beta) = \mathrm{Tr}[\frac{1}{1 - \beta\Delta}] \equiv R(\beta) \ . \tag{17}
$$

In the last equation, we introduced the resolvent $R(z)$, which will constitute a major ingredient in the study of the more complicated Spherical and XY models. Of course the integral appearing in the partition function is well defined only when the operator $(1 - \beta\Delta)$ is positive definite. This will be transparent in the explicit form of $R(\beta)$, and

[4]This result is quite natural if we remember the correspondence between the $q = 0$ frustrated model and the gauge glass.

reflects the trivial fact that the Gaussian Model is defined only in the high temperature phase. $R(z)$ is originally well-defined for real z in the interval $|z| < z_c = (\sqrt{1-q})/2$ and is then analytically extended to the whole complex plane of z. From the discussion of the previous Section we can now calculate $R(\beta)$:

$$\begin{aligned}
R(\beta) &= \int_{-\frac{2}{\sqrt{1-q}}}^{\frac{2}{\sqrt{1-q}}} \frac{dx}{1-\beta x} \; \nu_q(x) \\
&= \int_0^\pi \frac{d\theta}{\pi} \frac{2\sin\theta}{1-2\beta\frac{1}{\sqrt{1-q}}\cos\theta} \; \sum_{n=0}^\infty (-1)^n q^{\frac{n^2+n}{2}} \sin[(2n+1)\theta] \\
&= \sum_{n=0}^\infty q^{\frac{n^2+n}{2}} \; [1-G(\frac{2\beta}{\sqrt{1-q}})]^n \; G(\frac{2\beta}{\sqrt{1-q}}) \; , \qquad (18)
\end{aligned}$$

where

$$G(x) = \frac{2}{x^2}[1-\sqrt{1-x^2}] \; . \qquad (19)$$

The most important feature of previous formulae is that the structure of singularities in the complex β-plane of the complicated function $R(\beta)$ is easily readable from the one of the simpler function $G(\beta)$. In particular, in the case of the Gaussian model, for any value of the frustration parameter q (in the interval $-1 < q < 1$) the critical point is at $\beta = \sqrt{1-q}/2$. Note that in the $q \to 1$ limit (unfrustrated situation) the critical point moves towards higher and higher temperatures; this contradiction with the well known result about the ferromagnetic gaussian model, where the transition occurs at finite temperature, is however only apparent: we recall that the $q \to 1$ limit is very subtle; in particular $c(D)$ is different for $q = 1$ and $q \neq 1$, the latter obviously including $q \to 1$.

An important test for our formulas is that they should reproduce the high temperature expansion. We have expanded $R(\beta)$ in series of β, up to $\mathcal{O}(\beta^{18})$ with Mathematica [10] and verified that we indeed recover the q-dependent coefficients of the internal energy high temperature expansion found in ref. [1, 2] by direct enumeration of the graphs through a computer algorithm.

For $q = 0$, the critical temperature is simply $\beta_c = 1/2$, and in the high temperature phase, $U(\beta) = [1 - G(2\beta)]/\beta$; the specific heat diverges as $(\beta_c - \beta)^{-\frac{1}{2}}$ at the transition point (*i.e.* the value of critical exponent α is $\frac{1}{2}$). It is just a matter of calculation to verify that this result still holds for generic q.

We are now ready to face the less trivial Spherical Model. Our approach essentially follows the derivation of the exact solution of the Model in $D = 1, 2, 3$ [5]. Indeed we shall see that the mechanism of the phase transition is the same, despite the complication introduced by the presence of frustration. The spherical constraint is enforced through a

Lagrange multiplier μ. The partition function is

$$\begin{aligned} Z &= \int \mathcal{D}\phi \ \exp[\beta \sum \bar{\phi}\Delta\phi] \int_{\mu_0 - i\infty}^{\mu_0 + i\infty} d\mu \quad \exp[-\mu(\sum |\phi|^2 - N)] \\ &= \int_{\mu_0 - i\infty}^{\mu_0 + i\infty} d\mu \quad \exp[-\mathrm{Tr}[\ln(\mu - \beta\Delta)] + \mu N] \ . \end{aligned} \tag{20}$$

The integration path over μ is along a straight line which runs in the complex plane parallel to the imaginary axis; a positive real part has been added in order to make the integration over the spin variables convergent. The Trace in the exponent is proportional to N (Δ is an N-dimensional operator). In the large N limit, we can apply the saddle point method to eveluate the integral over μ. The saddle point equation is:

$$\mathrm{Tr}[\frac{1}{\mu - \beta\Delta}] = 1 \ , \tag{21}$$

and must be solved for real, positive values of μ. Introducing the variable $z = \frac{\beta}{\mu}$, the saddle point equation may be rewritten in the more convenient form:

$$\mathrm{Tr}[\frac{1}{1 - z\Delta}] = \frac{\beta}{z} \ . \tag{22}$$

Let us now restrict ourselves to the case $q = 0$. The saddle point equation takes the form

$$\frac{1}{2z^2}(1 - \sqrt{1 - 4z^2}) = \frac{\beta}{z} \ , \tag{23}$$

whose solution is

$$z = \frac{\beta}{1 + \beta^2}, \tag{24}$$

corresponding to $\mu = 1 + \beta^2$. The validity of this solution is however restricted by the condition $z < \frac{1}{2}$, *i.e.* $\beta < 1$: this solution describes the high temperature phase. Starting from $\beta = 0$, where $z = 0$ too, and increasing β, z increases till reaching the critical value $z = \frac{1}{2}$. Now z cannot increase anymore because it finds the cut in the complex plane coming form the trace of the logarithm of operator $1 - z\Delta$, and therefore "sticks to the singularity" [5]. Hence, for $\beta > 1$, the saddle point equation has no more an acceptable solution; the leading contribution to the integral in (20) however still comes from the neighbourhood of $z = \frac{1}{2}$ (*i.e.* $\mu = 2\beta$)

The free energy and the internal one are therefore given by

$$\beta F = \mathrm{Tr}[\ln(\mu - \beta\Delta)] - \mu \tag{25}$$

$$U(\beta) = (\partial_\beta + (\partial_\beta \mu)\partial_\mu)(\beta F) \tag{26}$$

$$= \frac{1}{\beta}(1 - \mathrm{Tr}[\frac{1}{1 - z\Delta}]) + (\partial_\beta \mu)\{\frac{1}{\mu}\mathrm{Tr}[\frac{1}{1 - z\Delta}] - 1\} \ . \tag{27}$$

In the high temperature regime the saddle point condition is satisfied and the last term of equation (27) is zero; on the other hand, in the low temperature regime all terms have to be taken into account properly. Anyway, in both cases, we end up with:

$$U(\beta) = \frac{1}{\beta} - \frac{1}{z(\beta)} \ . \tag{28}$$

Of course the dependence of z from β is different in the two regimes; in particular, in the low temperature phase, z is a constant: $z = z_c = \frac{1}{2}$. We therefore get:

$$U(T) = \begin{cases} -\frac{1}{T} & T > 1 \\ T - 2 & T < 1 \end{cases} \qquad (q = 0) \ . \tag{29}$$

The internal energy is therefore continuous at the transition, together with its first derivative. But the specific heat presents a discontinuity in its first derivative. These results are qualitatively the same of the Berlin-Kac analysis of the Spherical Model [5]. But we would like to stress that nevertheless their model is quite different, in the sense that frustration is there completely absent.

The solution for generic q is at this point straightforward: following step by step what we have done for the $q = 0$ case, we must solve the saddle point equation:

$$R(z) = \frac{\beta}{z} \ , \tag{30}$$

where $R(z)$ has been defined in (18). From the form of $G(2z/\sqrt{1-q})$ we immediately find the critical point located at $z = z_c \equiv \frac{\sqrt{1-q}}{2}$, from which we obtain

$$\beta_c = z_c R(z_c) = \sqrt{1-q} \sum_{n=0}^{\infty} (-1)^n q^{\frac{n^2+n}{2}} \ . \tag{31}$$

In the low temperature regime,

$$U(\beta) = \frac{1}{\beta} - \frac{2}{\sqrt{1-q}} \ , \tag{32}$$

while in the high temperature phase

$$U(\beta) = \frac{1}{\beta}[1 - R(z(\beta))] = \frac{1}{\beta} - \frac{1}{z(\beta)} \ . \tag{33}$$

A small β, small q expansion of the saddle point equations gives

$$z(\beta) = \frac{\beta}{1+\beta^2}[1 - q\beta^4 + (q - 3q^2 - q^3)\beta^6 + \ \dots\ ] \ . \tag{34}$$

A plot of the internal energy is shown in Fig. 1 for the three values $q = (-1/\sqrt{2},\ 0,\ 1/\sqrt{2})$. This can be easily obtained without solving the saddle-point equation, by using $z \in [0, z_c]$

as a parameter for both $U(\beta)$ in eq. (27) and β in eq. (22). Note that for $q \neq 0$, the first derivative of the internal energy (*i.e.* essentially the specific heat) is not continuous at the transition. The critical temperature goes to infinity for $q \to 1$ and to zero for $q \to -1$, respectively. In these limiting cases, we can sum the series in $R(z)$ (18) and obtain the explicit form of $z(\beta)$ and the internal energy. For $q = 1$, we find the meaningless results $z = \beta$ and $U(T) = 0$ in the high-temperature phase, which collapses to a point ($T_c = \infty$): this case has a pathologic $D \to \infty$ limit, as discussed in Ref. [1], and a sensible theory should include $1/D$ corrections.

The limit $q = -1$ is well defined and corresponds to the fully frustrated model at $D = \infty$[5]. Here, the high-temperature phase extends down to $T = T_c = 0$, i.e. there is no transition. Actually, we find that the resummed $R(z)$ no longer has the square-root branch cut of $K(z)$ and that the mapping $\beta = \beta(z)$ is invertible on the whole positive β axis:

$$\begin{aligned} \beta &= \frac{z}{1-z^2}\,, \qquad\qquad (q=-1) \\ U &= -z = \frac{1}{2\beta}\left(1-\sqrt{1+4\beta^2}\right)\,. \end{aligned} \tag{35}$$

Let us remark that the absence of a phase transition has also been observed in the fully-frustrated long-range XY model [11].

In conclusion, the qualitative behaviour of the (infinite-dimensional) frustrated Spherical Model in the low-temperature phase is rather smooth and standard: indeed, this model does not present the potential features of: (i) glassy behaviour (many ground states), which is usually found in systems with quenched disorder; (ii) commensurability in the spectrum for $B = 2\pi r/s$, which was observed in the $D = 2$ hopping model in a magnetic field. Regarding point (i), we would like to remark that the corresponding model with quenched disorder, *i.e.* the Spherical Gauge Glass, does not exhibit a glassy phase either. Actually, this model corresponds diagrammatically to the $q = 0$ frustrated model. In intuitive terms, the spin variables are only loosely constrained by the spherical condition, $\sum_i |\phi_i|^2 = N$, and thus can globally adapt themselves to any complex coupling configuration. As far as point (ii) is concerned, the commensurability effects are probably washed out by the $D \to \infty$ limit, which oversimplifies the geometry of hopping on the lattice.

On the other hand, we expect a glassy phase in the frustrated XY Model [2]. In this model, the Lagrange multiplier μ becomes a local field μ_j and the saddle-point equation (21) is functional. Nothing changes very much in the high-temperature phase, where the solution $\mu_j = \mu =$ const. is correct. However, the solution(s) in the low-temperature

[5]Note that the limits $D \to \infty$ and $q \to -1$ do not commute, as discussed in Ref. [1].

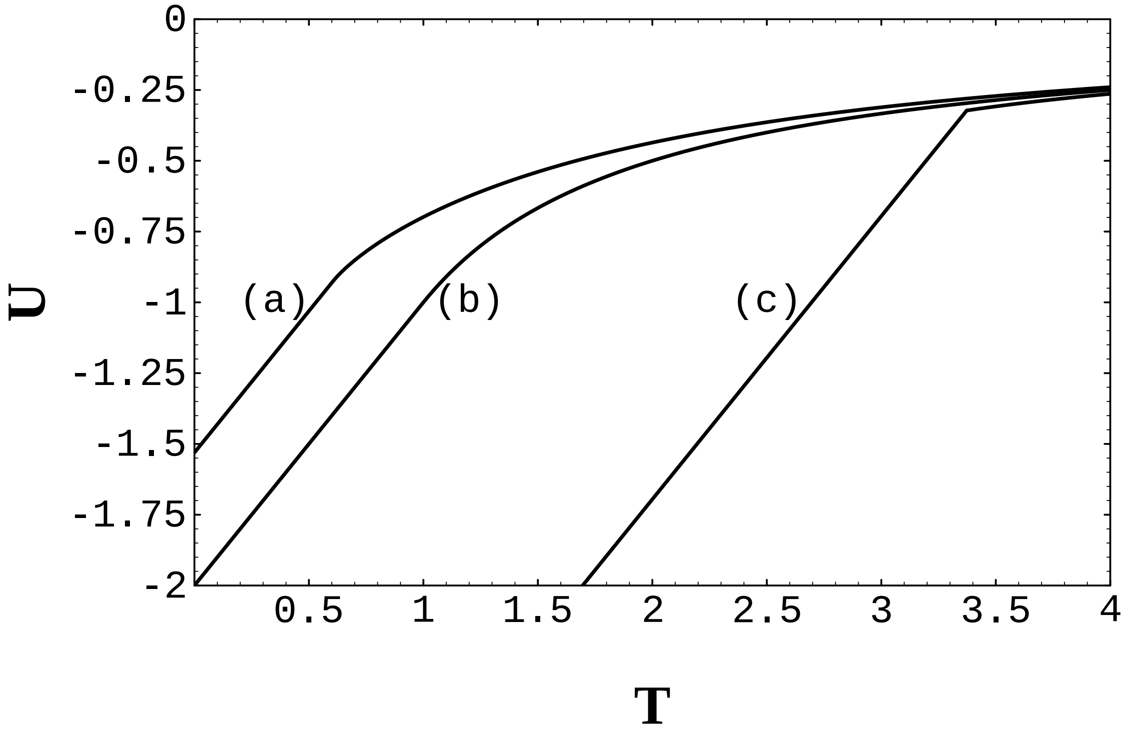

Figure 1: Internal energy of the Spherical Model $U(T)$ versus temperature for the three values: (a) $q = -0.707$; (b) $q = 0$; (c) $q = 0.707$.

phase can be rather different from the one of the Spherical Model. This more difficult problem (and more interesting, too) is currently under investigation.

Let us conclude by observing that we can build a Matrix Model [12, 13] related to our frustrated Spherical Model, in the sense that it has the same diagrammatic expansion in the planar approximation, or, equivalently, the same saddle point equation. The potential of this Matrix Model [14] is oscillating at infinity , where it has infinitely many lower and lower minima, at approximate distance $\log q$. This periodicity is a direct consequence of the q-periodicity of the Jacobi Theta function, *i.e.* of the lattice Laplacian. Once the corrections to the planar limit are considered, its states become metastable by tunnelling to these minima: therefore this model could have a band spectrum and possibly behave as a spin glass. On the other hand this q-periodicity does not affect the low-temperature phase of the frustrated Spherical Model, which is rather standard and non-glassy, the effect of frustration beeing quantitative only.

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