Classical Solutions
of the Many-time Functional Equations of Motion
of the Nambu String

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Abstract

In this paper the many-time approach for the open Nambu string is developed. This allows the discussion of the model in a general gauge. The corresponding functional equations of motion are solved; the restriction to the orthonormal gauge gives the standard results. These solutions are a preliminary and necessary step in the search of a complete set of observables, which will be the argument of a subsequent paper.

1. Introduction.

String and superstring theories[1] are one of the most interesting laboratories for the study of several subjects of the current theoretical investigation, as relativistic extended objects, two dimensional conformal theories, dual models and their S-matrix elements with the use of the theory of Riemann surfaces of arbitrary genus, and again anomalies and their cancellation. This apart from its interest as a possible unified theory of gravity.

In the present paper we shall develop the many-time [2] approach for the open Nambu string, which in principle allows to discuss the theory in a general gauge, and we shall solve the corresponding functional equations of motion as a preliminary step in the search of a complete set of observables, which will be the argument of a subsequent paper.

The problem of finding a complete set of observables is of very general interest, since it is not yet completely solved for gauge theories nor general relativity[3]

In order to clarify the argument of the present paper, it is useful to summarize what is known about the theory of the string models.

When a classical formulation is available, all string models are described in terms of 1^{th}-class constraints[4] in the bosonic case[5,6], and in terms of 1^{th} and 2^{th}-class constraints in the case of fermionic extensions[7]. The Euler-Lagrange equations of the bosonic string become the usual (linear) wave equation in the orthonormal gauge (o.g.), and the reparametrization invariance reduces to the two-dimensional conformal group. Two extra conditions are required in order to completely fix the gauge.

The solutions in an arbitrary gauge are given in reference [7], where they are obtained starting from the Brink-Di Vecchia-Howe[8] reformulation of the Nambu action. This form of the action is more suitable for a path integral approach[9]; it has extra variables given by the elements of the metric $g_{\alpha\beta}(\sigma,\tau)$, $\alpha = 0, 1$, of the (σ,τ) space. In this approach the constraints of the string are secondary constraints.

The usual phase-space approach to the Nambu string [6,10] is performed in the o.g., with an extension from the range $\sigma \in (0,\pi)$ to $(-\pi,\pi)$.

This is the standard covariant formalism. If one wishes to completely fix the gauge, one has to add two other gauge fixings for each point of the string, and then one has to evaluate the Dirac brackets [10]: in this way the usual non covariant light-cone gauge is obtained, in which only physical degrees of freedom are present.

In the non covariant quantization the critical dimension $D = 26$ and the value of the Regge intercept $\alpha(0) = 1$ arise from the requirement of the cancellation of the anomaly, which, at the quantum level, is present in the Lorentz algebra. Correspondingly, in the covariant quantization, the Virasoro algebra requires a c-number Schwinger term, which corresponds to a central charge extension of the conformal algebra. The presence of the Lorentz anomaly in the non covariant approach is here substituted by the presence of ghosts.

The no ghost theorem[11] has a solution for $D = 26$, like in the non covariant approach, and this is the only solution consistent with the BRST approach to the string theory[12].

However, in the open string case, the no-ghost theorem has other solutions, in particular for $D = 4$, where extra longitudinal modes are present, the so-called Brower modes [13]. Extra modes, the Liouville modes, are also present in the Polyakov approach [9], for $D = 4$ (see references[14,15] for attempts to identify them with the Brower modes, and reference[16] for the consistency of the BRST approach in this case). On the other hand there are indications that in the non covariant (lightcone transverse) approach some longitudinal mode could be lost (the Patrascioiu solutions[17]). Finally, dimensional reduction from $D = 26$ to $D = 4$ also introduces extra modes besides the purely transverse ones.

All these facts are an indication of a possible incompleteness of the non covariant quantization in $D < 26$.

The present paper was stimulated by these problems. If we apply the many-time approach [2] in order to find the classical solutions of the Hamilton equations in an arbitrary gauge, and not only in the o.g., these solutions can be only found in local charts on the manifold of the constraints.

The many-time approach was developed for the case of 1^{th}-class constraints in strong involution in nonrelativistic and relativistic dynamics[18]. As the constraints of the Nambu string model are in weak involution, they must be substituted by an equivalent set of constraints in strong involution (abelianization), in order to have integrable equations of motion. This very circumstance compels us to the choice of local charts. It should be observed that the existence of this set of equivalent abelian constraints is not ensured by general theorems as in the finite dimensional case. Nevertheless, for the case of the Nambu string model, the existence of this set seems to be implied by the feasibility of the BRST approach[19].

It turns out that the Nambu constraints manifold is covered by two main overlapping charts plus a set of other disconnected charts with longitudinal modes only. The obstruction in getting global solutions is such that in each of the two main charts one looses just the analogous of the Patrascioiu modes, which live in the other chart

The local quantization in a chart gives the usual critical dimension $D = 26$. It remains open the problem of a global quantization, and the comparison with the $D = 4$ solution of the covariant quantization. Even if we do not know yet how to do a global quantization (reference[20] could be relevant in this respect), we hope to have clarified some aspects of the gauge theory of the Nambu string model, and in particular the relevance of the Patrascioiu modes, which could explain the discrepancy between the non covariant quantization and the covariant one in $D < 26$.

The solutions in a general gauge given by Hwang and Marnelius [7] are also local; it is worth observing that the method of this last reference is peculiar to the string model, while the many-time approach is a general one.

In a future paper we will present a complete set of observables, giving the generalization of the transverse modes of Del Giudice-Di Vecchia-Fubini [21] to an arbitrary gauge. It will then be possible to find a canonical transformation from the original phase-space variables to a new set, in which the gauge degrees of freedom are separated from the observables. This is the canonical basis for a set of Dirac brackets, as the ones given in reference [10].

In Section 2 we give a review of the Nambu string model and of the o.g. treatment, and a discussion of the boundary conditions and of the Patrascioiu modes.

In Section 3 we discuss the canonical formulation and we define the center of mass and relative variables.

Finally, in Section 4 the functional equations of motion are solved. With a proper choice of the parameters, the usual results of the o.g. are recovered.

In Appendix A we discuss the transition from regular to singular parametrizations of the world-sheet of the string.

The consistency of the boundary conditions is checked in Appendix B, through an expansion near the end points of the string.

Some useful formulas are listed in Appendix C.

2. The Classical Bosonic String.

Let us consider the action for the open Nambu string ($\hbar = c = 1$)

$$S = -N \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \sqrt{-h(\sigma,\tau)}, \qquad\qquad L = -N\sqrt{-h}, \tag{2.1}$$

where $N = \frac{1}{2\pi\alpha'}$, and

$$-h = -\det \parallel h_{\alpha\beta} \parallel = (\dot{x}\cdot x')^2 - \dot{x}^2 x'^2 \geq 0, \tag{2.2}$$

$$\parallel h_{\alpha\beta} \parallel = \parallel \partial_\alpha x^\mu \partial_\beta x_\mu \parallel = \begin{pmatrix} \dot{x}^2 & \dot{x}\cdot x' \\ \dot{x}\cdot x' & x'^2 \end{pmatrix}, \quad \alpha = 0, 1, \quad \partial_0 = \frac{\partial}{\partial\tau}, \quad \partial_1 = \frac{\partial}{\partial\sigma}, \tag{2.3}$$

and where $-h \geq 0$ means that the surface swept by the string in the spacetime is everywhere timelike or null (i.e. it is a causal surface)[22].

The strip $0 < \sigma < \pi$ is mapped in the world-sheet spanned by the string in the Minkowski space, which is described by the coordinates $x^\mu(\sigma,\tau)$. $h_{\alpha\beta}(\sigma,\tau)$ is the induced metric, whose inverse is

$$\parallel h^{\alpha\beta} \parallel = \frac{1}{h} \begin{pmatrix} x'^2 & -(\dot{x}\cdot x') \\ -(\dot{x}\cdot x') & \dot{x}^2 \end{pmatrix}. \tag{2.4}$$

Let us define the following two quantities

$$\begin{cases} P^\mu(\sigma,\tau) = -\dfrac{\partial L}{\partial \dot{x}_\mu(\sigma,\tau)} = N\sqrt{-h}h^{0\alpha}\partial_\alpha x^\mu = \dfrac{N}{\sqrt{-h}}((\dot{x}\cdot x')x'^\mu - x'^2\dot{x}^\mu), \\ \Pi^\mu(\sigma,\tau) = -\dfrac{\partial L}{\partial {x'}_\mu(\sigma,\tau)} = N\sqrt{-h}h^{1\alpha}\partial_\alpha x^\mu = \dfrac{N}{\sqrt{-h}}((\dot{x}\cdot x')\dot{x}^\mu - \dot{x}^2 x'^\mu), \end{cases} \tag{2.5}$$

where P^μ is the canonical momentum, which satisfies the identities

$$\begin{cases} P^2(\sigma,\tau) + N^2 x'^2(\sigma,\tau) = 0, \\ (P(\sigma,\tau)\cdot x'(\sigma,\tau)) = 0. \end{cases} \tag{2.6}$$

Other identities are

$$\begin{cases} \Pi^2 + N^2\dot{x}^2 = 0, \\ (\Pi\cdot\dot{x}) = 0, \\ (\Pi\cdot x') = N\sqrt{-h}, \\ (\Pi\cdot P) = N^2(\dot{x}\cdot x'), \\ (P\cdot\dot{x}) = N\sqrt{-h}. \end{cases} \tag{2.7}$$

The Hessian is

$$\begin{aligned} \parallel W^{\mu\nu}(\sigma,\tau) \parallel &= \parallel \frac{\partial^2 L}{\partial\dot{x}_\mu \partial\dot{x}_\nu} \parallel = \\ &= \parallel \frac{Nx'^2}{(-h)^{\frac{3}{2}}}[-h\eta^{\mu\nu} + x'^2\dot{x}^\mu\dot{x}^\nu + \dot{x}^2 x'^\mu x'^\nu - (\dot{x}\cdot x')(\dot{x}^\mu x'^\nu + x'^\mu\dot{x}^\nu)] \parallel . \end{aligned} \tag{2.8}$$

$\dot{x}^\mu(\sigma,\tau)$ and $x'^\mu(\sigma,\tau)$ are the null eigenvectors of the Hessian for every value of σ except the end values $\sigma = 0,\pi$. The non-null eigenvalues are degenerate for $\sigma \neq 0,\pi$, and are equal to

$$\frac{Nx'^2(\sigma,\tau)}{\sqrt{-h(\sigma,\tau)}}.$$

The non-null eigenvectors $\varepsilon^\mu_\lambda(\sigma.\tau)$, with $\lambda = 1,2$, are orthogonal to $\dot{x}^\mu$ and x'^μ, i.e. to the world-sheet, and so they are spacelike.

The variational principle for the action (2.1), with the variations $\delta_0 x^\mu(\sigma,\tau)$ vanishing at $\tau = \tau_1, \tau_2$, is

$$\delta S = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma L_\mu \delta_0 x^\mu - \int_{\tau_1}^{\tau_2} d\tau \Pi_\mu \delta_0 x^\mu|_0^\pi = 0, \tag{2.9}$$

and gives the following Euler-Lagrange equations and boundary conditions

$$\begin{aligned} L^\mu(\sigma,\tau) = \dot{P}^\mu + \Pi'^\mu = -W^{\mu\nu}[\ddot{x}_\nu + \frac{1}{x'^2}\left(\dot{x}^2 x''{}_\nu - 2(\dot{x}\cdot x')\dot{x}'_\nu\right)] = \\ = N\partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta x^\mu\right) = 0, \end{aligned} \tag{2.10}$$

$$\Pi_\mu \delta_0 x^\mu|_0^\pi = 0. \tag{2.11}$$

These boundary conditions have been studied in detail in references [22] and[23], where it is shown that in regular coordinates, corresponding to a parametrization of the string world-sheet such that the tangent vectors $x'^\mu(\sigma,\tau)$ and $\dot{x}^\mu(\sigma,\tau)$ do not vanish and are independent, it amounts in requiring

$$\Pi_\mu x'^\mu|_0^\pi = N\sqrt{-h}|_0^\pi = 0. \tag{2.12}$$

As stressed by the authors of references [22] and [23], the requirement of regular coordinates is crucial for a consistent action principle.

The restriction from (2.11) to (2.12) is due to the requirement that a variation of the boundaries must not violate the condition $-h \geq 0$; nevertheless, in the o.g. the usual conditions

$$x'^\mu(0) = x'^\mu(\pi) = 0 \tag{2.13}$$

can be used.

In particular, in reference [22] it is shown that only the contribution to δx^μ proportional to x'^μ contributes to the boundary term (2.11).

If we only choose coordinates such that

$$\dot{x}^2 \geq 0, x'^2 \leq 0, \tag{2.14}$$

the condition $h = 0$ implies two possible situations at the end points:

i) ${x'}^2 < 0$, $\dot{x}^2 = 0$, $(\dot{x} \cdot x') = 0$, with $\dot{x}^\mu$ and x'^μ independent and $\dot{x}^\mu \neq 0$. This is a regular case (the jacobian of the map $(\sigma, \tau) \to x^\mu$ has maximal rank 2). In this case the rank of the induced metric $\parallel h_{\alpha\beta} \parallel$ is 1, and the end points of the string describe null surfaces [22]. There is the possible case $\dot{x}^\mu = 0$, which is a singular case (the jacobian of the map $(\sigma, \tau) \to x^\mu$ has rank 1).

ii) ${x'}^2 = 0$, $\dot{x}^2 = 0$, $(\dot{x} \cdot x') = 0$, with x'^μ collinear to $\dot{x}^\mu$. This is a singular case (where we may have $\Pi^\mu \neq 0$ as well as $\Pi^\mu = 0$).

The case $x'^\mu = 0$, corresponding to the o.g., may be considered as a particular case of (ii), since it is again a singular case.

Since we want to describe the solutions of the classical equations of motion in a class of gauges including as a special case the o.g., we want to work with the class (ii), that is necessarily with singular coordinates.

Let us check if the boundary condition $h|_0^\pi = 0$ is preserved in a singular case (since it was deduced in the regular one). We may perform a transformation from regular coordinates to those which will become singular at the end points, in the interior of the interval $(0, \pi)$, that is from the class (i) to the class (ii). As shown in reference [23], and more explicity in the Appendix A, the jacobian of the transformation vanishes as $\sqrt{\sigma}$ in $\sigma = 0$ (and in an analogous way in $\pm\pi$), so ensuring, *a fortiori*, the vanishing of the new h at the end points.

So we will assume the boundary conditions (2.12), with a choice of coordinates falling into class (ii).

In order to completely define the physical hypotheses, we will assume that the total momentum of the string P^μ be such that $P^2 \geq 0$, with $P^\mu \neq 0$. As shown in the Appendix B, this ensures a unique solution at the end points of the string with P^μ and Π^μ finite.

Let us stress that, with this kind of boundary conditions, the function $x'^\mu(\sigma, \tau)$, extended to all the real axis, may be discontinuos in $\sigma = 0, \pi$.

Let us recall that the Hessian $W^{\mu\nu}$ has two null eigenvectors. Their existence is connected to the σ, τ reparametrization invariance of the action; that is the action is invariant under the following transformations

$$\begin{cases} \delta\tau = \tilde{\tau}(\sigma, \tau) - \tau, \\ \delta\sigma = \tilde{\sigma}(\sigma, \tau) - \sigma, \qquad \tilde{\sigma}(0, \tau) = 0, \quad \tilde{\sigma}(\pi, \tau) = \pi, \end{cases} \tag{2.15}$$

$$\delta x^\mu(\sigma, \tau) = \tilde{x}^\mu(\tilde{\sigma}, \tilde{\tau}) - x^\mu(\sigma, \tau) = \delta_0 x^\mu(\sigma, \tau) + \dot{x}^\mu(\sigma, \tau)\delta\tau + x'^\mu(\sigma, \tau)\delta\sigma = 0, \tag{2.16}$$

where the last formula expresses the fact that $x^\mu(\sigma, \tau)$ is scalar under reparametrization. It must also remembered that only two of the equations (2.10) are independent, for $\sigma \neq 0, \pi$, as

$$\dot{x}_\mu L^\mu = 0, \text{and} \quad {x'}_\mu L^\mu = 0.$$

Let us now recall how the theory is developed in the o.g. case. The o.g. is defined by a choice of parameters which satisfy, beside

$$-h(\sigma, \tau) \geq 0, \tag{2.17}$$

the conditions

$$\dot{x}^2 + {x'}^2 = (\dot{x} \cdot x') = 0, \tag{2.18}$$

that is

$$h_{\alpha\beta} = \dot{x}^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{with} \quad \dot{x}^2 \geq 0. \tag{2.19}$$

This implies the following conditions

$$\begin{cases} L^\mu = N(\ddot{x}^\mu - x''^\mu) = 0, \\ P^\mu = N\dot{x}^\mu, \\ \Pi^\mu = -Nx'^\mu, \\ W^{\mu\nu} = -\dfrac{N}{\dot{x}^2}(\dot{x}^2\eta^{\mu\nu} - \dot{x}^\mu\dot{x}^\nu + x'^\mu x'^\nu). \end{cases} \tag{2.20}$$

The usual o.g. boundary conditions are

$$x'^\mu(0,\tau) = x'^\mu(\pi,\tau) = 0, \tag{2.21}$$

so we have

$$\dot{x}^2(0,\tau) = \dot{x}^2(\pi,\tau) = 0. \tag{2.22}$$

This in particular means that at the end points the induced metric $h_{\alpha\beta}$ has zero rank. This is peculiar of a set of singular choices of parameters, like in the o.g., where the singular character is shown by the fact that the tangent field x' vanishes at the end points.

These boundary conditions suggest the following extension from the interval $(0,\pi)$ to $(-\pi,\pi)$

$$x^\mu(\sigma,\tau) = x^\mu(-\sigma,\tau), \tag{2.23}$$

and to the real line with 2π periodicity.

The solutions of the equations (2.20) satisfying (2.21) are

$$\begin{aligned} x^\mu(\sigma,\tau) =& q^\mu + \frac{P^\mu}{\pi N}\tau + f^\mu(\tau+\sigma) + f^\mu(\tau-\sigma) = \\ =& q^\mu + \frac{P^\mu}{\pi N}\tau + \frac{i}{\sqrt{\pi N}} \sum_{n\neq 0} \alpha_n^\mu \exp(-in\tau)\cos n\sigma = \\ =& \frac{1}{2}[Q^\mu(\tau+\sigma) + Q^\mu(\tau-\sigma)], \end{aligned} \tag{2.24}$$

with

$$\left(\frac{P^\mu}{2\pi N} + \frac{df^\mu(u)}{du}\right)^2 = 0, \qquad f^\mu(u) = f^\mu(u+2n\pi), \tag{2.25}$$

where

$$u = \tau \pm \sigma,$$

and where

$$P^\mu = \int_0^\pi d\sigma P^\mu(\sigma,\tau)$$

is the conserved total momentum. Equation (2.25) is a consequence of the o.g. conditions, and

$$\begin{cases} Q^\mu(\tau) = x^\mu(0,\tau) = q^\mu + \dfrac{P^\mu}{\pi N}\tau + 2f^\mu(\tau), \\ Q^\mu(\tau+2\pi) = Q^\mu(\tau) + 2\dfrac{P^\mu}{N}, \end{cases} \tag{2.26}$$

is the coordinate of the end point at $\sigma = 0$.

In terms of Q^μ equation (2.25) becomes

$$\frac{1}{4}\dot{Q}^2(\tau) = 0.$$

The coordinates of the other end point are

$$x^\mu(\pi,\tau) = x^\mu(0,\tau+\pi) - \frac{P^\mu}{N} = Q^\mu(\tau+\pi) - \frac{P^\mu}{N},$$

see reference [24]. In this reference it is shown that the end points suffer a constant translation of $\frac{2P^\mu}{N}$ for every $\Delta\tau = 2\pi$, and that for $\Delta\tau = \pi$ the distance between them is $\frac{P^\mu}{N}$. Their motion is given by a double helix with these periods. $Q^\mu(\tau)$ is a relevant function, because the transverse conformal invariant oscillators defined by Del Giudice, Di Vecchia and Fubini [21]

$$\mathbf{A}_n = \sqrt{\frac{N}{2\pi}} \int_{-\pi}^{\pi} d\rho \frac{d\mathbf{Q}(\rho)}{d\rho} \exp{(i\pi n \frac{Q^+(\rho)}{P^+})} \tag{2.27}$$

and the vertex of the dual models

$$\exp{(iQ^+(z))}$$

are defined in terms of it[25]. The Cauchy problem for the equations (2.20) is defined in reference [26]. The residual invariance group in the o.g. is given by the conformal transformations holding the end points $\sigma = 0, \pi$ fixed[27]:

$$\begin{aligned} \tilde{\tau} &= \tau_1 + \tau + g(\tau+\sigma) + g(\tau-\sigma) = \tau_1 + \tau + \sum_{n\neq 0} a_n \cos n\sigma \cdot \exp{(-in\tau)}, \\ \tilde{\sigma} &= \sigma + g(\tau+\sigma) - g(\tau-\sigma) = \sigma - i\sum_{n\neq 0} a_n \sin n\sigma \cdot \exp{(-in\tau)}, \end{aligned} \tag{2.28}$$

and the Jacobian is

$$J = \frac{\partial(\tilde{\tau},\tilde{\sigma})}{\partial(\tau,\sigma)} = [1 + 2g'(\tau+\sigma)][1 + 2g'(\tau-\sigma)] \neq 0,$$

or

$$1 + 2\frac{dg(u)}{du} \neq 0.$$

These transformations leave invariant in form the wave equation (2.20) and the conditions (2.18).

To completely fix the gauge one has to add a further condition, for instance of the kind [10]

$$t_\mu[x^\mu(\sigma,\tau) - q^\mu - \frac{P^\mu\tau}{\pi N}] = 0, \tag{2.29}$$

where t_μ is a constant vector. In the usual light-cone gauge one has

$$t^\mu = (1;0,0,1),$$

so that

$$x^+(\sigma,\tau) - q^+ - \frac{P^+\tau}{\pi N} = 0, \quad A^+ = A^0 + A^3, \quad \text{implying} \quad f^+(u) = 0,$$

(compare equation (2.24).

Patrascioiu [17] noticed that, while in a timelike gauge, $t^\mu = (1;\mathbf{0})$ and $f^0(u) = 0$, every solution of equation (2.20) can be made to satisfy equation (2.29) because $P^0 \neq 0$, in the light-cone gauge solutions exist which cannot satisfy equation (2.29). Let us review his argument, on which we will come back in the following, when discussing the solutions of the classical equations of motion.

Let us start from an arbitrary o.g., with a solution $x^\mu(\sigma,\tau)$ not satisfying equation (2.29), and let us look for a conformal transformation (2.28) to a new o.g. $(\tilde{x}^\mu(\tilde{\sigma},\tilde{\tau}))$, where

$$\tilde{x}^+ - \tilde{q}^+ - \frac{P^+\tilde{\tau}}{\pi N} = 0.$$

From $x^\mu(\sigma,\tau) = \tilde{x}^\mu(\tilde{\sigma},\tilde{\tau})$ and $P^\mu = \tilde{P}^\mu$ we get

$$\begin{cases} q^\mu = \tilde{q}^\mu + \dfrac{P^\mu\tau_1}{\pi N}, \\ f^\mu(u) = \dfrac{P^\mu}{\pi N} g(u) + \tilde{f}^\mu[\tau_1 + u + 2g(u)]. \end{cases} \tag{2.30}$$

Equation (2.29) implies $\tilde{f}^+(u) = 0$ or $g(u) = \dfrac{\pi N}{P^+} f^+(u)$. Now equations (2.28) and (2.25) imply

$$\begin{cases} 1 + \dfrac{2\pi N}{P^+}\dfrac{df^+(u)}{du} \neq 0, \quad (J \neq 0), \\ 2\left(\dfrac{P^+}{2\pi N} + \dfrac{df^+(u)}{du}\right)\left(\dfrac{P^-}{2\pi N} + \dfrac{df^-(u)}{du}\right) - \left(\dfrac{\mathbf{p}}{2\pi N} + \dfrac{d\mathbf{f}(u)}{du}\right)^2 = 0. \end{cases} \tag{2.31}$$

For the class of solutions of equation (2.24) such that

$$\mathbf{f}(u) = \mathbf{f}_0 - \frac{\mathbf{p}u}{2\pi N}, \quad f^+(u) = f_0^+ - \frac{P^+u}{2\pi N},$$

with $f^-(u)$ arbitrary, equations (2.31) are not compatible: the second one implies $J = 0$. This form of $f^\mu(u)$, when substituted in equation (2.24), shows that this class of solutions has $P^+ = \mathbf{p} = 0$, so that these longitudinal modes do not satisfy equation (2.29) and are lost in the light-cone gauge where $P^+ \neq 0$.

We will see in the following how to recover all these modes.

3. The Hamilton Approach.

In order to define a Poisson structure in the phase-space we will use the following extension of the coordinates outside the interval $(0, \pi)$, mainly in order to make contact with the usual extension used in the literature on string models,

$$\begin{cases} x^\mu(\sigma,\tau) = x^\mu(-\sigma,\tau) = x^\mu(\sigma+2n\pi,\tau), \\ \\ P^\mu(\sigma,\tau) = P^\mu(-\sigma,\tau) = P^\mu(\sigma+2n\pi,\tau), \end{cases} \tag{3.1}$$

where n is an integer, with the points $\sigma = 2n\pi$, for any n (or $\sigma = (2n+1)\pi$, for any n), corresponding to the boundary values $x^\mu(0,\tau)$ and $P^\mu(0,\tau)$ (or $x^\mu(\pi,\tau)$ and $P^\mu(\pi,\tau)$) as limit values from the open interval $(0,\pi)$.

With the kind of boundary conditions we have chosen, and with this kind of extension to the real axis, the functions $x'^\mu(\sigma,\tau)$ and $P'^\mu(\sigma,\tau)$ will have in σ_i ($i = 1,2; \sigma_1 = 0, \sigma_2 = \pi$) a jump, not present in the o.g. This means that we must define the physical values of these functions as limit value from the open interval $(0,\pi)$, since the Fourier series will converge pointwise to the mean value of the left and right limits at the end points, that is to unphysical values.

Following reference [10] we introduce an even and odd delta function with period 2π:

$$\begin{aligned} \Delta_\pm(\sigma,\sigma') &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(e^{in(\sigma-\sigma')} \pm e^{-in(\sigma+\sigma')}\right) = \\ &= \sum_{n=-\infty}^{\infty}\left(\delta(\sigma-\sigma'+2n\pi) \pm \delta(\sigma+\sigma'+2n\pi)\right) \longrightarrow \delta(\sigma-\sigma'), \quad \text{for } \sigma,\sigma' \in (0,\pi). \end{aligned} \tag{3.2}$$

$\Delta_\pm$ have the following properties

$$\begin{cases} \Delta_+(\sigma,\sigma') = \Delta_+(-\sigma,\sigma') = \Delta_+(\sigma',\sigma) = \Delta_+(\sigma+2n\pi,\sigma'), \\ \\ \Delta_-(\sigma,\sigma') = -\Delta_-(-\sigma,\sigma') = \Delta_-(\sigma',\sigma) = \Delta_-(\sigma+2n\pi,\sigma'), \\ \frac{\partial}{\partial\sigma}\Delta_\pm(\sigma,\sigma') = -\frac{\partial}{\partial\sigma'}\Delta_\mp(\sigma,\sigma'), \\ \int_{-\pi}^{\pi} d\sigma' f(\sigma')\Delta_\pm(\sigma',\sigma) = f(\sigma) \pm f(-\sigma). \end{cases} \tag{3.3}$$

We introduce the following Poisson structure ($\eta^{\mu\nu} = (1; -1, -1, -1)$):

$$\{x^\mu(\sigma,\tau), P^\nu(\sigma',\tau)\} = -\eta^{\mu\nu}\Delta_+(\sigma,\sigma') \quad \longrightarrow \quad -\eta^{\mu\nu}\delta(\sigma-\sigma'), \quad \text{for } \sigma,\sigma' \in (0,\pi). \tag{3.4}$$

This definition implies a suitable definition of the functional derivative

$$\frac{\delta x_\mu(\sigma)}{\delta x^\nu(\sigma')} = \frac{\delta P_\mu(\sigma)}{\delta P^\nu(\sigma')} = \eta_{\mu\nu}\Delta_+(\sigma,\sigma').$$

More generally we must define the Poisson bracket for two canonical observables $A(\sigma)$, $B(\sigma)$

$$\{A(\sigma), B(\sigma')\} = \int_0^\pi d\bar\sigma \left[\frac{\delta A(\sigma)}{\delta P_\mu(\bar\sigma)}\frac{\delta B(\sigma')}{\delta x^\mu(\bar\sigma)} - \frac{\delta A(\sigma)}{\delta x^\mu(\bar\sigma)}\frac{\delta B(\sigma')}{\delta P_\mu(\bar\sigma)}\right]. \tag{3.5}$$

Since there can be a dependence on $x'(\sigma)$, we must check if some boundary term is present [28]. We will discuss this point in the following of this Section.

Some identities useful in the calculation of such Poisson brackets are given in Appendix C.

The center-of-mass coordinates of the string are

$$\begin{cases} X^\mu(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} d\sigma x^\mu(\sigma,\tau), \\ P^\mu = \dfrac{1}{2}\displaystyle\int_{-\pi}^{\pi} d\sigma P^\mu(\sigma,\tau), \end{cases} \tag{3.6}$$

where P^μ is the conserved generator of the space-time translations.
Let us introduce the following relative coordinates

$$\begin{cases} y^\mu(\sigma,\tau) = -x'^\mu(\sigma,\tau) = -y^\mu(-\sigma,\tau), \\ \mathcal{P}^\mu(\sigma,\tau) = \int_0^\sigma d\sigma' P^\mu(\sigma',\tau) - \dfrac{\sigma}{\pi}P^\mu = -\mathcal{P}^\mu(-\sigma,\tau) = \mathcal{P}^\mu(\sigma+2n\pi,\tau), \end{cases} \tag{3.7}$$

with the following properties

$$\begin{cases} \int_{-\pi}^{\pi} d\sigma y^\mu(\sigma,\tau) = \int_{-\pi}^{\pi} d\sigma \mathcal{P}^\mu(\sigma,\tau) = 0, \\ \\ \mathcal{P}^\mu(0) = \mathcal{P}^\mu(\pm\pi) = 0 \to \int_{-\pi}^{\pi} d\sigma \mathcal{P}'^\mu(\sigma,\tau) = 0. \end{cases} \tag{3.8}$$

It may be checked that the coordinates (3.6) and (3.7) constitute a basis of canonical variables

$$\begin{cases} \{X^\mu, P^\nu\} = -\eta^{\mu\nu}, \\ \\ \{y^\mu(\sigma,\tau), \mathcal{P}^\nu(\sigma',\tau)\} = -\eta^{\mu\nu}\Delta_-(\sigma,\sigma'), \end{cases} \tag{3.9}$$

with all the other Poisson brackets vanishing.

The inverse formulae are

$$\begin{cases} x^\mu(\sigma,\tau) = X^\mu(\tau) + \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 y^\mu(\sigma_2,\tau) - \int_0^\sigma d\sigma_2 y^\mu(\sigma_2,\tau), \\ P^\mu(\sigma,\tau) = \dfrac{1}{\pi}P^\mu + \mathcal{P}'^\mu(\sigma,\tau). \end{cases} \tag{3.10}$$

Useful variables are

$$\begin{aligned} A^\mu_\pm(\sigma,\tau) &= A^\mu_\mp(-\sigma,\tau) = P^\mu(\sigma,\tau) \pm N x'^\mu(\sigma,\tau) = \\ &= \frac{1}{\pi} P^\mu + \mathcal{P}'^\mu(\sigma,\tau) \mp N y^\mu(\sigma,\tau) = \frac{\partial}{\partial\sigma} B^\mu_\pm(\sigma,\tau), \end{aligned} \tag{3.11}$$

$$B^\mu_\pm(\sigma,\tau) = -B^\mu_\mp(-\sigma,\tau) = \frac{\sigma}{\pi} P^\mu + \mathcal{P}^\mu(\sigma,\tau) \pm N x^\mu(\sigma,\tau), \tag{3.12}$$

where

$$\begin{cases} B^\mu_\pm(\sigma + 2n\pi,\tau) = B^\mu_\pm(\sigma,\tau) + 2nP^\mu, \\ B^\mu_\pm(\pi,\tau) - B^\mu_\pm(-\pi,\tau) = \int_{-\pi}^{\pi} d\sigma A^\mu_\pm(\sigma,\tau) = 2P^\mu. \end{cases}$$

In this definition of the $B^\mu_\pm$ we have put to zero the function of τ left arbitrary by the defining equation (3.11). The Poisson brackets of the $A^\mu_\pm$ are

$$\begin{cases} \{A^\mu_\pm(\sigma_1,\tau), A^\nu_\pm(\sigma_2,\tau)\} = \mp N\eta^{\mu\nu} \left(\Delta'_+(\sigma_1,\sigma_2) + \Delta'_-(\sigma_1,\sigma_2)\right), \\ \{A^\mu_\pm(\sigma_1,\tau), A^\nu_\mp(\sigma_2,\tau)\} = \mp N\eta^{\mu\nu} \left(\Delta'_+(\sigma_1,\sigma_2) - \Delta'_-(\sigma_1,\sigma_2)\right), \end{cases} \tag{3.13}$$

where

$$\Delta'_\pm(\sigma_1,\sigma_2) = \frac{\partial}{\partial\sigma_1} \Delta_\pm(\sigma_1,\sigma_2).$$

The constraints implied by equation (2.6) are

$$\begin{cases} \chi_1(\sigma,\tau) = \chi_1(-\sigma,\tau) = P^2(\sigma,\tau) + N^2 x'^2(\sigma,\tau) \approx 0, \\ \chi_2(\sigma,\tau) = -\chi_2(-\sigma,\tau) = P(\sigma,\tau)\cdot x'(\sigma,\tau) \approx 0, \end{cases} \tag{3.14}$$

or, equivalently,

$$\chi_\pm(\sigma,\tau) = \chi_\mp(-\sigma,\tau) = A^2{}_\pm(\sigma,\tau) \approx 0, \tag{3.15}$$

where

$$\begin{cases} \chi_1 = \frac{1}{2}\left(\chi_+ + \chi_-\right), \\ \chi_2 = \frac{1}{4N}\left(\chi_+ - \chi_-\right). \end{cases}$$

They satisfy the algebra

$$\begin{cases} \{\chi_\pm(\sigma_1,\tau), \chi_\pm(\sigma_2,\tau)\} = \mp 2N \left(\chi_\pm(\sigma_1,\tau) + \chi_\pm(\sigma_2,\tau)\right)\cdot\left(\Delta'_+(\sigma_1,\sigma_2) + \Delta'_-(\sigma_1,\sigma_2)\right), \\ \{\chi_+(\sigma_1,\tau), \chi_-(\sigma_2,\tau)\} = -2N \left(\chi_+(\sigma_1,\tau) + \chi_-(\sigma_2,\tau)\right)\cdot\left(\Delta'_+(\sigma_1,\sigma_2) - \Delta'_-(\sigma_1,\sigma_2)\right). \end{cases} \tag{3.16}$$

Therefore the constraints are 1^{th}-class, but they are in weak involution; the algebra (3.16) is the universal Dirac algebra of reparametrization [4,29]. In the many-time approach of the next Section a set of 1^{th}-class constraints in strong involution are needed.

In the finite-dimensional case their existence is ensured (at least locally) by theorems about function groups [19][30], which however have no general extension to the infinite-dimensional case. However, constraints in strong involution always exist when the original constraints can be solved in terms of a subset of the canonical variables, and this is equivalent to the use of the BRST method [19].

Here we will consider two possible solutions of the constraints. They are defined with lightcone variables, as we wish to make contact with the non covariant approach to the string. Other solutions are possible, and we will explore them elsewhere. In terms of the following lightcone variables

$$\begin{cases} A^+_\pm(\sigma,\tau) = \dfrac{1}{\sqrt{2}}\left(A^0_\pm(\sigma,\tau) + A^3_\pm(\sigma,\tau)\right), \\ A^-_\pm(\sigma,\tau) = \dfrac{1}{\sqrt{2}}\left(A^0_\pm(\sigma,\tau) - A^3_\pm(\sigma,\tau)\right), \end{cases} \tag{3.17}$$

we may define the new constraints

$$\tilde{\chi}^{(+)}_\pm(\sigma,\tau) = \frac{\chi_\pm(\sigma,\tau)}{2A^+_\pm(\sigma,\tau)} = A^-_\pm(\sigma,\tau) - \frac{\vec{A}^2_\pm(\sigma,\tau)}{2A^+_\pm(\sigma,\tau)} \approx 0, \qquad \text{if} \qquad A^+_\pm(\sigma,\tau) \neq 0, \tag{3.18}$$

or

$$\tilde{\chi}^{(-)}_\pm(\sigma,\tau) = \frac{\chi_\pm(\sigma,\tau)}{2A^-_\pm(\sigma,\tau)} = A^+_\pm(\sigma,\tau) - \frac{\vec{A}^2_\pm(\sigma,\tau)}{2A^-_\pm(\sigma,\tau)} \approx 0, \qquad \text{if} \qquad A^-_\pm(\sigma,\tau) \neq 0 \tag{3.18$'$}$$

where $\vec{A}^2 = (A^1)^2 + (A^2)^2$.

These constraints are only weakly Poincaré invariant and are only locally defined, in those regions of the constraints manifold $\chi_\pm(\sigma,\tau) \approx 0$ where the denominators do not vanish. They are now in strong involution:

$$\begin{cases} \left\{\tilde{\chi}^{(a)}_\pm(\sigma,\tau), \tilde{\chi}^{(a)}_\pm(\sigma',\tau)\right\} = 0 \\ \\ \left\{\tilde{\chi}^{(a)}_+(\sigma,\tau), \tilde{\chi}^{(a)}_-(\sigma',\tau)\right\} = 0. \end{cases} \qquad (a = +,-) \tag{3.19}$$

Let us remember that in order to fix an o.g. uniquely, usually one adds the following gauge-fixing constraints (transverse light-cone gauge):

$$\begin{cases} \phi_1(\sigma,\tau) = x^+(\sigma,\tau) - q^+ - \dfrac{P^+}{\pi N}\tau \approx 0, \\ \phi_2(\sigma,\tau) = P^+(\sigma,\tau) - \dfrac{P^+}{\pi} \approx 0, \end{cases} \tag{3.20}$$

one may define the Dirac brackets [10] and select the physical transverse degrees of freedom.

The classical Virasoro generators are

$$L_n = \frac{1}{4N}\int_{-\pi}^{\pi} d\sigma e^{in\sigma}\chi_+(\sigma) = \frac{1}{4N}\int_{-\pi}^{\pi} d\sigma e^{-in\sigma}\chi_-(\sigma) \qquad \Longrightarrow$$
$$\Longrightarrow \qquad \chi_\pm(\sigma,\tau) = \frac{2N}{\pi}\sum_{n=-\infty}^{+\infty} e^{\mp in\sigma}L_n(\tau), \tag{3.21}$$

with

$$\{L_m, L_n\} = i(n-m)L_{n+m}.$$

Solving equation (3.11) in terms of $x^\mu(\sigma,\tau)$, we get

$$x^\mu(\sigma,\tau) = \frac{1}{2N}\left(B_+^\mu(\sigma,\tau) - B_-^\mu(\sigma,\tau)\right) = \frac{1}{2N}\left(B_+^\mu(\sigma,\tau) + B_+^\mu(-\sigma,\tau)\right), \tag{3.22}$$

and, by comparison with equation (2.24), we may assert that $\frac{1}{N}B_+^\mu(\sigma,\tau)$ are the generalization to an arbitrary gauge of the o.g. functions $Q^\mu(\tau) = x^\mu(0,\tau)$, which describe the $\sigma = 0$ end point.

As the canonical Hamiltonian of the string vanishes, the Dirac Hamiltonian is

$$\begin{aligned} H_D(\tau) &= \int_0^\pi d\sigma\left[\lambda_1(\sigma,\tau)\chi_1(\sigma,\tau) + \lambda_2(\sigma,\tau)\chi_2(\sigma,\tau)\right] = \\ &= \int_0^\pi d\sigma\left[\lambda_+(\sigma,\tau)\chi_+(\sigma,\tau) + \lambda_-(\sigma,\tau)\chi_-(\sigma,\tau)\right] = \\ &= \int_{-\pi}^{\pi} d\sigma\lambda_+(\sigma,\tau)\chi_+(\sigma,\tau) = \\ &= \int_{-\pi}^{\pi} d\sigma\lambda_-(\sigma,\tau)\chi_-(\sigma,\tau), \end{aligned} \tag{3.23}$$

with the following properties for the Dirac multipliers:

$$\begin{cases} \lambda_1(\sigma,\tau) = \lambda_+(\sigma,\tau) + \lambda_-(\sigma,\tau) = \lambda_1(-\sigma,\tau) = \lambda_1(\sigma+2n\pi,\tau), \\ \\ \lambda_2(\sigma,\tau) = 2N\left(\lambda_+(\sigma,\tau) - \lambda_-(\sigma,\tau)\right) = -\lambda_2(-\sigma,\tau) = \lambda_2(\sigma+2n\pi,\tau), \end{cases} \tag{3.24}$$

The Hamilton equations are

$$\begin{cases} \delta x^\mu(\sigma,\tau) = \{x^\mu(\sigma,\tau), H_D\}\,\delta\tau \approx \left[-2\lambda_1(\sigma,\tau)P^\mu(\sigma,\tau) - \lambda_2(\sigma,\tau)x'^\mu(\sigma,\tau)\right]\delta\tau, \\ \delta P^\mu(\sigma,\tau) = \{P^\mu(\sigma,\tau), H_D\}\,\delta\tau \approx -\dfrac{\partial}{\partial\sigma}\left[2N^2\lambda_1(\sigma,\tau)x'^\mu(\sigma,\tau) + \lambda_2(\sigma,\tau)P^\mu(\sigma,\tau)\right]\delta\tau. \end{cases} \tag{3.25}$$

Carefully performing the calculation, we may verify that boundary terms are absent, as far as the physical values of δx and δP at $\sigma = 0, \pi$ are taken as limit from the interior of the interval $(0,\pi)$, that is as $\sigma \to \pi-$ and $\sigma \to 0+$.

To be more precise, in the calculation of the r.h.s of the second equation (3.25), we have a term like

$$\int_0^\pi d\sigma' \Pi^\mu(\sigma') \frac{\partial}{\partial \sigma'} \Delta_+(\sigma, \sigma'),$$

which correctly gives $-\frac{\partial}{\partial \sigma} \Pi^\mu(\sigma)$, if we remember that this last derivative must be interpreted as a distributional derivative. This means that the possible jumps of x'^μ, on which Π^μ depends (see below), will give δ-like contributions at $\sigma = n\pi$ for any integer n. This is the price for having chosen the extension (3.1), with boundary conditions different from those of the o.g. case.

In any case, these δ-like terms do not contribute to the physical values, since the latters are obtained as limiting values from the inside of the $(0, \pi)$ interval.

From the first equation (3.25) we get for λ_i

$$\begin{cases} \lambda_1(\sigma, \tau) \approx \dfrac{\sqrt{-h(\sigma, \tau)}}{2N x'^2(\sigma, \tau)}, \\ \lambda_2(\sigma, \tau) \approx -\dfrac{\dot{x}(\sigma, \tau) \cdot x'(\sigma, \tau)}{x'^2(\sigma, \tau)}. \end{cases} \tag{3.26}$$

Inserting these in the definition (2.5) of $\Pi^\mu(\sigma, \tau)$ we get

$$\Pi^\mu(\sigma, \tau) = 2N^2 \lambda_1(\sigma, \tau) x'^\mu(\sigma, \tau) + \lambda_2(\sigma, \tau) P^\mu(\sigma, \tau), \tag{3.27}$$

The variation of any canonical observable A, induced by H_D is

$$\delta A(\sigma) = \{A(\sigma), H_D\} \delta\tau,$$

or

$$\delta A(\sigma) = \int_0^\pi d\bar{\sigma} \left[\frac{\delta A(\sigma)}{\delta x^\mu(\bar{\sigma})} \delta x^\mu(\bar{\sigma}) + \frac{\delta A(\sigma)}{\delta P^\mu(\bar{\sigma})} \delta P^\mu(\bar{\sigma}) \right], \tag{3.28}$$

as can be verified by inserting the expressions (3.25) for δx and δP.

Observe that the λ_i given by equations (3.26) appear undetermined at the end points. Actually, they have a well defined limit, which can be recovered from a careful analysis of the end points behaviour as discussed in the Appendix B.

In the o.g., we have $\lambda_1(\sigma_i, \tau) < \infty$ and $\lambda_2(\sigma_i, \tau) = 0$, or $\lambda_+(\sigma_i, \tau) = \lambda_-(\sigma_i, \tau)$.

With the kind of boundary conditions we have chosen, the values at the end points of the λ_i are not independent, as shown in Appendix B. This means that at these points the two constraints (3.25) degenerate in a unique constraint. This fact can be seen as due to the collinearity of x'^μ and $\dot{x}^\mu$ (or x'^μ and P^μ in the hamiltonian formalism) in $\sigma = 0, \pi$.

If we define

$$G[\varepsilon_i] = \int_0^\pi d\sigma \left(\varepsilon_1(\sigma, \tau) \chi_1(\sigma, \tau) + \varepsilon_2(\sigma, \tau) \chi_2(\sigma, \tau) \right),$$

with the $\varepsilon_i(\sigma, \tau)$ not satisfying the relation between λ_1 and λ_2 at the end points, this is an improper constraint, *i.e.* it generates transformations which turn out to be not gauge transformations [23]: they change the physical state of the system, mapping one physical solution onto a different physical solution.

Conversely, if the $\varepsilon_i(\sigma,\tau)$ satisfy equations (3.24), we get the following transformations

$$\begin{cases} \delta\tau = \delta\sigma = 0 \\ \delta_0 x^\mu(\sigma,\tau) = -2\varepsilon_1 P^\mu(\sigma,\tau) - \varepsilon_2(\sigma,\tau) x'^\mu(\sigma,\tau), \end{cases} \tag{3.29}$$

If we use equation (2.5) to express $P^\mu(\sigma,\tau)$, the variation of the action (2.1) induced by this transformation turns out to be quasi-invariant under the canonical gauge transformations generated by the first-class constraints, that is

$$\delta S = \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \partial_\alpha F^\alpha(\sigma,\tau), \qquad \begin{cases} F^0(\sigma,\tau) = -2N^2\varepsilon(\sigma,\tau)x'^2(\sigma,\tau), \\ F^1(\sigma,\tau) = +2N^2\varepsilon_1(\sigma,\tau)(\dot{x}(\sigma,\tau)\cdot x'(\sigma,\tau)) - \\ \qquad -N\varepsilon_2(\sigma,\tau)\sqrt{-h(\sigma,\tau)}. \end{cases} \tag{3.30}$$

These transformations are not the standard (σ,τ)-reparametrization transformations (2.15), but a possible reformulation of the latter as canonical transformations. In the case we would prefer to use the constraints $\tilde{\chi}^{(a)}_\pm(\sigma,\tau)$, $a=\pm$, of equations (3.17), it is easy to define an analogous of H_D, at least locally, over an interval of values $\sigma \in (\sigma_1,\sigma_2)$ where the denominators $A^a_\pm(\sigma,\tau)$ do not vanish:

$$\tilde{H}_{D(\sigma_1,\sigma_2)}(\tau) = \int_{\sigma_1}^{\sigma_2} d\sigma \left[\tilde{\lambda}^{(a)}_+(\sigma,\tau)\tilde{\chi}^{(a)}_+(\sigma,\tau) + \tilde{\lambda}^{(a)}_-(\sigma,\tau)\tilde{\chi}^{(a)}_-(\sigma,\tau)\right]. \tag{3.31}$$

Let us remark again that the Poisson structure we have defined is only oriented to the o.g. and to the gauges which can be reached from the orthonormal ones with (σ,τ)-reparametrizations with non-vanishing Jacobian. Therefore the Patrascioiu longitudinal modes do exist in this formulation, which is incompatible with regular coordinates $x^\mu(\sigma,\tau)$ associated to an embedding $(\sigma,\tau) \longmapsto x^\mu(\sigma,\tau)$ [23], and actually they are contained in the zeros of the functions $A^{(a)}_\pm(\sigma,\tau)$, $a=\pm$, as we shall see in the following sections. The locality of the many-time approach associated to the constraints $\tilde{\chi}^{(a)}_\pm(\sigma,\tau)$ of the next section is intimately connected to these modes with this Poisson structure.

4. The Many-time Approach. and the Solutions of the Equations of Motion in an Arbitrary Gauge.

This method was developed in reference [2] in order to study systems of n nonrelativistic or relativistic particles with action-at-a-distance interactions described by n first-class constraints in strong involution

$$\{\chi_a, \chi_b\} = 0 \qquad a,b = 1,...,n, \tag{4.1}$$

and for the general problem of a constrained system in [31].

Following the Dirac's approach [4], if $x^{\mu a}$, $P_{\mu a}$, $a = 1, ...n$, are the canonical variables, with $\{x^{\mu a}, P_b^\nu\} = -\eta^{\mu\nu}\delta_b^a$, $\eta^{\mu\nu} \equiv (1, -1, -1, -1)$, the Hamilton equations with respect to the Dirac Hamiltonian are

$$\begin{cases} \dfrac{d}{d\tau}x^{\mu a}(\tau) = \{x^{\mu a}, H_D\} \approx \sum_{b=1}^n \lambda^b(\tau)\,\{x^{\mu a}, \chi_b\}\,, \\ \dfrac{d}{d\tau}P_{\mu a}(\tau) = \{P_{\mu a}, H_D\} \approx \sum_{b=1}^n \lambda^b(\tau)\,\{P_{\mu a}, \chi_b\}\,. \end{cases} \tag{4.2}$$

These equations can be solved only after the first step in fixing a gauge, *i.e.* after assigning a set of multipliers $\lambda^a(\tau)$.

Another way to approach the problem of solving the system (4.2) is to introduce n "times" τ^a formally defined through the equations

$$d\tau^a = \lambda^a(\tau)d\tau, \tag{4.3}$$

and by redefining $x^{\mu a}(\tau) = \tilde{x}^{\mu a}(\tau^1, ..., \tau^n)$, $P_{\mu a}(\tau) = \tilde{P}_{\mu a}(\tau^1, ..., \tau^n)$, equations (4.2) are replaced by the n-times Hamilton equations

$$\begin{cases} \dfrac{\partial x^{\mu a}}{\partial \tau^b} = \{x^{\mu a}, \chi_b\}\,, \\ \dfrac{\partial P_{\mu a}}{\partial \tau^b} = \{P_{\mu a}, \chi_b\}\,, \end{cases} \tag{4.4}$$

whose integrability conditions are just equations (4.1). Each "time" τ^a has its own Hamiltonian. This formal derivation of the equations of motion can be justified in a more rigorous way, obtaining them as characteristic equations of the constraint's equations as in [31]. Since the system (4.4) is autonomous, apart from an initial constant each parameter τ^a is defined by the system (4.4) itself, and can eventually be eliminated in terms of some physical coordinate.

The physical coordinates $q^{\mu a}(\tau^a)$ in the configuration space are recovered [2] through the Droz-Vincent conditions [32]

$$\begin{cases} \{q^{\mu a}, \chi_b\} = 0, & \text{when } \mathrm{a} \neq b, \\ q^{\mu a} = x^{\mu a}, & \text{when } \tau^{(a)} = \tau^{(a)}(\tau). \end{cases} \tag{4.5}$$

The no-interaction theorem[33] stems from the requirement $q^{\mu a} = x^{\mu a}$ when the times do not satisfy $\tau^{(a)} = \tau^{(a)}(\tau)$.

When we have a set of first-class constraints in weak involution

$$\{\chi_a, \chi_b\} = C_{ab}^c(x, P)\chi_c \approx 0, \tag{4.6}$$

equations (4.4), which are a direct consequence of definition (4.3), are no more integrable. To recover integrable equations we have to replace the constraints χ_a with equivalent constraints $\tilde{\chi}_a$ satisfying equations (4.1). In the finite-dimensional case this is always possible, at least locally, by solving the constraints with respect to some of the canonical variables.

In our case equations (3.16) solve the problem in light-cone coordinates, but the price we pay is the loss of manifest Lorentz covariance and the fact that the theory is defined only locally, where the denominators do not vanish.

We have two first class constraints for each value of $\sigma \neq \sigma_i$ given by equation (3.18),so that, according to equation (4.3), we have to introduce two “times” for each $\sigma \in (0, \pi)$, $\tau_{\pm}^{(a)}(\sigma)$, with associated Hamiltonians $\tilde{\chi}_{\pm}^{(a)}(\sigma)$. A 1-1 correspondence must be assumed between any of these ”time” and the parameter τ.

The connection with the $\lambda_{\pm}(\sigma, \tau)$ is

$$\delta\tau_{\pm}^{(a)}(\sigma) = \tilde{\lambda}_{\pm}^{(a)}(\sigma, \tau)\delta\tau = 2A_{\pm}^{(a)}(\sigma, \tau)\lambda_{\pm}(\sigma, \tau)\delta\tau, \qquad a = \pm, \tag{4.7}$$

where $a = \pm$ are two possible choices of a pair of constraints equivalent to the pair $\chi_{\pm}$.

The “time functions” $\tau_{\pm}^{(a)}(\sigma)$, can be chosen such to satisfy equations (3.24):

$$\tau_{\pm}^{(a)}(\sigma) = \tau_{\mp}^{(a)}(-\sigma) = \tau_{\pm}^{(a)}(\sigma + 2n\pi). \tag{4.8}$$

This is not compulsory; we may require it in order to agree with the usual form of the solutions (2.24), where such extension is assumed.

With (4.8), when we work with $\sigma \in (-\pi, \pi)$, we only need

$$\tau_{+}^{(a)}(\sigma), \quad \tilde{\chi}_{+}^{(a)}(\sigma|\tau_{\pm}^{(a)}(\sigma)]$$

or

$$\tau_{-}^{(a)}(\sigma), \quad \tilde{\chi}_{-}^{(a)}(\sigma|\tau_{\pm}^{(a)}(\sigma)],$$

according to equation (3.15). Observe that at the end points the functions $\tau_{\pm}^{(a)}(\sigma)$ can be discontinuous. As for the physical coordinates, we will define the values of the parameters in $\sigma = 0, \pi$ as the limit from the open interval $(0, \pi)$. This means that, by definition

$$\begin{cases} \tau_{\pm}(0) \equiv \tau_{\pm}(0+), \\ \tau_{\pm}(\pi) \equiv \tau_{\pm}(\pi-), \end{cases} \tag{4.9}$$

so the equation (4.8) implies

$$\begin{cases} \tau_{\pm}(0-) \equiv \tau_{\mp}(0), \\ \tau_{\pm}(-\pi) \equiv \tau_{\mp}(\pi). \end{cases} \tag{4.10}$$

With these definitions it turns out that $\tau_{\pm}(0)$ are two different values, in spite of what could be apparent from (4.8).

Accordingly we redefine the canonical variables

$$x^{\mu}(\sigma, \tau) \longrightarrow x^{\mu}(\sigma|\tau_{\pm}^{(a)}(\sigma)], \qquad P^{\mu}(\sigma, \tau) \longrightarrow P^{\mu}(\sigma|\tau_{\pm}^{(a)}(\sigma)], \qquad \sigma \in (0, \pi),$$

The new Hamilton equations become the following functional equations (where the constraints $\tilde{\chi}^{\pm}_{\pm}$ are defined in eq. (3.18):

$$\begin{cases} \dfrac{\delta x^\mu(\sigma|\tau^{(a)}_\pm(\sigma)]}{\delta\tau^{(a)}_\pm(\sigma')} = \left\{x^\mu(\sigma|\tau^{(a)}_\pm(\sigma)], \tilde{\chi}^{(a)}_\pm(\sigma'|\tau^{(a)}_\pm(\sigma)]\right\} \approx \\ \qquad\qquad \approx - \dfrac{P^\mu(\sigma'|\tau^{(a)}_\pm(\sigma)] \pm N x'^\mu(\sigma'|\tau^{(a)}_\pm(\sigma)]}{A^{(a)}_\pm(\sigma'|\tau^{(a)}_\pm(\sigma)]}\Delta_+(\sigma,\sigma'), \\ \dfrac{\delta P^\mu(\sigma|\tau^{(a)}_\pm(\sigma)]}{\delta\tau^{(a)}_\pm(\sigma')} = \left\{P^\mu(\sigma|\tau^{(a)}_\pm(\sigma)], \tilde{\chi}^{(a)}_\pm(\sigma'|\tau^{(a)}_\pm(\sigma)]\right\} \approx \\ \qquad\qquad \approx \pm N \dfrac{P^\mu(\sigma'|\tau^{(a)}_\pm(\sigma)] \pm N x'^\mu(\sigma'|\tau^{(a)}_\pm(\sigma)]}{A^{(a)}_\pm(\sigma'|\tau^{(a)}_\pm(\sigma)]}\dfrac{\partial}{\partial\sigma'}\Delta_+(\sigma,\sigma'), \end{cases} \tag{4.11}$$

whose integrability is guaranteed by equations (3.17). When $\sigma' \in (-\pi,\pi)$, the equations in $\tau^{(a)}_-(\sigma')$ become those in $\tau^{(a)}_+(\sigma')$, with $\sigma' \longrightarrow -\sigma'$.

We shall study only those equations with $a = +$, with the constraints (3.18), since it is possible to obtain the solutions for $a = -$ merely interchanging the light-cone indices $+ \leftrightarrow -$. From now on we shall therefore drop the suffix (a).

The first line of equations (4.11) implies $\left\{x^\mu(\sigma), \tilde{\chi}^{(a)}_\pm(\sigma')\right\} = 0$ for $\sigma \neq \sigma'$; if we compare this result with the Droz-Vincent equations (4.5), we see that in this infinitely-dimensional case the $x^\mu(\sigma,\tau)$ can be identified with the physical coordinates $q^\mu(\sigma,\tau)$, so there is no room for the No-Interaction theorem [33].

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Instead of equations (4.11) it is convenient to integrate the equations for $A^\mu(\sigma|\tau_\pm(\sigma)]$. If we restrict σ, σ' to the interval $(0,\pi)$, so that $\Delta_\pm(\sigma,\sigma') = \delta(\sigma \mp \sigma')$, we have:

$$\begin{cases} \dfrac{\delta A^\mu_+(\sigma|\tau_\pm]}{\delta\tau_-(\sigma')} = 0 \\ \dfrac{\delta A^\mu_-(\sigma|\tau_\pm]}{\delta\tau_+(\sigma')} = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} A^\mu_+ = A^\mu_+(\sigma|\tau_+] \\ A^\mu_- = A^\mu_-(\sigma|\tau_-]. \end{cases} \tag{4.12}$$

The equations for $A^+_+(\sigma|\tau_+]$ and $A^+_-(\sigma|\tau_-]$ are

$$\begin{cases} \dfrac{\delta A^+_+(\sigma|\tau_+]}{\delta\tau_+(\sigma')} = -2N\delta'(\sigma-\sigma') \\ \dfrac{\delta A^+_-(\sigma|\tau_-]}{\delta\tau_-(\sigma')} = 2N\delta'(\sigma-\sigma'), \end{cases} \tag{4.13}$$

and their solution are

$$\begin{cases} A^+_+(\sigma|\tau_+] = -2N\dfrac{d}{d\sigma}\left[\tau_+(\sigma) - c_+(\sigma)\right] \\ A^+_-(\sigma|\tau_-] = 2N\dfrac{d}{d\sigma}\left[\tau_-(\sigma) - c_-(\sigma)\right], \end{cases} \tag{4.14}$$

where $c_\pm(\sigma)$ are a double infinity of "integration constants" which do not depend on $\tau_\pm$. The remaining equations are

$$\begin{cases} \dfrac{\delta \vec{A}_+(\sigma|\tau_+]}{\delta\tau_+(\sigma')} = -2N\dfrac{\vec{A}_+(\sigma'|\tau_+]}{A^+_+(\sigma'|\tau_+]}\delta'(\sigma-\sigma') \\ \dfrac{\delta A^-_+(\sigma|\tau_+]}{\delta\tau_+(\sigma')} \approx -2N\dfrac{A^-_+(\sigma'|\tau_+]}{A^+_+(\sigma'|\tau_+]}\delta'(\sigma-\sigma'), \end{cases} \tag{4.15$'$}$$

and

$$\begin{cases} \dfrac{\delta \vec{A}_-(\sigma|\tau_-]}{\delta\tau_-(\sigma')} = 2N\dfrac{\vec{A}_-(\sigma'|\tau_-]}{A^+_-(\sigma'|\tau_-]}\delta'(\sigma-\sigma') \\ \dfrac{\delta A^-_-(\sigma|\tau_-]}{\delta\tau_-(\sigma')} \approx 2N\dfrac{A^-_-(\sigma'|\tau_-]}{A^+_-(\sigma'|\tau_-]}\delta'(\sigma-\sigma'), \end{cases} \tag{4.15”}$$

where in the equations for $A^-_\pm$ we have used the constraints $\tilde{\chi}_\pm \approx 0$. If we use equations (4.13), we can write all the equations (4.14) under the same form

$$\frac{\delta f(\sigma|\tau]}{\delta\tau(\sigma')} = \frac{f(\sigma'|\tau]}{\tau'(\sigma')-c'(\sigma')}\delta'(\sigma-\sigma').$$

Introducing the notation $T(\sigma)=\tau(\sigma)-c(\sigma)$, the solution of this equation is

$$f(\sigma|\tau] = T'(\sigma)G(T(\sigma)) = T'(\sigma)\left.\frac{dF(T)}{dT}\right|_{T=T(\sigma)} = \frac{d}{d\sigma}F(T(\sigma)),$$

where G is an arbitrary function, then written as the derivative of another arbitrary function F. Indeed we have

$$\begin{aligned} \frac{\delta f(\sigma|\tau]}{\delta\tau(\sigma')} &= \frac{\delta}{\delta\tau(\sigma')}\left[T'(\sigma)G(T(\sigma))\right] = G(T(\sigma))\delta'(\sigma-\sigma') + T'(\sigma)\left.\frac{dG(T)}{dT}\right|_{T=T(\sigma)}\delta(\sigma-\sigma') = \\ &= \frac{d}{d\sigma}\left[G(T(\sigma))\delta(\sigma-\sigma')\right] = G(T(\sigma'))\delta'(\sigma-\sigma') = \frac{f(\sigma'|\tau]}{T'(\sigma')}\delta'(\sigma-\sigma'). \end{aligned}$$

Therefore the solutions for $A^\mu_\pm = \dfrac{d}{d\sigma}B^\mu_\pm$, see equation (3.10), are

$$\begin{aligned} &\begin{cases} A^+_+(\sigma|\tau_+] = -2NT'_+(\sigma), \\ \vec{A}_+(\sigma|\tau_+] = -2N\dfrac{d}{d\sigma}\vec{F}_+(T_+(\sigma)), \\ A^-_+(\sigma|\tau_+] = -2N\dfrac{d}{d\sigma}F^-_+(T_+(\sigma)), \end{cases} \\ &\begin{cases} A^+_-(\sigma|\tau_-] = 2NT'_-(\sigma), \\ \vec{A}_-(\sigma|\tau_-] = 2N\dfrac{d}{d\sigma}\vec{F}_-(T_-(\sigma)), \\ A^-_-(\sigma|\tau_-] = 2N\dfrac{d}{d\sigma}F^-_-(T_-(\sigma)), \end{cases} \end{aligned} \tag{4.16}$$

or

$$
\begin{cases}
B^+_+(\sigma|\tau_+] = -2NT_+(\sigma), \\
\vec{B}_+(\sigma|\tau_+] = -2N\vec{F}_+(T_+(\sigma)), \\
B^-_+(\sigma|\tau_+] = -2NF^-_+(T_+(\sigma)),
\end{cases}
\qquad
\begin{cases}
B^+_-(\sigma|\tau_-] = 2NT_-(\sigma), \\
\vec{B}_-(\sigma|\tau_-] = 2N\vec{F}_-(T_-(\sigma)), \\
B^-_-(\sigma|\tau_-] = 2NF^-_-(T_-(\sigma)),
\end{cases}
\tag{4.17}
$$

where we readsorbed an integration constant in the $c_\pm$'s and the $F_\pm$'s in the expressions for $B^\mu_\pm$, and we have defined

$$
T_\pm(\sigma) = \tau_\pm(\sigma) - c_\pm(\sigma) = \mp\frac{1}{2N}B^+_\pm(\sigma). \tag{4.18}
$$

The constraints $\tilde{\chi}_\pm \approx 0$ impose the following restrictions over the functions $F^-_\pm(T_\pm(\sigma))$:

$$
F'^-_\pm(T_\pm(\sigma)) = \frac{d}{dT_\pm}F^-_\pm(T_\pm)\Big|_{T_\pm=T_\pm(\sigma)} = \frac{1}{2}\left(\frac{d}{dT_\pm}\vec{F}_\pm(T_\pm)\right)^2\Big|_{T_\pm=T_\pm(\sigma)} = \frac{1}{2}\left[\vec{F}'_\pm(T_\pm(\sigma))\right]^2 \tag{4.19}
$$

From $P^\mu(\sigma) = \frac{1}{2}\left[A^\mu_+(\sigma) + A^\mu_-(\sigma)\right]$ and $x^\mu(\sigma) = \frac{1}{2N}\left[B^\mu_+(\sigma) - B^\mu_-(\sigma)\right]$, we get the final solutions:

$$
\begin{cases}
x^+(\sigma|\tau_+,\tau_-] = -[T_+(\sigma) + T_-(\sigma)], \\
\\
\vec{x}\ (\sigma|\tau_+,\tau_-] = -[\vec{F}_+(T_+(\sigma)) + \vec{F}_-(T_-(\sigma))], \\
\\
x^-(\sigma|\tau_+,\tau_-] = -[F^-_+(T_+(\sigma)) + F^-_-(T_-(\sigma))] = \\
\qquad = x^-(0|\tau_+,\tau_-] - \frac{1}{2}\int_0^\sigma d\bar{\sigma}[T'_+(\bar{\sigma})\vec{F}'^2_+(T_+(\bar{\sigma})) + T'_-(\bar{\sigma})\vec{F}'^2_-(T_-(\bar{\sigma}))];
\end{cases}
\tag{4.20'}
$$

$$
\begin{cases}
P^+(\sigma|\tau_+,\tau_-] = -N[T'_+(\sigma) - T'_-(\sigma)], \\
\\
\vec{P}\ (\sigma|\tau_+,\tau_-] = -N\frac{d}{d\sigma}[\vec{F}_+(T_+(\sigma)) - \vec{F}_-(T_-(\sigma))], \\
\\
P^-(\sigma|\tau_+,\tau_-] = -N\frac{d}{d\sigma}[F^-_+(T_+(\sigma)) - F^-_-(T_-(\sigma))] = \\
\qquad = -\frac{N}{2}[T'_+(\sigma)\vec{F}'^2_+(T_+(\sigma)) - T'_-(\sigma)\vec{F}'^2_-(T_-(\sigma))].
\end{cases}
\tag{4.20”}
$$

The total momentum is

$$
\begin{cases}
P^+ = -N\left(T_+(\pi) - T_+(0) - [T_-(\pi) - T_-(0)]\right), \\
\vec{P}\ = -N\left(\vec{F}_+(T_+(\pi)) - \vec{F}_+(T_+(0)) - \left[\vec{F}_-(T_-(\pi)) - \vec{F}_-(T_-(0))\right]\right), \\
P^- = -\frac{N}{2}\int_0^\pi d\sigma\left[T'_+(\sigma)\vec{F}'^2_+(T_+(\sigma)) - T'_-(\sigma)\vec{F}'^2_-(T_-(\sigma))\right].
\end{cases}
\tag{4.21}
$$

Let us define

$$\begin{cases} T_1(\sigma) = \frac{1}{2}(T_+(\sigma) + T_-(\sigma)), \\ T_2(\sigma) = \frac{1}{2}(T_+(\sigma) - T_-(\sigma)), \end{cases} \tag{4.22}$$

so that from equation (4.20') and (4.20")

$$\begin{cases} x^+(\sigma) = -2T_1(\sigma), \\ P^+(\sigma) = -2NT_2'(\sigma). \end{cases} \tag{4.23}$$

We have to choose a class of functions $T_{1,2}(\sigma)$ consistent with the various assumed properties of the canonical variables. In particular with those defined in equations (3.1), (3.8), (3.12). This means that we must choose

$$\begin{cases} T_1(-\sigma) = T_1(\sigma), \qquad T_2(-\sigma) = -T_2(\sigma), \\ T_1(\sigma + 2n\pi) = T_1(\sigma), \\ T_2(\sigma + 2n\pi) = T_2(\sigma) - n\frac{P^+}{N}, \end{cases} \tag{4.24}$$

and in particular from equation (3.8)

$$\lim_{\sigma \to 0+} T_2(\sigma) = 0, \tag{4.25}$$

since $\mathcal{P}^+(0) = 0$ implies, from equation (3.11)

$$(B_+^+(\sigma) + B_-^+(\sigma))|_{\sigma=0} = 0. \tag{4.26}$$

From the conditions (4.24) it follows that $T_1(\sigma)$ is an even continuous periodic function of σ (with period 2π), with derivatives possibly discontinuous in $\sigma = n\pi$. Instead $T_2(\sigma)$ is a continuous function, with continuous derivative, but quasi periodic as shown in equation (4.24).

In particular we get

$$T_2(\pi) - T_2(-\pi) = -\frac{P^+}{N}, \tag{4.27}$$

and, since $T_2(0) = 0$, and $T_2(-\sigma) = -T_2(\sigma)$, we get

$$T_2(\pi) = -\frac{P^+}{2N}. \tag{4.28}$$

With this condition we may check that the expression for P^+ given by equation (4.21) is correct.

This in turn implies that $\tau_2(0)$ and $\tau_2(\pi)$ are fixed, that is they are not "free times":

$$\delta\tau_2(\sigma)|_{\sigma=0,\pi} = 0. \tag{4.29}$$

Since $\delta\tau_1$ and $\delta\tau_2$ determine the multipliers λ_1 and λ_2 in the Dirac hamiltonian, see equation (3.23), this fact means that not both the λ_1, λ_2 are free at the end points.

Taking into account the relation between the $\tau_{1,2}$ and the $\lambda_{1,2}$ as given in equation (4.7), it is possible to verify that $\delta\tau_2(0)|_{\sigma=0,\pi}$ is exactly the relation (B.21) between λ_1 and λ_2 of Appendix B.

The use of the light-cone variables requires $P^+ \neq 0$ (for $a = -$, $P^- \neq 0$) and this fact puts restrictions on $c_\pm(0)$, $c_\pm(\pi)$. From equation (4.18), (4.8) and (3.11) we get (with the already explained meaning of the limit in $\sigma = 0$)

$$\begin{cases} c_\pm(\sigma) = c_\mp(-\sigma) & \Longrightarrow \quad c_+(0) = c_-(0); \\ c_\pm(\sigma + 2n\pi) = c_\pm(\sigma) \pm \dfrac{n}{N} P^+. & \end{cases} \tag{4.30}$$

From equations (4.21) and (4.30) we get

$$\frac{P^+}{N} = c_+(\pi) - c_-(\pi). \tag{4.31}$$

Therefore $T_\pm(\sigma)$ satisfies

$$\begin{aligned} &\begin{cases} T_\pm(\sigma) = T_\mp(-\sigma), \\ T_\pm(\sigma + 2n\pi) = T_\pm(\sigma) \mp \dfrac{n}{N} P^+, \end{cases} \qquad \Longrightarrow \\ &\qquad \Longrightarrow \qquad \begin{cases} T_+(0) = T_-(0), \\ T_+(\pi) = T_-(-\pi) = T_+(-\pi) - \dfrac{P^+}{N} = T_-(\pi) - \dfrac{P^+}{N}, \end{cases} \end{aligned} \tag{4.32}$$

where the equation (4.9) was used. Then equations (4.32), (4.16) and (3.11) imply

$$\begin{aligned} \vec{F}_\pm(T_\pm(\sigma + 2n\pi)) =& \vec{F}_\pm(T_\pm(\sigma) \mp \frac{n}{N} P^+) = \mp \frac{1}{2N} \vec{B}_\pm(\sigma + 2n\pi) = \\ &= \mp \frac{1}{2N} \vec{B}_\pm(\sigma) \mp \frac{n}{N} \vec{P} = \vec{F}_\pm(T_\pm(\sigma)) \mp \frac{n}{N} \vec{P}, \end{aligned} \tag{4.33$'$}$$

$$\begin{aligned} F^-_\pm(T_\pm(\sigma + 2n\pi)) =& F^-_\pm(T_\pm(\sigma) \mp \frac{n}{N} P^+) = \mp \frac{1}{2N} B^-_\pm(\sigma + 2n\pi) = \\ &= \mp \frac{1}{2N} B^-_\pm(\sigma) \mp \frac{n}{N} P^- = F^-_\pm(T_\pm(\sigma)) \mp \frac{n}{N} P^-, \end{aligned} \tag{4.33$''$}$$

so that

$$\frac{d}{d\sigma} \vec{F}_\pm \left(T_\pm(\sigma) \mp \frac{n}{N} P^+ \right) = \frac{d}{d\sigma} \vec{F}_\pm(T_\pm(\sigma)).$$

Through equality $B^\mu_+(\sigma) = -B^\mu_-(-\sigma)$ and equation (4.16), at last we obtain

$$\begin{cases} \vec{F}_+(T_+(\sigma)) = \vec{F}_-(T_-(-\sigma)) = \vec{F}_-(T_+(\sigma)), \\ \\ F^-_+(T_+(\sigma)) = F^-_-(T_-(-\sigma)) = F^-_-(T_+(\sigma)), \end{cases} \quad \Longrightarrow \quad \begin{cases} \vec{F}_+ = \vec{F}_- \equiv \vec{F}, \\ \\ F^-_+ = F^-_- \equiv F^-. \end{cases} \tag{4.34}$$

We are now able to rewrite the solutions (4.20), (4.21) in the following final form:

$$\left\{\begin{array}{l} x^+(\sigma|\tau_+,\tau_-] = -[T_+(\sigma)+T_-(\sigma)],\\ \vec{x}\ (\sigma|\tau_+,\tau_-] = -[\vec{F}(T_+(\sigma))+\vec{F}(T_-(\sigma))],\\ x^-(\sigma|\tau_+,\tau_-] = -[F^-(T_+(\sigma))+F^-(T_-(\sigma))] = \\ \qquad\qquad = x^-(0|\tau_+,\tau_-] - \frac{1}{2}\int_0^\sigma d\bar{\sigma}[T'_+(\bar{\sigma})\vec{F}'^2(T_+(\bar{\sigma}))+T'_-(\bar{\sigma})\vec{F}'^2(T_-(\bar{\sigma}))]; \end{array}\right. \tag{4.35$'$}$$

$$\left\{\begin{array}{l} P^+(\sigma|\tau_+,\tau_-] = -N[T'_+(\sigma)-T'_-(\sigma)],\\ \vec{P}\ (\sigma|\tau_+,\tau_-] = -N\frac{d}{d\sigma}[\vec{F}(T_+(\sigma))-\vec{F}(T_-(\sigma))],\\ P^-(\sigma|\tau_+,\tau_-] = -N\frac{d}{d\sigma}[F^-(T_+(\sigma))-F^-(T_-(\sigma))] = \\ \qquad\qquad = -\frac{N}{2}[T'_+(\sigma)\vec{F}'^2(T_+(\sigma))-T'_-(\sigma)\vec{F}'^2(T_-(\sigma))]; \end{array}\right. \tag{4.35$''$}$$

$$\left\{\begin{array}{l} P^+ = N\left[c_+(\pi)-c_-(\pi)\right],\\ \vec{P}\ = -N\left[\vec{F}(T_+(\pi))-\vec{F}(T_-(\pi))\right],\\ P^- = -\frac{N}{2}\int_0^\pi d\sigma\left[T'_+(\sigma)\vec{F}'^2(T_+(\sigma))-T'_-(\sigma)\vec{F}'^2(T_-(\sigma))\right], \end{array}\right. \tag{4.36}$$

where $T_\pm(\sigma)=\tau_\pm(\sigma)-c_\pm(\sigma)$, $\vec{F}$ is an arbitrary function and $F'^-(u)=\frac{1}{2}\vec{F}'^2(u)$. Moreover, using equations (4.32) and (4.33) we get consistently

$$\begin{array}{r} P^2(\sigma|\tau_+,\tau_-] = N^2T'_+(\sigma)T'_-(\sigma)\left[2\vec{F}'(T_+(\sigma))\cdot\vec{F}'(T_-(\sigma))-\right.\qquad\qquad\\ \left.-\vec{F}'^2(T_+(\sigma))-\vec{F}'^2(T_-(\sigma))\right]\quad\longrightarrow\quad 0, \end{array} \tag{4.37}$$

for $\sigma\longrightarrow\sigma_i$.

In order to define a Cauchy problem for the many-time equations (4.11), let us put

$$c_\pm(\sigma)=\bar{\tau}_\pm(\sigma)\pm\frac{1}{2N}\bar{B}^+_\pm(\sigma)\qquad\Longrightarrow\qquad T_\pm(\sigma)=\tau_\pm(\sigma)-\bar{\tau}_\pm(\sigma)\mp\frac{1}{2N}\bar{B}^+_\pm(\sigma). \tag{4.38}$$

If we assign the following initial data at the "times" $\bar{\tau}_\pm(\sigma)$:

$$\left\{\begin{array}{l} x^\mu(\sigma|\tau_+\tau_-]|_{\tau_\pm=\bar{\tau}_\pm} = \bar{x}^\mu(\sigma)\\ P^\mu(\sigma|\tau_+\tau_-]|_{\tau_\pm=\bar{\tau}_\pm} = \bar{P}^\mu(\sigma), \end{array}\right. \tag{4.39}$$

with $\bar{x}^\mu(\sigma)$, $\bar{P}^\mu(\sigma)$ satisfying the constraints $\tilde{\chi}_\pm(\sigma)\approx 0$, we obtain for $\tau_\pm\to\bar{\tau}_\pm$: $B^+_\pm(\sigma)\to\bar{B}^+_\pm(\sigma)$, where

$$\bar{B}^\mu_\pm(\sigma)=\frac{\sigma}{\pi}\bar{P}^\mu+\bar{\mathcal{P}}^\mu(\sigma)\pm N\bar{x}^\mu(\sigma)$$

and

$$\bar{A}^\mu(\sigma)=\frac{d}{d\sigma}\bar{B}^\mu(\sigma)=\bar{P}^\mu(\sigma)\pm N\bar{x}'^\mu(\sigma).$$

If we remember that

$$\begin{cases} x^\mu(\sigma) = \frac{1}{2N}(B_+^\mu(\sigma) - B_-^\mu(\sigma)), \\ P^\mu(\sigma) = \frac{1}{2}(A_+^\mu(\sigma) + A_-^\mu(\sigma)), \end{cases}$$

we get

$$\begin{aligned} x^\mu(\sigma|\tau_+,\tau_-]\Big|_{\tau_\pm=\bar\tau_\pm} =& \frac{1}{2N}\left\{\bar B_+^\mu[\bar B_+^{+\,-1}(-2NT_+(\sigma))] - \bar B_-^\mu[\bar B_-^{+\,-1}(2NT_-(\sigma))]\right\}\Big|_{\tau_\pm=\bar\tau_\pm} = \\ =& \bar x^\mu(\sigma), \\ P^\mu(\sigma|\tau_+,\tau_-]\Big|_{\tau_\pm=\bar\tau_\pm} =& \frac{1}{2}\frac{d}{d\sigma}\left\{\bar B_+^\mu[\bar B_+^{+\,-1}(-2NT_+(\sigma))] + \bar B_-^\mu[\bar B_-^{+\,-1}(2NT_-(\sigma))]\right\}\Big|_{\tau_\pm=\bar\tau_\pm} = \\ =& \bar P^\mu(\sigma). \end{aligned} \tag{4.40}$$

Remembering equation (4.35), we make the following identification:

$$\begin{cases} \vec F(T_\pm(\sigma)) = \mp\frac{1}{2N}\vec{\bar B}_\pm\left[\bar B_\pm^{+\,-1}(\mp 2NT_\pm(\sigma))\right], \\ F^-(T_\pm(\sigma)) = \mp\frac{1}{2N}\bar B_\pm^-\left[\bar B_\pm^{+\,-1}(\mp 2NT_\pm(\sigma))\right]. \end{cases} \tag{4.41}$$

As shown in equation (3.25), in the usual o.g., fixed by the constraints (3.19), the Dirac multipliers assume the value $\lambda_\pm(\sigma,\tau) = -\frac{1}{4N}$. Equation (4.7) then implies $\tilde\lambda_\pm^{(+)}(\sigma,\tau) = -\frac{1}{2N}A_\pm^+(\sigma,\tau)$. With the $\tilde\lambda_\pm^{(+)}(\sigma,\tau)$ fixed only the evolution in τ is left. This means that $\tau_\pm(\sigma)$ must become function of τ,

$$\tau_\pm^{o.g.}(\sigma) = f_\pm(\sigma,\tau),$$

with $f_\pm(\sigma,\tau)$ a given function, and equation (4.7) becomes a condition for the $f_\pm$s:

$$\begin{aligned} \frac{\delta\tau_\pm^{o.g.}(\sigma)}{\delta\tau} = \frac{\partial f_\pm(\sigma,\tau)}{\partial\tau} &= -\frac{1}{2N}A_\pm^+(\sigma|\tau_\pm^{o.g.}] = \pm\frac{\partial}{\partial\sigma}T_\pm^{o.g.}(\sigma,\tau) = \\ &= \pm\frac{\partial}{\partial\sigma}\left(f_\pm(\sigma,\tau) - c_\pm(\sigma)\right) = \pm\frac{\partial f_\pm(\sigma,\tau)}{\partial\sigma} \mp c'_\pm(\sigma), \end{aligned} \tag{4.42}$$

where we have used the solutions (4.16) in order to express $A_\pm^+$ in terms of $f_\pm$. The solutions of equation (4.42) may be written, for later convenience, in the form

$$T_\pm^{o.g.}(\sigma,\tau) = f_\pm(\sigma,\tau) - c_\pm(\sigma) = c_\pm + d_\pm\cdot(\tau\pm\sigma) + g_\pm(\tau\pm\sigma), \tag{4.43}$$

where $c_\pm$, $d_\pm$ are constants. Equations (4.32) imply

$$\begin{cases} T_\pm^{o.g.}(\sigma,\tau) = T^{o.g.}(\tau\pm\sigma) = c - \frac{P^+}{2\pi N}(\tau\pm\sigma) + g(\tau\pm\sigma), \\ g(u+2\pi) = g(u). \end{cases} \tag{4.44}$$

Equations (2.24) may be recovered from equations (4.35) with the positions

$$\begin{cases} c = -\dfrac{q^+}{2N}, \\ f^+(u) = -g(u), \\ \vec{f}(u) = -\dfrac{\vec{q}}{2N} - \dfrac{\vec{P}u}{2\pi N} - \vec{F}\left[-\dfrac{q^+}{2N} - \dfrac{P^+u}{2\pi N} + g(u)\right], \\ f^-(u) = -\dfrac{q^-}{2N} - \dfrac{P^-u}{2\pi N} - F^-\left[-\dfrac{q^+}{2N} - \dfrac{P^+u}{2\pi N} + g(u)\right], \end{cases} \tag{4.45}$$

where $f^\mu(u+2\pi) = f^\mu(u)$. We may similarly recover the o.g. condition (2.25),

$$\left(\frac{P^\mu}{2\pi N} + \frac{df^\mu(u)}{du}\right)^2 = 0,$$

from $F'^-(u) = \frac{1}{2}\vec{F}'^2(u)$, which follows directly from imposing the constraints $\tilde{\chi}_\pm^{(+)} \approx 0$. Moreover we see that

$$\frac{1}{N}B_+^\mu(\sigma|\tau_+^{o.g.}] = \frac{1}{N}P^\mu(\sigma|\tau_+^{o.g.}, \tau_-^{o.g.}] + x'^\mu(\sigma|\tau_+^{o.g.}, \tau_-^{o.g.}]$$

evaluated by means of the solutions (4.35) coincides with $Q^\mu(\tau+\sigma) = x^\mu(0, \tau+\sigma)$ and that $P^\mu(\sigma|\tau_+^{o.g.}, \tau_-^{o.g.}] \longrightarrow N\dot{x}^\mu(\sigma,\tau)$, as expected.

If we only fix the two multipliers $\lambda_\pm(\sigma,\tau) = -\frac{1}{4N}$, without adding the gauge-fixing constraints (3.19) the gauge is not completely fixed yet. The residual gauge-freedom consists in the conformal transformations connecting every possible orthonormal parametrization. Here this residual gauge-freedom is represented by the arbitrariness of the function g in equation (4.44). As usually done in gauge field theory, we fix the residual gauge through boundary conditions; we get from equations (4.35) and (4.44)

$$\begin{cases} x^+(\sigma,\tau) = q^+ + \dfrac{P^+\tau}{\pi N} + g(\tau+\sigma) + g(\tau-\sigma), \\ P^+(\sigma,\tau) = \dfrac{P^+}{N} - N\dfrac{\partial}{\partial\sigma}[g(\tau+\sigma) - g(\tau-\sigma)] = \dfrac{P^+}{N} - N\dfrac{\partial}{\partial\tau}[g(\tau+\sigma) + g(\tau-\sigma)], \end{cases}$$

and it is therefore sufficient to assign as a boundary condition for instance:

$$x^+(0,\tau) = Q^+(\tau) = q^+ + \frac{P^+\tau}{\pi N} + 2g(\tau) = h(\tau).$$

This implies the following gauge-fixing constraints with well defined associated Dirac brackets:

$$\begin{cases} x^+(\sigma,\tau) - \dfrac{1}{2}[h(\tau+\sigma) + h(\tau-\sigma)] = 0, \\ P^+(\sigma,\tau) - \dfrac{2P^+}{\pi} + \dfrac{N}{2}\dfrac{\partial}{\partial\tau}[h(\tau+\sigma) + h(\tau-\sigma)] = 0. \end{cases} \tag{4.46}$$

We recover equations (3.19) if we choose $h(\tau) = q^+ + \dfrac{P^+\tau}{\pi N}$, *i.e.* $g(\tau) = 0$. In this particular gauge we get

$$\begin{cases} \dfrac{1}{N}B^+_\pm(\sigma|\tau_\pm] \longrightarrow \pm Q^+(\tau \pm \sigma) = \pm \left[q^+ + \dfrac{P^+}{\pi N}(\tau \pm \sigma)\right], \\ P^+(\sigma|\tau_\pm] \longrightarrow \dfrac{P^+}{\pi}, \\ A^+_\pm(\sigma|\tau_\pm] \longrightarrow \dfrac{P^+}{\pi}. \end{cases} \tag{4.47}$$

Now let us look when the denominators of the constraints $\tilde{\chi}^{(+)}_\pm(\sigma,\tau)$ vanish. We start with the solutions (4.35)

$$A^+_\pm(\sigma|\tau^{(+)}_\pm] = \mp\, 2N\frac{d}{d\sigma}T^{(+)}_\pm(\sigma) = \mp 2N\left[\frac{d\tau^{(+)}_\pm(\sigma)}{d\sigma} - \frac{dc^{(+)}_\pm(\sigma)}{d\sigma}\right] = 0, \tag{4.48$'$}$$

$$A^-_\pm(\sigma|\tau^{(-)}_\pm] = \mp\, 2N\frac{d}{d\sigma}T^{(-)}_\pm(\sigma) = \mp 2N\left[\frac{d\tau^{(-)}_\pm(\sigma)}{d\sigma} - \frac{dc^{(-)}_\pm(\sigma)}{d\sigma}\right] = 0, \tag{4.48$''$}$$

where equation (4.48") corresponds to the vanishing of the denominators of $\tilde{\chi}^{(-)}_\pm(\sigma,\tau)$. The solution of equations (4.48) is $\tau^{(a)}_\pm(\sigma) = c^{(a)}_\pm(\sigma) + K^{(a)}_\pm$ (this is why in equation (4.38) we could not put $\bar{\tau}_\pm = c_\pm$). If we now consider the Patrascioiu modes related to equation (2.31), *i.e.*

$$\begin{cases} x^-(\sigma,\tau) = q^- + \dfrac{P^-}{\pi N}\tau + f^-(\tau+\sigma) + f^-(\tau-\sigma), \\ x^+(\sigma,\tau) = q^+ + 2f^+_0, \\ \vec{x}(\sigma,\tau) = \vec{q} + 2\vec{f}_0, \end{cases}$$
$$\begin{cases} P^-(\sigma,\tau) = N\dot{x}^-(\sigma,\tau) = \dfrac{P^-}{\pi N} + \dot{f}^-(\tau+\sigma) + \dot{f}^-(\tau-\sigma), \\ P^+(\sigma,\tau) = N\dot{x}^+(\sigma,\tau) = 0, \\ \vec{P}(\sigma,\tau) = N\dot{\vec{x}}(\sigma,\tau) = 0, \end{cases} \tag{4.49}$$

we get $A^+_\pm(\sigma,\tau) = P^+(\sigma,\tau) \pm Nx'^+(\sigma,\tau) = 0$ (similarly, $A^-_\pm = 0$ for the modes with $P^- = \vec{P} = 0$). In our case, from equation (4.49) we get $A^-_\pm(\sigma,\tau) = \frac{1}{\pi}P^- + 2N\dot{f}^-(\tau\pm\sigma) \neq 0$ and we can use the constraints $\tilde{\chi}^{(-)}_\pm(\sigma,\tau)$ to describe this solution.

The conclusion is that the open manifold of the string needs more than two charts to be described with the many-time approach. The two main charts are *i)* $A^+_\pm(\sigma,\tau) \neq 0$, *ii)* $A^-_\pm(\sigma,\tau) \neq 0$. In the former we use $\tilde{\chi}^{(+)}_\pm$, in the latter $\tilde{\chi}^{(-)}_\pm$ and, where the two charts overlap, it is possible to make the transition from one map to the other by means of a canonical transformation. Once we have chosen one of the two main charts, all the Patrascioiu modes can be described in the nonoverlapping part of the other chart.

But there are cases in which the denominators of both $\tilde{\chi}^{(+)}_\pm$, $\tilde{\chi}^{(-)}_\pm$ vanish, so that neither of the two charts are suitable. We may study these sectors by means of the original constraints $\chi_\pm \approx 0$:

1) $A^-_\pm(\sigma,\tau) = A^+_\pm(\sigma,\tau) = 0$; then $\chi_\pm = 0$ implies $x'^\mu(\sigma,\tau) = P^\mu(\sigma,\tau) = 0$. The solutions are

$$\begin{cases} x^\mu(\sigma,\tau) = f^\mu(\tau), & \text{with } f^\mu \text{ arbitrary} \\ P^\mu(\sigma,\tau) = 0. \end{cases} \tag{4.50}$$

2) $A^+_\pm(\sigma,\tau) = A^-_-(\sigma,\tau) = 0$; together with $\chi_\pm = 0$ they imply $P^+(\sigma,\tau) = x'^+(\sigma,\tau) = \vec{P}(\sigma,\tau) = \vec{x}'(\sigma,\tau) = P^-(\sigma,\tau) - Nx'^-(\sigma,\tau) = 0$, while $P^-(\sigma,\tau) + Nx'^-(\sigma,\tau) = 2P^-(\sigma,\tau) = 2Nx'^-(\sigma,\tau)$ is arbitrary. The solutions are

$$\begin{cases} x^+(\sigma,\tau) = f^+(\tau) \\ \vec{x}(\sigma,\tau) = \vec{f}(\tau) \\ x^-(\sigma,\tau) = f^-(\sigma,\tau) & \text{with } f^\mu \text{ arbitrary} \\ P^+(\sigma,\tau) = \vec{P}(\sigma,\tau) = 0 \\ P^-(\sigma,\tau) = Nf'^-(\sigma,\tau). \end{cases} \tag{4.51}$$

3) $A^+_\pm(\sigma,\tau) = A^-_+(\sigma,\tau) = 0$; with $\chi_\pm = 0$, this imply $P^+(\sigma,\tau) = x'^+(\sigma,\tau) = \vec{P}(\sigma,\tau) = \vec{x}'(\sigma,\tau) = P^-(\sigma,\tau) + Nx'^-(\sigma,\tau) = 0$, $P^-(\sigma,\tau) - Nx'^-(\sigma,\tau) = 2P^-(\sigma,\tau) = -2Nx'^-(\sigma,\tau)$ arbitrary. The solutions are

$$\begin{cases} x^+(\sigma,\tau) = f^+(\tau) \\ \vec{x}(\sigma,\tau) = \vec{f}(\tau) \\ x^-(\sigma,\tau) = f^-(\sigma,\tau) & \text{with } f^\mu \text{ arbitrary} \\ P^+(\sigma,\tau) = \vec{P}(\sigma,\tau) = 0 \\ P^-(\sigma,\tau) = -Nf'^-(\sigma,\tau). \end{cases} \tag{4.52}$$

4) $A^-_\pm(\sigma,\tau) = A^+_-(\sigma,\tau) = 0$; with $\chi_\pm = 0$, this imply $P^-(\sigma,\tau) = x'^-(\sigma,\tau) = \vec{P}(\sigma,\tau) = \vec{x}'(\sigma,\tau) = P^+(\sigma,\tau) - Nx'^+(\sigma,\tau) = 0$, $P^+(\sigma,\tau) + Nx'^+(\sigma,\tau) = 2P^+(\sigma,\tau) = 2Nx'^+(\sigma,\tau)$ arbitrary. The solutions are

$$\begin{cases} x^+(\sigma,\tau) = f^+(\sigma,\tau) \\ \vec{x}(\sigma,\tau) = \vec{f}(\tau) \\ x^-(\sigma,\tau) = f^-(\tau) & \text{with } f^\mu \text{ arbitrary} \\ P^-(\sigma,\tau) = \vec{P}(\sigma,\tau) = 0 \\ P^+(\sigma,\tau) = +Nf'^+(\sigma,\tau). \end{cases} \tag{4.53}$$

5) $A^-_\pm(\sigma,\tau) = A^+_+(\sigma,\tau) = 0$; with $\chi_\pm = 0$, this imply $P^-(\sigma,\tau) = x'^-(\sigma,\tau) = \vec{P}(\sigma,\tau) = \vec{x}'(\sigma,\tau) = P^+(\sigma,\tau) + Nx'^+(\sigma,\tau) = 0$, $P^+(\sigma,\tau) - Nx'^+(\sigma,\tau) = 2P^+(\sigma,\tau) = -2Nx'^+(\sigma,\tau)$ arbitrary. The solutions are

$$\begin{cases} x^+(\sigma,\tau) = f^+(\sigma,\tau) \\ \vec{x}(\sigma,\tau) = \vec{f}(\tau) \\ x^-(\sigma,\tau) = f^-(\tau) & \text{with } f^\mu \text{ arbitrary} \\ P^-(\sigma,\tau) = \vec{P}(\sigma,\tau) = 0 \\ P^+(\sigma,\tau) = -Nf'^+(\sigma,\tau). \end{cases} \tag{4.54}$$

6) $A^+_-(\sigma,\tau) = A^-_-(\sigma,\tau) = 0$; with $\chi_\pm = 0$, this imply $P^\mu(\sigma,\tau) = Nx'^\mu(\sigma,\tau)$; $P^\mu(\sigma,\tau) = +\ Nx'^\mu(\sigma,\tau) = 2P^\mu(\sigma,\tau) = 2Nx'^\mu(\sigma,\tau)$ arbitrary. The solutions are

$$\begin{cases} x^\mu(\sigma,\tau) = f^\mu(\sigma,\tau) & \text{with } f^\mu \text{ arbitrary} \\ P^\mu(\sigma,\tau) = Nf'^\mu(\sigma,\tau). \end{cases} \tag{4.55}$$

7) $A^+_+(\sigma,\tau) = A^-_+(\sigma,\tau) = 0$; with $\chi_\pm = 0$, this imply $P^\mu(\sigma,\tau) = -Nx'^\mu(\sigma,\tau)$; $P^\mu(\sigma,\tau) -\ Nx'^\mu(\sigma,\tau) = 2P^\mu(\sigma,\tau) = -2Nx'^\mu(\sigma,\tau)$ arbitrary. The solutions are

$$\begin{cases} x^\mu(\sigma,\tau) = f^\mu(\sigma,\tau) & \text{with } f^\mu \text{ arbitrary} \\ P^\mu(\sigma,\tau) = -Nf'^\mu(\sigma,\tau). \end{cases} \tag{4.56}$$

8) $A^+_-(\sigma,\tau) = A^-_+(\sigma,\tau) = 0$; with $\chi_\pm = 0$, this imply $P^+(\sigma,\tau) = Nx'^+(\sigma,\tau)$; $P^-(\sigma,\tau) = -Nx'^-(\sigma,\tau)$; $\vec{P}(\sigma,\tau) = \vec{x}'(\sigma,\tau) = 0$; $P^+(\sigma,\tau) + Nx'^+(\sigma,\tau) = 2P^+(\sigma,\tau) = 2Nx'^+(\sigma,\tau)$ and $P^-(\sigma,\tau) - Nx'^-(\sigma,\tau) = 2P^-(\sigma,\tau) = -2Nx'^-(\sigma,\tau)$ arbitrary. The solutions are

$$\begin{cases} x^+(\sigma,\tau) = f^+(\sigma,\tau) & \\ \vec{x}(\sigma,\tau) = \vec{f}(\tau) & \\ x^-(\sigma,\tau) = f^-(\sigma,\tau) & \text{with } f^\mu \text{ arbitrary} \\ P^+(\sigma,\tau) = Nf'^+(\sigma,\tau) & \\ \vec{P}(\sigma,\tau) = 0 & \\ P^-(\sigma,\tau) = -Nf'^-(\sigma,\tau). & \end{cases} \tag{4.57}$$

9) $A^+_+(\sigma,\tau) = A^-_-(\sigma,\tau) = 0$; with $\chi_\pm = 0$, this imply $P^+(\sigma,\tau) = -Nx'^+(\sigma,\tau)$; $P^-(\sigma,\tau) = Nx'^-(\sigma,\tau)$; $\vec{P}(\sigma,\tau) = \vec{x}'(\sigma,\tau) = 0$; $P^+(\sigma,\tau) - Nx'^+(\sigma,\tau) = 2P^+(\sigma,\tau) = -2Nx'^+(\sigma,\tau)$ and $P^-(\sigma,\tau) + Nx'^-(\sigma,\tau) = 2P^-(\sigma,\tau) = 2Nx'^-(\sigma,\tau)$ arbitrary. The solutions are

$$\begin{cases} x^+(\sigma,\tau) = f^+(\sigma,\tau) & \\ \vec{x}(\sigma,\tau) = \vec{f}(\tau) & \\ x^-(\sigma,\tau) = f^-(\sigma,\tau) & \text{with } f^\mu \text{ arbitrary} \\ P^+(\sigma,\tau) = -Nf'^+(\sigma,\tau) & \\ \vec{P}(\sigma,\tau) = 0 & \\ P^-(\sigma,\tau) = Nf'^-(\sigma,\tau). & \end{cases} \tag{4.58}$$

These exceptional cases correspond to the following kinds of motion: i) case (1) is the exceptional Lorentz orbit $P^\mu = 0$; ii) cases (2)-(5) correspond to $P^2 = 0$, $P^\mu(\sigma)//x'(\sigma)$; i.e. they are massless longitudinal motions; iii) cases (6)-(9) correspond to $P^2 \neq 0$, $P^\mu(\sigma)//x'(\sigma)$, i.e. they are massive longitudinal motions.

We have completed the analysis of the solutions of the functional equations of motion, and of the various charts where they are defined. As already said in the introduction this is a preliminary and necessary step towards the search of a complete set of observables, which will be the argument of a subsequent paper.

One point which would deserve a separate analysis is whether the set of charts previously found does constitute an atlas of the constraint manyfold; another point of interest is how to make a right identification of how many kinds of not diffeomorphic gauge orbits exist, taking also into account the Lorentz orbits structure of the manifold.

Appendix A.

Let us look for a reparametrization

$$\zeta^{\alpha} \to \tilde{\zeta}^{\alpha}(\zeta), \quad (\zeta^0 = \tau,\ \zeta^1 = \sigma),$$

with

$$J = \det \parallel \frac{\partial \tilde{\zeta}^{\alpha}}{\partial \zeta^{\beta}} \parallel \neq 0,$$

which change a given induced metric $h_{\alpha\beta}(\zeta)$ into an orthonormal gauge metric

$$\tilde{h}_{\alpha\beta}(\tilde{\zeta}) = \dot{\tilde{x}}^2(\tilde{\zeta})\eta_{\alpha\beta} = e^{\tilde{\phi}}\eta_{\alpha\beta}.$$

As we have

$$\tilde{h}^{\alpha\beta}(\tilde{\zeta}) = \frac{\partial \tilde{\zeta}^{\alpha}}{\partial \zeta^{\gamma}} \frac{\partial \tilde{\zeta}^{\beta}}{\partial \zeta^{\delta}} h^{\gamma\delta}(\zeta), \tag{A.1}$$

from

$$\tilde{h}^{00} + \tilde{h}^{11} = \tilde{h}^{01} = 0$$

we get the two equations

$$\begin{cases} \left(\dfrac{\partial \tilde{\zeta}^0}{\partial \zeta^{\alpha}} \dfrac{\partial \tilde{\zeta}^0}{\partial \zeta^{\beta}} + \dfrac{\partial \tilde{\zeta}^1}{\partial \zeta^{\alpha}} \dfrac{\partial \tilde{\zeta}^1}{\partial \zeta^{\beta}} \right) h^{\alpha\beta} = 0, \\ \dfrac{\partial \tilde{\zeta}^0}{\partial \zeta^{\alpha}} \dfrac{\partial \tilde{\zeta}^1}{\partial \zeta^{\beta}} h^{\alpha\beta} = 0. \end{cases} \tag{A.2}$$

By taking their sum and difference and if

$$\partial_1 \tilde{\zeta}^0 \neq \pm \partial_1 \tilde{\zeta}^1 \quad (\partial_0 = \frac{\partial}{\partial \zeta^0}, \quad \partial_1 = \frac{\partial}{\partial \zeta^1}) :$$

we get

$$\begin{cases} h^{00} \left(\dfrac{\partial_0 \tilde{\zeta}^0 + \partial_0 \tilde{\zeta}^1}{\partial_1 \tilde{\zeta}^0 + \partial_1 \tilde{\zeta}^1} \right)^2 + 2h^{01} \left(\dfrac{\partial_0 \tilde{\zeta}^0 + \partial_0 \tilde{\zeta}^1}{\partial_1 \tilde{\zeta}^0 + \partial_1 \tilde{\zeta}^1} \right) + h^{11} = 0, \\ h^{00} \left(\dfrac{\partial_0 \tilde{\zeta}^0 - \partial_0 \tilde{\zeta}^1}{\partial_1 \tilde{\zeta}^0 - \partial_1 \tilde{\zeta}^1} \right)^2 + 2h^{01} \left(\dfrac{\partial_0 \tilde{\zeta}^0 - \partial_0 \tilde{\zeta}^1}{\partial_1 \tilde{\zeta}^0 - \partial_1 \tilde{\zeta}^1} \right) + h^{11} = 0. \end{cases} \tag{A.3}$$

The equation

$$h^{00} z^2 + 2h^{01} z + h^{11} = 0$$

has the solutions

$$\begin{cases} z_1 = \dfrac{1}{h^{00}} \left(-h^{01} - \sqrt{(h^{01})^2 - h^{00}h^{11}} \right) = \dfrac{1}{h_{11}} \left(h_{01} - \sqrt{-h} \right) = \dfrac{(\dot{x} \cdot x') - \sqrt{-h}}{x'^2}, \\ z_2 = \dfrac{1}{h^{00}} \left(-h^{01} + \sqrt{(h^{01})^2 - h^{00}h^{11}} \right) = \dfrac{1}{h_{11}} \left(h_{01} + \sqrt{-h} \right) = \dfrac{(\dot{x} \cdot x') + \sqrt{-h}}{x'^2}. \end{cases} \tag{A.4}$$

Therefore equations (A.3) have two sets of possible solutions with $J \neq 0$

$$\begin{cases} \partial_0\tilde{\zeta}^0 + \partial_0\tilde{\zeta}^1 = z_1(\partial_1\tilde{\zeta}^0 + \partial_1\tilde{\zeta}^1), \\ \partial_0\tilde{\zeta}^0 - \partial_0\tilde{\zeta}^1 = z_2(\partial_1\tilde{\zeta}^0 - \partial_1\tilde{\zeta}^1), \end{cases} \qquad (A.5')$$

$$\begin{cases} \partial_0\tilde{\zeta}^0 + \partial_0\tilde{\zeta}^1 = z_2(\partial_1\tilde{\zeta}^0 + \partial_1\tilde{\zeta}^1), \\ \partial_0\tilde{\zeta}^0 - \partial_0\tilde{\zeta}^1 = z_1(\partial_1\tilde{\zeta}^0 - \partial_1\tilde{\zeta}^1). \end{cases} \qquad (A.5'')$$

If we put

$$\begin{cases} \tilde{\zeta}^0 + \tilde{\zeta}^1 = F^+(\zeta), \\ \tilde{\zeta}^0 - \tilde{\zeta}^1 = F^-(\zeta), \end{cases} \qquad (A.6)$$

equations (A.5) become

$$\begin{cases} \partial_0 F^+ - z_1(\zeta)\partial_1 F^+ = 0, \\ \partial_0 F^- - z_2(\zeta)\partial_1 F^- = 0, \end{cases} \qquad (A.7')$$

$$\begin{cases} \partial_0 F^+ - z_2(\zeta)\partial_1 F^+ = 0, \\ \partial_0 F^- - z_1(\zeta)\partial_1 F^- = 0, \end{cases} \qquad (A.7'')$$

Let the associated characteristic equations $d\zeta^0 = -\dfrac{d\zeta^1}{z_i(\zeta)}$ have the solutions

$$\rho_i(\zeta) = c_i$$

so that

$$\partial_0\rho_i = z_i\partial_1\rho_i.$$

This implies that the solutions of equations (A.7') (or, respectively, (A.7")) are $\partial_0 F^+(\rho_1(\zeta))$ and $\partial_0 F^-(\rho_2(\zeta))$ (or $\partial_0 F^+(\rho_2(\zeta))$ and $\partial_0 F^-(\rho_1(\zeta))$) with $F^\pm$ arbitrary functions. Therefore we have in general two sets of solutions to our original problem:

$$\begin{cases} \tilde{\tau} = \tilde{\zeta}^0(\zeta) = \dfrac{1}{2}\left[F^+(\rho_1(\zeta)) + F^-(\rho_2(\zeta))\right], \\ \tilde{\sigma} = \tilde{\zeta}^1(\zeta) = \dfrac{1}{2}\left[F^+(\rho_1(\zeta)) - F^-(\rho_2(\zeta))\right], \end{cases} \qquad (A.8')$$

$$\begin{cases} \tilde{\tau} = \tilde{\zeta}^0(\zeta) = \dfrac{1}{2}\left[F^+(\rho_2(\zeta)) + F^-(\rho_1(\zeta))\right], \\ \tilde{\sigma} = \tilde{\zeta}^1(\zeta) = \dfrac{1}{2}\left[F^+(\rho_2(\zeta)) - F^-(\rho_1(\zeta))\right]. \end{cases} \qquad (A.8'')$$

If for instance the original $h_{\alpha\beta}$ was already in an orthonormal gauge ($h_{\alpha\beta} = \dot{x}^2\eta_{\alpha\beta} = e^\phi\eta_{\alpha\beta}$), we have $z_1 = 1$, $z_2 = -1$ and $\rho_1(\zeta) = \tau + \sigma$, $\rho_2(\zeta) = \tau - \sigma$. Equations (A.8') and (A.8") give the reparametrizations that leave invariant a metric of the form $e^\phi\eta_{\alpha\beta}$, apart from a conformal rescaling $\phi \to \tilde{\phi}$. These are the conformal transformations: (A.8') gives the transformations connected to the identity, while (A.8"), containing the spatial inversion $\tilde{\tau} = \tau$, $\tilde{\sigma} = -\sigma$, gives the other connected component of the group.

Let us remark that if we have $\tilde{h}_{\alpha\beta} = e^{\tilde{\phi}}\eta_{\alpha\beta}$ the reparametrization (A.8') gives the following form for a general metric $h_{\alpha\beta}$

$$h_{\alpha\beta}(\zeta) = \frac{\partial\tilde{\zeta}^\gamma}{\partial\zeta^\alpha}\frac{\partial\tilde{\zeta}^\delta}{\partial\zeta^\beta}\tilde{h}_{\gamma\delta}(\tilde{\zeta}) = \frac{1}{2}e^{[\phi(F^+(\zeta),F^-(\zeta))]}\left[\partial_\alpha F^+(\zeta)\partial_\beta F^-(\zeta) + \partial_\beta F^+(\zeta)\partial_\alpha F^-(\zeta)\right]. \tag{A.9}$$

Equation (A.9), when applied to the intrinsic metric $g_{\alpha\beta}$ of the Brink-Di Vecchia-Howe lagrangian, is the starting point of reference [7] to find the string solutions in an arbitrary gauge. Indeed, from

$$(\tilde{\partial}_0^2 - \tilde{\partial}_1^2)\tilde{x}^\mu = 0,$$

we get

$$\tilde{x}^\mu(\tilde{\sigma},\tilde{\tau}) = \alpha^\mu(\tilde{\tau}+\tilde{\sigma}) + \beta^\mu(\tilde{\tau}-\tilde{\sigma}) = x^\mu(\sigma,\tau) = \alpha^\mu(F^+[\rho_1(\sigma,\tau)])) + \beta^\mu(F^-[\rho_2(\sigma,\tau)]), \tag{A.10}$$

where we have used equation (A.8') and where $\rho_i(\zeta)$ are the solutions of the equations

$$\partial_0\rho_i = z_i\partial_1\rho_i,$$

with the z_i given by equations (A.4) in terms of the metric $h_{\alpha\beta}(\sigma,\tau)$.

If now $x^\mu(\sigma,\tau)$ are regular coordinates, with an induced metric having the limit for $\sigma \to 0$ or π as given by the equation (B.27), that is

$$\begin{cases} h_{\alpha\beta}(\sigma,\tau) \xrightarrow[\sigma\to 0]{} \begin{pmatrix} 0 & 0 \\ 0 & -D^2 \end{pmatrix} + \sigma\begin{pmatrix} A & B \\ B & C \end{pmatrix} + \sigma^2\begin{pmatrix} A' & B' \\ B' & C' \end{pmatrix} + O(\sigma^3) \\ h^{\alpha\beta}(\sigma,\tau) \xrightarrow[\sigma\to 0]{} \dfrac{1}{A\sigma}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \dfrac{1}{AD^2}\begin{pmatrix} -\frac{B^2+A'D^2}{A} & +B \\ +B & -A \end{pmatrix} + O(\sigma), \end{cases} \tag{A.11}$$

we have the associated z_i

$$\begin{cases} z_1 = \dfrac{\sqrt{A}}{D}\sqrt{\sigma} - \dfrac{B}{D^2}\sigma - \dfrac{B^2 - AC - A'D^2}{2\sqrt{A}D^3}\sigma^{\frac{3}{2}} + O(\sigma^2), \\ z_2 = -\dfrac{\sqrt{A}}{D}\sqrt{\sigma} - \dfrac{B}{D^2}\sigma + \dfrac{B^2 - AC - A'D^2}{2\sqrt{A}D^3}\sigma^{\frac{3}{2}} + O(\sigma^2), \end{cases} \tag{A.12}$$

and the equations $\partial_0\rho_i = z_i\partial_1\rho_i$ have the following solutions in the neighbourhood of $\sigma = 0$:

$$\rho_i(\sigma,\tau) \xrightarrow[\sigma\to 0]{} \int_0^\tau \frac{\sqrt{A(\lambda)}}{D(\lambda)}d\lambda + 2(-1)^{i+1}\sqrt{\sigma} + O(\sigma). \tag{A.13}$$

Therefore the parametrization we are looking for has the form (A.8') for $\sigma \to 0$:

$$\begin{cases} \tilde{\tau}(\sigma,\tau) \to \alpha(\tau) + \beta(\tau)\sqrt{\sigma} + O(\sigma), \\ \tilde{\sigma}(\sigma,\tau) \to \gamma(\tau)\sqrt{\sigma} + O(\sigma). \end{cases} \tag{A.14}$$

The Jacobian is

$$J = \frac{\partial\tilde{\tau}}{\partial\tau}\frac{\partial\tilde{\sigma}}{\partial\sigma} - \frac{\partial\tilde{\tau}}{\partial\sigma}\frac{\partial\tilde{\sigma}}{\partial\tau} = (\dot{\alpha} + \dot{\beta}\sqrt{\sigma})\frac{\gamma}{2\sqrt{\sigma}} - \frac{\beta}{2\sqrt{\sigma}}\dot{\gamma}\sqrt{\sigma} + O(\sqrt{\sigma}) = \\ = \frac{\dot{\alpha}(\tau)\gamma(\tau)}{2\sqrt{\sigma}} + \frac{1}{2}\left(\dot{\beta}(\tau)\gamma(\tau) - \beta(\tau)\dot{\gamma}(\tau)\right) + O(\sqrt{\sigma}). \tag{A.15}$$

Therefore J diverges at $\sigma = 0$ to transform regular coordinates into singular ones.

Appendix B.

Let us assume that the coordinates $x^\mu(\sigma,\tau)$, either regular or singular, may be expanded in Taylor series in the neighbourhood of $\sigma = \sigma_i$, $i = 1,2$. We shall only consider $\sigma_1 = 0$, for the results for $\sigma_2 = \pi$ are just the same. Moreover we shall consider parametrization such that $\dot{x}^2(\sigma,\tau) \geq 0$, ${x'}^2(\sigma,\tau) \leq 0$. Then, by assumption we have:

$$x^\mu(\sigma,\tau) = \sum_{n=0}^{\infty}\frac{\sigma^n}{n!}x^\mu_n(\tau) \qquad \begin{cases} x^\mu_0(\tau) = x^\mu(0,\tau), & \\ x^\mu_n(\tau) = \left.\dfrac{\partial^n x^\mu(\sigma,\tau)}{\partial\sigma^n}\right|_{\sigma=0} & n > 0. \end{cases} \tag{B.1}$$

The two tangent vectors in $\sigma = 0$ are $\dot{x}^\mu_0(\tau)$ and $x^\mu_1(\tau) = x'^\mu(0,\tau)$. From equation (B.1) we get

$$\begin{aligned} \dot{x}^2(\sigma,\tau) &= \sum_{n=0}^{\infty}\sigma^n\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\dot{x}_m\cdot\dot{x}_{n-m}, \\ {x'}^2(\sigma,\tau) &= \sum_{n=0}^{\infty}\sigma^n\sum_{m=0}^{n}\frac{1}{m!(n-m)!}x_{m+1}\cdot x_{n-m+1}, \\ \dot{x}(\sigma,\tau)\cdot x'(\sigma,\tau) &= \sum_{n=0}^{\infty}\sigma^n\sum_{m=0}^{n}\frac{1}{m!(n-m)!}\dot{x}_{n-m}\cdot x_{m+1}. \end{aligned} \tag{B.2}$$

Equations (B.2) imply

$$\begin{aligned} -h(\sigma,\tau) = &\sum_{n=0}^{\infty}\sigma^n\sum_{m=0}^{n}\sum_{k=0}^{n-m}\sum_{h=0}^{m}\frac{1}{k!(n-m-k)!h!(m-h)!}\cdot \\ &\cdot\left[(\dot{x}_{n-m-k}\cdot x_{k+1})(\dot{x}_{m-h}\cdot x_{h+1}) - (\dot{x}_{n-m-k}\cdot\dot{x}_k)(x_{m-h+1}\cdot x_{h+1})\right] = \\ \equiv &\sum_{n=0}^{\infty}g_n\sigma^n, \\ \frac{\sqrt{-h(\sigma,\tau)}}{N}P^\mu(\sigma,\tau) = &\sum_{n=0}^{\infty}\sigma^n\sum_{m=0}^{n}\sum_{k=0}^{n-m}\frac{(\dot{x}_{n-m-k}\cdot x_{k+1})x^\mu_{m+1} - (x_{n-m-k+1}\cdot x_{k+1})\dot{x}^\mu_m}{k!m!(n-m-k)!}, \\ \frac{\sqrt{-h(\sigma,\tau)}}{N}\Pi^\mu(\sigma,\tau) = &\sum_{n=0}^{\infty}\sigma^n\sum_{m=0}^{n}\sum_{k=0}^{n-m}\frac{(\dot{x}_{n-m-k}\cdot x_{k+1})\dot{x}^\mu_m - (\dot{x}_{n-m-k}\cdot\dot{x}_k)x^\mu_{m+1}}{k!m!(n-m-k)!}. \end{aligned} \tag{B.3}$$

We are looking for the minimal set of conditions we have to require on $x_n^\mu(\tau)$, in order to ensure $P^\mu(0,\tau)$ and $\Pi^\mu(0,\tau)$ to be finite (and obviously $h(\sigma,\tau)\leq 0$), with a total momentum $P^\mu\neq 0$, and $P^2\geq 0$.

It is easily seen that a consistent solution only exists for $h=0$ up to second order in σ. So we will have

$$g_0 = (\dot{x}_0, x_1)^2 - \dot{x}_2^2 {x_1}^2 = 0, \tag{B.4}$$

$$\begin{aligned} g_1 =& 2(\dot{x}_0, x_1)[(\dot{x}-0, x_2) + (x_1, \dot{x}_1)] - \\ &- 2[\dot{x}_0^2 (x_1, x_2) + {x_1}^2 (\dot{x}_0, \dot{x}_1)] = 0, \end{aligned} \tag{B.5}$$

$$\begin{aligned} g_2 =& (\dot{x}_0, x_1)[(\dot{x}_2, x_1) + 2(\dot{x}_1, x_2) + (\dot{x}_0, x_3)] + \\ &+ [(\dot{x}-0, x_2) + (x_1, \dot{x}_1)]^2 - \dot{x}_0^2[(x_1, x_3) + {x_2}^2] - \\ &- 4(x_1, x_2)(\dot{x}_0, \dot{x}_1) - {x_1}^2[(\dot{x}_0, \dot{x}_2) + \dot{x}_1^2] > 0. \end{aligned} \tag{B.6}$$

Moreover, the finiteness of $P^\mu(0,\tau)$ and $\Pi^\mu(0,\tau)$ requires

$$(\dot{x}_0, x_1){x_1}^\mu - {x_1}^2 \dot{x}_0^\mu = 0, \tag{B.7}$$
$$(\dot{x}_0, x_1)\dot{x}_0^\mu - \dot{x}_0^2 {x_1}^\mu = 0. \tag{B.8}$$

Since we are interested in the class (ii) of Section 2, we find first of all that

$${x_1}^2 = \dot{x}_0^2 = (\dot{x}_0, x_1) = 0, \tag{B.9}$$

and, up to second order in σ for $h(\sigma,\tau)$, we get a solution with

$$-h(\sigma,\tau) = g_2(\tau)\sigma^2 + O(\sigma^3), \tag{B.10}$$

where

$$g_2 = (\dot{x}_0, x_2)^2 - 4(x_1, x_2)(\dot{x}_0, x_1) > 0. \tag{B.11}$$

For $P^\mu(\sigma,\tau)$ and $\Pi^\mu(\sigma,\tau)$ (finite in $\sigma=0$), we get

$$P^\mu(\sigma,\tau) = N\frac{(\dot{x}_0, x_2)x_1^\mu - 2(x_1, x_2)\dot{x}_0^\mu}{\sqrt{g_2}} + O(\sigma), \tag{B.12}$$

$$\Pi^\mu(\sigma,\tau) = N\frac{(\dot{x}_0, x_2)\dot{x}_0^\mu - 2(x_1, x_2)x_1^\mu}{\sqrt{g_2}} + O(\sigma). \tag{B.13}$$

The $\lambda_{1,2}$ are given by

$$\lambda_1(\sigma,\tau) = \frac{\sqrt{g_2}}{4N(x_1, x_2)} + O(\sigma), \tag{B.14}$$

$$\lambda_2(\sigma,\tau) = -\frac{(\dot{x}_0, x_2)}{2(x_1, x_2)} + O(\sigma). \tag{B.15}$$

Nevertheless, since in this case (class (ii)) we have that $\dot{x}_0^\mu$ and x_1^μ are collinear, it follows from $\dot{x}_0^2 = 0$ that $(\dot{x}_0, \ddot{x}_0) = 0$, and from this $(\ddot{x}_0, x_1) = 0$, so that, from $(\dot{x}_0, x_1) = 0$ it follows $(\ddot{x}_0, x_1) + (\dot{x}_0, \dot{x}_1) = 0$, that is $(\dot{x}_0, \dot{x}_1) = 0$.

So we have the following simplification

$$
\begin{aligned}
g_2 &= (\dot{x}_0, x_2)^2, && (B.16)\\
P^\mu &= N\frac{(\dot{x}_0, x_2){x_1}^\mu - 2(x_1, x_2)\dot{x}_0^\mu}{|(\dot{x}_0, x_2)|}, && (B.17)\\
\Pi^\mu &= N\frac{(\dot{x}_0, x_2)\dot{x}_0^\mu}{|(\dot{x}_0, x_2)|}, && (B.18)
\end{aligned}
$$

and

$$
\begin{aligned}
\lambda_1 &= \frac{|(\dot{x}_0, x_2)|}{4N(x_1, x_2)}, && (B.19)\\
\lambda_1 &= -\frac{(\dot{x}_0, x_2)}{2(x_1, x_2)}. && (B.20)
\end{aligned}
$$

We see that between λ_1 and λ_2 the following relation holds

$$
2N\lambda_1 \pm \lambda_2 = 0, \qquad (B.21)
$$

according to the sign of $(\dot{x}_0, x_2) >$ or < 0.

In the o.g. case, since $x'^\mu(0, \tau) = 0$ for any τ, we have $x_1^\mu = 0$, and a consistent solution can be recovered to fourth order in σ for $h(\sigma, \tau)$.

If we look for a solution in the regular case (class (i) of Section 2), without necessarily requiring the finiteness of $P^\mu(0, \tau)$ and $\Pi^\mu(0, \tau)$, we find that a solution exists at first order in σ for $-h$.

This means

$$
g_0 = 0, \qquad g_1 > 0, \qquad (B.22)
$$

with

$$
{x_1}^2 < 0, \qquad \dot{x}_0^2 = 0, \qquad (\dot{x}_0, x_1) = 0. \qquad (B.23)
$$

We get

$$
g_1 = -2{x_1}^2(\dot{x}_0, \dot{x}_1), \qquad (B.24)
$$

$$
P^\mu = N\sqrt{-\frac{{x_1}^2}{2(\dot{x}_0, \dot{x}_1)}}\frac{1}{\sqrt{\sigma}}\dot{x}_0^\mu + O(\sqrt{\sigma}), \qquad (B.25)
$$

$$
\Pi^\mu = N\frac{[((\dot{x}_0, x2) + (x_1, \dot{x}_1))\,\dot{x}_0^\mu - 2(\dot{x}_0, \dot{x}_1){x_1}^\mu]}{\sqrt{-2{x_1}^2(\dot{x}_0, \dot{x}_1)}}\sqrt{\sigma} + O(\sigma^{\frac{3}{2}}). \qquad (B.26)
$$

The induced metric $h_{\alpha\beta}$ and its determinant h are given by

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & {x_1}^2 \end{pmatrix} + \sigma \begin{pmatrix} 2(\dot{x}_0, \dot{x}_1) & [(\dot{x}_0, x_2) + (\dot{x}_1, x_1)] \\ [(\dot{x}_0, x_2) + (\dot{x}_1, x_1)] & 2(x_1, x_2) \end{pmatrix} +$$
$$+ \sigma^2 \begin{pmatrix} [(\dot{x}_0, \dot{x}_2) + \dot{x}_1^2] & \frac{1}{2}[(\dot{x}_0, x_1) + 2(\dot{x}_1, x_2) + (\dot{x}_0, x_3)] \\ \frac{1}{2}[(\dot{x}_0, x_1) + 2(\dot{x}_1, x_2) + (\dot{x}_0, x_3)] & [(x_1, x_3) + {x_2}^2] \end{pmatrix} +$$
$$+ O(\sigma^3), \qquad (B.27)$$

$$h = \det \parallel h_{\alpha\beta} \parallel = \sigma[2{x_1}^2(\dot{x}_0, \dot{x}_1)] +$$
$$+ \sigma^2[4(x_1, x_2)(\dot{x}_0, \dot{x}_1) + {x_1}^2\left((\dot{x}_0, \dot{x}_2) + \dot{x}_1^2\right) - ((\dot{x}_0, x_2) + (x_1, \dot{x}_1))^2] +$$
$$+ O(\sigma^3). \qquad (B.28)$$

Appendix C.

We give here some formulas for the distributions $\Delta_\pm(\sigma, \sigma')$, used in the evaluation of some Poisson brackets.

If $f(x)$ and $g(x)$ are periodic functions, with period 2π, and with definite parity given by

$$P_f = \pm 1, \qquad \text{if} \qquad f(x) = \pm f(-x),$$

where, of course, $P_{f'} = -P_f$, we have the following identities

$$f(x)g(y)\Delta'_\pm(x, y) = f(y)g(y)\Delta'_{\pm P_f}(x, y) - f'(x)g(x)\Delta_{\pm P_g}(x, y) \qquad (C.1)$$

$$f(x)g(y)\Delta'_+(x, y) \pm f(y)g(x)\Delta'_-(x, y) = \pm f(x)g(x)\Delta'_{-P_f}(x, y) +$$
$$+ f(y)g(y)\Delta'_{+P_f}(x, y) - f'(x)g(x)\left[\Delta_{+P_g}(x, y) \mp \Delta_{-P_f}(x, y)\right] \qquad (C.2)$$

$$[f(x)g(y) - f(y)g(x)]\,\Delta'_\pm(x, y) = [f(y)g(y) - f(x)g(x)]\,\Delta'_{\pm P_f}(x, y) -$$
$$- f'(x)g(x)\left[\Delta_{\pm P_g}(x, y) + \Delta_{\pm P_f}(x, y)\right] \qquad (C.3)$$

$$[f(x)g(y) + f(y)g(x)]\,\Delta'_\pm(x, y) = [f(y)g(y) + f(x)g(x)]\,\Delta'_{\pm P_f}(x, y) -$$
$$- f'(x)g(x)\left[\Delta_{\pm P_g}(x, y) - \Delta_{\pm P_f}(x, y)\right]. \qquad (C.4)$$

References.

[1] J.Scherk, Rev.Mod.Phys. **47**, 123 (1975);
J.H.Schwarz, Phys.Reports **89**, 223 (1982);
M.B.Green, J.H.Schwarz and E.Witten, *Superstring Theory*, Cambridge University Press, Cambridge, 1987.

[2] L.Lusanna, in Proc. IV M.Grossmann Meeting, Roma 1985, pg.1163, ed. R.Ruffini, Elsevier, Amsterdam 1986.
G.Longhi, *Multitime approach to Nonrelativistic and Relativistic Quantum Mechanics*, talk given at the Encuentros Espaņolos, St.Ander 1984;
G.Longhi, L.Lusanna and J.M.Pons, in Proc. Workshop *Constraint's Theory and Relativistic Dynamics*, eds. G.Longhi and L.Lusanna, Firenze 1986, pg.252; World Scientific, Singapore, 1986.

[3] A.Ashtekar, *New Perspectives in Canonical Gravity*, Bibliopolis, Napoli 1986.

[4] P.A.M.Dirac, Can. J. Math. **2**, 129 (1950);
P.A.M.Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Monographs Series, Yeshiva University, New York, 1964;
J.L.Anderson and P.G.Bergmann, Phys.Rev. **83**, 1018 (1951);
P.G.Bergmann and J.Goldberg, Phys.Rev. **98**, 531 (1955);
E.C.G. Sudarshan and N.Mukunda, *Classical Mechanics: a Modern Perspective*, Wiley and Sons Pub., New York, 1974;
A. Lichnerowicz, C.R.Acad.Sci., Paris **A280**, 523 (1975);
A.J.Hanson, T.Regge and C.Teitelboim, *Constrained Hamiltonian Systems*, Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche, Fisiche e loro Applicazioni, n. 22, Accademia Nazionale dei Lincei, Roma, 1975;
K. Sundermeyer, *Constrained Dynamics with Applications to Yang-Mills Theory, General Relativity, Classical Spin, Dual String Model*, Lecture Notes in Physics, Vol. 169, Springer, Berlin, 1982.

[5] Y.Nambu, Proc.Int.Conf. on *Symmetries and Quark Models*, 1969, Wayne State Univ.;
H.B.Nielsen, Proc. 15-th Conf. in *High Energy Physics*, Kiev Conference, 1970;
L.Susskind, Nuovo Cimento **69**, 457 (1970);
T.Goto, Prog.Theor.Phys. **46**, 1560 (1971).

[6] P.Goddard, J.Goldstone, C.Rebbi and C.B.Thorn, Nucl.Phys. **B56**, 109 (1972);
M.B.Green and J.H.Schwarz, Phys.Lett. **136B**, 367 (1984);
M.B.Green and J.H.Schwarz, Nucl.Phys. **B243**, 285 (1984);
T.Hori and K.Kamimura, Prog.Theor.Phys. **73**, 476 (1985);
M.Kaku and J.Lykken, in *Anomalies, Geometry and Topology*, Argonne 1985, pg.360, eds. W.A.Bardeen and A.R.White,
World Scientific, Singapore, 1985.

[7] S.Hwang and R.Marnelius, Nucl.Phys. **B272**, 389 (1986);
A.P.Isaev, Theor.Math.Phys. **54**, 134 (1983).

[8] L.Brink, P.Di Vecchia and P.Howe, Phys.Lett. **65B**,
L.Brink, P.Di Vecchia and P.Howe, 471 (1976); Nucl.Phys. **B118**, 76 (1977);

S.Deser and B.Zumino, Phys.Lett. **65B**, 369 (1976);
L.Brink and J.H.Schwarz, Nucl.Phys. **B121**, 285 (1977).

[9] A.M.Polyakov, Phys.Lett. **103B**, 207 (1981);
B.Durhuus, H.B.Nielsen, P.Olesen and J.L.Petersen, Nucl.Phys. **B196**, 448 (1982);
B.Durhuus, P.Olesen and J.L.Petersen, Nucl.Phys. **B198**, 157 (1982); **B201**, 176 (1982).

[10] A.J.Hanson, T.Regge and C.Teitelboim, *Constrained Hamiltonian Systems*, Contributi del Centro Linceo Interdisciplinare di Scienze Matematiche, Fisiche e loro Applicazioni, N.22, Accademia Nazionale dei Lincei, Roma, 1975.

[11] R.C.Brower and C.B.Thorn, Nucl.Phys. **B31**, 163 (1971);
R.C.Brower, Phys.Rev. **D6**, 1655 (1972);
P.Goddard and C.B.Thorn, Phys.Lett. **40B**, 2, 235 (1972);
P.Goddard, C.Rebbi and C.B.Thorn, Nuovo Cimento **12A**, 425 (1972);
L.Brink and D.Olive, Nucl.Phys. **B56**, 256 (1973);
L.Brink, D.Olive and J.Scherk, Nucl. Phys. **B61**, 173 (1973);
P.Goddard, A.J.Hanson and G.Ponzano, Nucl.Phys. **B89**, 76 (1975);
C.B.Thorn, Nucl.Phys. **B248**, 551 (1984).

[12] M.Kato and K.Ogawa, Nucl.Phys. **B212**, 443 (1983).

[13] R.C.Brower, Phys.Rev. **D6**, 1655 (1972).

[14] R.Marnelius, Nucl.Phys. **B211**, 14 (1983);
R.Marnelius, Nucl.Phys. **B104**, 477 (1976);
R.Marnelius, Phys.Lett. **B172**, 337 (1986); in Proc. Workshop *Constraint's Theory and Relativistic Dynamics*, Firenze 1986, pg.277, eds. G.Longhi and L.Lusanna, World Scientific, Singapore, 1986.

[15] J.L.Gervais and A.Neveu, Nucl.Phys. **B199**, 59 (1982);
T.L.Curtright and C.B.Thorn, Phys.Rev.Lett. **48**, 1309 (1982).

[16] S.Hwang, Phys.Rev. **D28**, 2614 (1983).

[17] A.Patrascioiu, Lett. Nuovo Cimento **10**, 676 (1974);
A.Patrascioiu, Nucl.Phys. **B81**, 525 (1974).

[18] For a review on the constraint's dynamics see:
T.Takabayasi, Suppl.Prog.Theor.Phys. **67**, 1 (1979);
I.T.Todorov, Sissa Report, Trieste 1980,; in *Quantum Theory, Groups , Fields and Particles*, ed. A.O.Barut, Reidel, Dordrecht 1983;
L.Lusanna, in Proc. *VII Seminar on Problems of High Energy Physics and Quantum Field Theory*, Vol.I, pg. 123, Protvino (USSR) 1984;
L.Lusanna, Nuovo Cimento **65B**, 135 (1985).

[19] M.Henneaux, Phys.Reports **126**, 1 (1985).

[20] R.Abraham and J.E.Marsden, *Foundations of Mechanics*, second edition, Benjamin, Reading 1978, p.425.
C.J.Isham, in *Relativity, Groups and Topology II*, pg.1059, Les Houches XL 1983, eds. B.S.De Witt and R.Stora, Elsevier, Amsterdam 1984.

[21] E.Del Giudice,P.Di Vecchia and S.Fubini, Ann.Phys.(NY) **70**, 378 (1972).

[22] D.C.Salisbury and K.Sundermeyer, Nucl.Phys. **B191**, 250 (1981).

[23] D.G.Boulware and E.T.Newman, Phys.Letters, **B174**, 378 (1986).

[24] X.Artru, Phys.Rev. **97**, 147 (1983).
[25] S.Fubini, D.Gordon and G.Veneziano, Phys. Letters **29B**, 679 (1969).
[26] B.M.Barbashov and V.V.Nesterenko, Sov.J.Part.Phys. **9**, 391 (1978).
[27] M.S.Marinov, Sov.Phys.Usp. **20**, 179 (1977).
[28] R.Benguria, P.Cordero and C.Teitelboim, Nucl.Phys. **B122**, 61 (1977); P.J.Steinhardt, Ann.Phys.(NY) **128**, 425 (1980).
[29] C.Teitelboim, Ann. of Phys. **79**, 542 (1973).
[30] L.E.P. Eisenhart, *Continuous Group of Transformations*, Dover, New York, 1961.
[31] D.Dominici, G.Longhi, J.Gomis and J.M.Pons, J.Math.Phys., **25**, 2439 (1984).
[32] Ph.Droz-Vincent, Rep.Math.Phys.,**8**,79(1075).
[33] D.G.Currie, T.F.Jordan and E.C.G.Sudarshan, Rev.Mod.Phys. **35**, 350 (1963); H.Leutwyler, Nuovo Cimento **37**, 556 (1965).