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# INFINITE CONFORMAL SYMMETRY IN TWO-DIMENSIONAL QUANTUM FIELD THEORY

A.A. Belavin and A.M. Polyakov
Landau Institute for Theoretical Physics, Moscow

and

A.B. Zamolodchikov
CERN -- Geneva
and
Landau Institute for Theoretical Physics, Moscow

## ABSTRACT

We present the investigation of the massless, two-dimensional, interacting field theories. Their basic property is the invariance under an infinite-dimensional group of conformal (analytic) transformations. It is shown that the local fields forming the operator algebra can be classified according to the irreducible representations of Virasoro algebra, and that the correlation functions are built up of the "conformal blocks" which are completely determined by conformal invariance. Exactly solvable conformal theories associated with the degenerate representations are analyzed. In these theories the anomalous dimensions are known exactly and the correlation functions satisfy the systems of linear differential equations.

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## 1. INTRODUCTION

Conformal symmetry was introduced into the quantum field theory about twelve years ago, mainly due to the scaling ideas in the second order phase transition theory (see Ref. [1] and references therein). According to the scaling hypothesis, the interaction of the fields of the order parameters at the critical point is invariant with respect to the scale transformations

$$\xi^a \to \lambda \xi^a \tag{1.1}$$

where $\xi^a$ are the co-ordinates; $a = 1,2,\ldots,D$. In quantum field theory, the scale symmetry (1.1) takes place provided the stress energy tensor is traceless,

$$T_a^a(\xi) = 0 \tag{1.2}$$

Under condition (1.2), the theory possesses not only the scale symmetry but is invariant with respect to all the co-ordinate transformations

$$\xi^a \to \eta^a(\xi) \tag{1.3}$$

having the property that the metric tensor transforms as

$$g_{ab} \to \frac{\partial \xi^{a'}}{\partial \eta^a} \frac{\partial \xi^{b'}}{\partial \eta^b} g_{a'b'} = \rho(\xi) g_{ab} \tag{1.4}$$

where $\rho(\xi)$ is any function. Co-ordinate transformations of this type constitute the <u>conformal group</u>. These transformations can be easily described, the properties of the conformal group being different for the cases $D > 2$ and $D = 2$. If $D > 2$ the conformal group is finite-dimensional and consists of translations, rotations, dilatations and special conformal transformations (see Refs. [2] and [3]). A kinematical manifestation of this symmetry and its dynamical realization in the quantum field theory has been investigated in many papers (see, e.g., Refs. [2]-[4]). In particular, it has been shown that the local fields $A_j(\xi)$, involved in the conformal theory, should possess anomalous scale dimensions $d_j$, i.e., should transform as follows under transformation (1.1)

$$A_j \rightarrow \lambda^{-d_j} A_j \tag{1.5}$$

where the parameters $d_j$ are non-negative. A computation of the spectrum $\{d_j\}$ of the anomalous dimensions is the most important problem of the theory because these quantities determine the critical exponents.

To solve this problem, a bootstrap approach based on the operator algebra hypothesis has been proposed in Ref. [4]. Let us describe it in some detail since it is most suitable for our purposes. The operator algebra is a strong version of the Wilson operator product expansion[5]. Namely, the existence of an infinite set of local fields $A_j(\xi)$ is assumed, then the set of operators $\{A_j(0)\}$ is assumed to be complete in the sense specified below. The set $\{A_j\}$ contains the identity operator I and also all the co-ordinate derivatives of each field involved. The completeness of the set $\{A_j(0)\}$ means that any state can be generated by a linear action of these operators. This condition is equivalent to the operator algebra

$$A_i(\xi)\, A_j(0) = \sum_k C_{ij}^k(\xi)\, A_k(0) \tag{1.6}$$

where the structure constants $C_{ij}^k(\xi)$ are the c-number functions which should be single-valued in order to take into account the locality. Relation (1.6) is understood as an exact expansion of the correlation functions

$$\langle A_i(\xi) A_j(0)\, A_{\ell_1}(\xi_1) \cdots A_{\ell_N}(\xi_N) \rangle =$$
$$= \sum_k C_{ij}^k(\xi) \langle A_k(0) A_{\ell_1}(\xi_1) \cdots A_{\ell_N}(\xi_N) \rangle$$

which is convergent in some finite domain of $\xi$, the domain being certainly dependent on the location of $\xi_1, \dots, \xi_N$. The most restrictive requirement, which is considered as the main dynamical principle in this approach, is the associativity of the operator algebra (1.6). This requirement leads to an infinite system of equations for the structure constants $C_{ij}^k(\xi)$. Since the conformal symmetry fixes the form of the functions $C_{ij}^k(\xi)$ up to some numerical parameters (which are the anomalous dimensions and numerical factors), this system of equations has to determine these parameters. However, in the multidimensional theory (D 2), this system turns out to be too complicated to be solved exactly, the main difficulty being the classification of the fields $A_j$ entering the algebra.

The situation seems to be much better in two dimensions. The main reason is that the conformal group is infinite-dimensional in this case; it consists of the conformal analytic transformations. To describe this group, it is convenient to introduce the complex co-ordinates

$$z = \xi^1 + i\xi^2 \quad ; \quad \bar{z} = \xi^1 - i\xi^2 \tag{1.7}$$

with the metric having the form

$$ds^2 = dz\, d\bar{z} \tag{1.8}$$

The conformal group of the two-dimensional space (which will be denoted by G) consists of all substitutions of the form

$$z \to \zeta(z) \quad ; \quad \bar{z} \to \bar{\zeta}(\bar{z}) \tag{1.9}$$

where $\zeta$ and $\bar{\zeta}$ are arbitrary analytic functions.

For our purposes, it will be convenient to consider the space co-ordinates $\xi^1$, $\xi^2$ as complex variables, i.e., to deal with the complex space $C^2$. Therefore, in general, we shall treat the co-ordinates (1.7) not as complex conjugated but as two independent complex variables; the same is supposed for the functions (1.9). The complex metric (1.8) belo gs to this space. The Euclidean plane and the Minkowski space-time could be obtained as appropriate real sections of this complex space.

In the complex case, it is evident from (1.9) that the conformal group G is a direct product

$$\mathcal{G} = \Gamma \otimes \bar{\Gamma} \tag{1.10}$$

where $\Gamma$ ($\bar{\Gamma}$) is a group of the analytic substitutions of a variable $z$ ($\bar{z}$). In what follows, we shall often concentrate on the properties of the group $\Gamma$, keeping in mind that the same properties hold for $\bar{\Gamma}$.

Infinitesimal transformations of the group $\Gamma$ have the form

$$z \to z + \varepsilon(z) \tag{1.11}$$

where $\varepsilon(z)$ is an infinitesimal analytic function. It can be represented as an infinite Laurent series

$$\varepsilon(z) = \sum_{n=-\infty}^{\infty} \varepsilon_n z^{n+1} \tag{1.12}$$

The Lie algebra of the group $\Gamma$ coincides therefore with that of the differential operators

$$\ell_n = z^{n+1} \frac{d}{dz} \; ; \quad n = 0, \pm 1, \pm 2, \ldots \tag{1.13}$$

the commutation relations having the form

$$[\ell_n, \ell_m] = (n-m)\,\ell_{n+m} \tag{1.14}$$

The generators $\bar{\ell}_n$ of the group $\bar{\Gamma}$ satisfy the same commutation relations, the operators $\ell_n$ and $\bar{\ell}_m$ being commutative. We shall denote the algebra (1.14) as $L_0$.

The generators $\ell_{-1}$, $\ell_0$, $\ell_{+1}$ form the algebra $s\ell(2c)$ $L_0$. The corresponding subgroup $SL(2c) \subset \Gamma$ consists of the projective transformations

$$z \to \zeta = \frac{az+b}{cz+d} \; ; \quad ad - bc = 1 \tag{1.15}$$

Note that the projective transformations are uniquely invertible mappings of the whole z-plane on itself and these are the only conformal transformations with this property.

This is the first paper of a series we intend to devote to the general properties of the two-dimensional quantum field theory invariant with respect to the conformal group G [1]. In this paper we give a general classification of the fields $A_j(\xi)$ entering the operator algebra (1.6) according to the representations of the conformal group and investigate the special exactly solvable cases of the conformal quantum field theory associated with degenerate representations. In more detail, we shall show that:

a) The components of the stress-energy tensor $T_{ab}(\xi)$ [which satisfy (1.2)] represent the generators of the conformal group G in quantum field theory. The algebra of these generators is the central extension of the algebra $L_0$ (1.14) and coincides with the <u>Virasoro algebra</u> $L_c$. The value of the central charge C is the parameter of the theory.

b) Among the fields $A_j(\xi)$ forming the operator algebra, there are some numbers of primary fields $\phi_n(\xi)$ which transform in a simple way

$$\phi_n(z,\bar{z}) \to \left(\frac{d\zeta}{dz}\right)^{\Delta_n}\left(\frac{d\bar{\zeta}}{d\bar{z}}\right)^{\bar{\Delta}_n}\phi_n(\zeta,\bar{\zeta}) \tag{1.16}$$

under the substitution (1.9). Here $\Delta_n$ and $\bar{\Delta}_n$ are real non-negative parameters. In fact, the combinations $d_n = \Delta_n+\bar{\Delta}_n$ and $S_n = \Delta_n-\bar{\Delta}_n$ are the anomalous scale dimension and the spin of the field $\phi_n$, respectively[2]. We shall often call these quantities $\Delta_n$ and $\bar{\Delta}_n$ the <u>dimensions</u> of the field $\phi_n$. The simplest example of a primary field is the identity operator I. A non-trivial theory containing more than one primary field and index n is introduced to distinguish between them.

c) The complete set of fields $A_j(\xi)$ consists of <u>conformal families</u> $[\phi_n]$ each corresponding to some primary field $\phi_n$. The primary field $\phi_n$ belongs to the conformal family $[\phi_n]$ and, in some sense, plays the role of the ancestor of the family. Each conformal family also contains infinitely many other fields ("descendants"). The dimensions of these descendant fields form the integer spaced series

$$\Delta_n^{(\kappa)} = \Delta_n + \kappa \quad ; \quad \bar{\Delta}_n^{(\bar{\kappa})} = \bar{\Delta}_n + \bar{\kappa} \tag{1.17}$$

where $\kappa$, $\bar{\kappa} = 0,1,2,\ldots$ . The variations of any descendant field $A \in [\phi_n]$ under the infinitesimal conformal transformations (1.11) are expressed linearly in terms of the representatives of the same conformal family $[\phi_n]$. So, each conformal family corresponds to some representation of the conformal group G. In accordance with (1.10), this representation is the direct product $[\phi_n] = V_n \otimes \bar{V}_n$, where $V_n$ and $\bar{V}_n$ are the representations of the Virasoro algebra $L_c$ [3]; in general, these representations are irreducible.

d) Correlation functions of any descendant fields can be expressed in terms of the correlators of the corresponding primary fields by means of special linear differential operators. Therefore, all the information about the conformal quantum field theory is collected in the correlators of the primary fields $\phi_n$.

e) The structure constants $C_{ij}^{k}(\xi)$ of the operator algebra (1.6) can be computed, in principle, in terms of the coefficients $C_{nm}^{\ell}$ of the primary field $\phi_\ell$ in the operator product expansion of $\phi_n \phi_m$. Therefore the bootstrap equations (i.e., the associativity condition for the operator algebra) can be reduced to the equations imposing constraints upon these coefficients and the dimensions $\Delta_n$ of the primary fields.

f) At a given value of the central charge c, there are (infinitely many) special values of dimension $\Delta$ such that the representation $[\phi_\Delta]$ proves to be degenerate. The most important property of the corresponding "degenerate" primary field $\phi_\Delta$ is that the correlation functions involving this field satisfy special linear differential equations, whose simplest example is the hypergeometric equation.

g) If the parameter c satisfies the equation

$$\frac{\sqrt{25-c}-\sqrt{1-c}}{\sqrt{25-c}+\sqrt{1-c}} = p/q \qquad (1.18)$$

where p and q are positive integers, then the "minimal" conformal quantum field theory can be constructed so that it is exactly solvable in the following sense: i) a finite number of conformal families $[\phi_n]$ is involved in the operator algebra, each of them being degenerate; ii) all the anomalous dimensions $\Delta_n$ are known exactly; iii) all the correlation functions of the theory can be computed as the solutions of the special systems of linear partial differential equations. There are infinitely many conformal quantum field theories of this type, each is associated with some solution of (1.18) and, the simplest non-trivial example ($c = \frac{1}{2}$) describes the critical theory of the two-dimensional Ising model. We suppose that other "minimal" conformal theories describe the second order phase transitions in some two-dimensional spin systems with discrete symmetry groups.

Apart from second order phase transitions in two dimensions, there is another important application of the conformal quantum field theory. This is the dual theory. From the mathematical point of view, dual models are none other than special kinds of the two-dimensional conformal quantum field theory. This is natural in view of their association with the string theory. The quantum fields describe the degrees of freedom associated with the string, the conformal symmetry being the manifestation of the reparametrization invariance of the world surface swept out by the string. In fact, the dual amplitudes are expressed in terms of correlation functions of some local fields (vertex operators). In standard models (like the Veneziano model), the vertex operators are related in a simple way to free massless fields. We suppose that incorporating the interacting fields into the theory might produce new types of dual models with more suitable physical properties.

## 2. STRESS-ENERGY TENSOR IN CONFORMAL QUANTUM FIELD THEORY

Consider an arbitrary correlation function of the form

$$\langle X \rangle = \langle A_{j_1}(\xi_1) A_{j_2}(\xi_2) \ldots A_{j_N}(\xi_N) \rangle \tag{2.1}$$

where $A_{j_k}(\xi)$ are some local fields, and perform an infinitesimal co-ordinate transformation

$$\xi^a \to \xi^a + \varepsilon^a(\xi) \tag{2.2}$$

As is well known in quantum field theory the following relation is valid

$$\sum_{k=1}^{N} \langle A_{j_1}(\xi_1) \ldots A_{j_{k-1}}(\xi_{k-1}) \delta_\varepsilon A_{j_k}(\xi_k) A_{j_{k+1}}(\xi_{k+1}) \ldots A_{j_N}(\xi_N) \rangle + \tag{2.3}$$
$$+ \int d^2\xi \, \partial^a \varepsilon^b(\xi) \langle T_{ab}(\xi) X \rangle = 0$$

where the field $T_{ab}(\xi)$ is the stress energy tensor and $\delta_\varepsilon A_j(\xi)$ denotes variations of the fields $A_j$ under the transformation (2.2). Due to the locality, these variations are linear combinations of a finite number of derivatives of the function $\varepsilon(\xi)$ taken at the point $\xi = \xi_k$, the coefficients being some local fields. It follows from (2.3) that

$$\partial_a \langle T^{ab}(\xi) X \rangle = 0 \tag{2.4}$$

everywhere, except at the points $\xi_1, \xi_2, \ldots, \xi_N$. In conformal quantum field theory, the trace of the stress energy tensor vanishes, $T_a^a = 0$. Therefore, in the two-dimensional case, this tensor has only two independent components which can be chosen as

$$\begin{aligned} T(\xi) &= T_{11} - T_{22} + 2i\,T_{12} \\ \bar{T}(\xi) &= T_{11} - T_{22} - 2i\,T_{12} \end{aligned} \tag{2.5}$$

Combining relations (1.2) and (2.4), one can easily find that these components satisfy the Cauchy-Riemann equations

$$\begin{aligned} \partial_{\bar z} \langle T(\xi)\, \mathcal{X} \rangle &= 0 \\ \partial_{z} \langle \bar T(\xi)\, \mathcal{X} \rangle &= 0 \end{aligned} \tag{2.6}$$

where z and $\bar z$ are defined by (1.7). So, each of the fields T and $\bar T$ is an analytic function of a single variable (z and $\bar z$ respectively) and we shall write

$$T = T(z) \quad ; \quad \bar T = \bar T(\bar z) \tag{2.7}$$

Let us concentrate on the correlation function[4)]

$$\langle T(z)\, \mathcal{X} \rangle \tag{2.8}$$

It is the analytic function of z which is single-valued (due to the locality) and regular everywhere but at the points $z = z_k$; $z_k = \xi_k^1 + i\xi_k^2$, where it has poles, the orders and residues of these poles being determined by the conformal properties of the fields $A_{jk}(\xi)$. Actually, for the conformal co-ordinate transformations (1.11) the relation can be reduced to the form

$$\langle \delta_\varepsilon \mathcal{X} \rangle = \oint_C d\zeta\, \varepsilon(\zeta) \langle T(\zeta)\, \mathcal{X} \rangle \tag{2.9}$$

where $\delta_\varepsilon X$ is a variation of the product $X = A_{j1}(\xi_1),\ldots,A_{jn}(\xi_N)$ under the transformation (1.11) and the contour C encloses all the singular points $z_k$, $k = 1,2,\ldots,N$. Equivalently, the following relation is valid

$$\delta_\varepsilon A_j(z,\bar z) = \oint_{C_z} d\zeta\, \varepsilon(\zeta) T(\zeta) A_j(z,\bar z) \tag{2.10}$$

where the contour $C_z$ surrounds the point z. The same formula (with the substitution $T \to \bar T$) holds for the variation $\delta_{\bar\varepsilon} A_j$ of the field $A_j$ under the infinitesimal transformation

$$\bar z \to \bar z + \bar\varepsilon(\bar z) \tag{2.11}$$

of the group $\bar{\Gamma}$. Therefore, the fields T(z) and $\bar{T}(\bar{z})$ represent the generators of the conformal group $\Gamma\otimes\bar{\Gamma}$ in quantum field theory.

Conformal transformation laws for the general fields $A_j$ will be considered in the next Section. We are now interested in the conformal properties of the fields T(z) and $\bar{T}(\bar{z})$ themselves which are obviously related to the algebra of the generators of the conformal group. The variations $\delta_\varepsilon T$ and $\delta_{\bar{\varepsilon}}\bar{T}$ should be expressed linearly in terms of the same fields T and $\bar{T}$ and their derivatives, and may also include c number Schwinger terms. Taking into account the tensor properties of the field T(z) and the locality condition, one can write down the following most general expression for the variation $\delta_\varepsilon T$ :

$$\delta_\varepsilon T(z) = \varepsilon(z)T'(z) + 2\varepsilon'(z)T(z) + \frac{c}{12}\varepsilon'''(z) \qquad (2.12)$$

where the prime denotes the z derivative[5]. For the variation $\delta_{\bar{\varepsilon}} T$ one can obtain, by the same consideration

$$\delta_{\bar{\varepsilon}} T(z) = 0 \qquad (2.13)$$

The numerical constant c in relation (2.12) is not determined by general principles; it should be treated as the parameter of the theory. The variation $\delta_{\bar{\varepsilon}}\bar{T}$ satisfies the same relation (2.12), the corresponding constant $\bar{c}$ being equal to c. The constant c can take real positive values. These statements follow from the reality condition for the stress energy tensor in Euclidean space and in Minkowski space-time.

If none of the points $z_k$, k = 1,2,...,N in (2.1) is equal to infinity, then the correlation function $\langle T(z)X\rangle$ should be regular at $z = \infty$. This means, as one can easily verify using the transformation law (2.12), that the function $\langle T(z)X\rangle$ decreases as

$$T(z) \sim 1/z^4 \qquad at \qquad z \to \infty . \qquad (2.14)$$

In quantum field theory the correlation functions (2.1) are represented as the vacuum expectation values of the time-ordered products of the local field operators $A_j(\xi)$. In our case it is convenient to introduce the co-ordinates $\sigma$ and $\tau$ according to

$$z = \exp(\tau + i\sigma) \; ; \; \bar{z} = \exp(\tau - i\sigma) \tag{2.15}$$

Choosing both $\sigma$ and $\tau$ to be real, $\sigma$ being the angular variable, $0 \quad \sigma \quad \pi$, one gets the Euclidean real section. The correlation functions in this Euclidean space can be represented as

$$\langle X \rangle = \langle 0 | T [ A_{j_1}(\sigma_1, \tau_1) \cdots A_{j_N}(\sigma_N, \tau_N) ] | 0 \rangle \tag{2.16}$$

where the chronological ordering should be performed with respect to the "Euclidean time" $\tau$. In the operator formalism, the variations $\delta_\varepsilon A_j$ can be expressed in terms of equal-time commutators

$$\delta_\varepsilon A_j(\sigma, \tau) = [ T_\varepsilon , A_j(\sigma, \tau) ] \tag{2.17}$$

where the generators $T_\varepsilon$ are defined by

$$T_\varepsilon = \oint_{\log|z| = \tau} \varepsilon(z) T(z) dz \tag{2.18}$$

Note that due to Eqs. (2.7) these operators are in fact $\tau$ independent.

Relation (2.12) becomes :

$$[ T_\varepsilon , T(z) ] = \varepsilon(z) T'(z) + 2\varepsilon'(z) T(z) + \frac{c}{12} \varepsilon'''(z) \tag{2.19}$$

It is useful to introduce the operators $L_n$, $\bar{L}_n$; $n = 0, \pm 1, \mp 2, \ldots$, as coefficients of the Laurent expansions

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \; ; \; \bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \frac{\bar{L}_n}{\bar{z}^{n+2}} \tag{2.20}$$

It follows from (2.19) that the operators $L_n$ satisfy the commutation relations

$$[ L_n , L_m ] = (n-m) L_{n+m} + \frac{c}{12}(n^3 - n) \delta_{n+m,0} \tag{2.21}$$

Clearly, the same relations are satisfied by $\bar{L}_n$'s, the operators $L_n$ and $\bar{L}_m$ being commutative. The algebra (2.21) of the conformal generators $L_n$ is the central extension of the algebra (1.14)[6]. This fact is well known in the dual theory and the algebra (2.21) is called the Virasoro algebra [6]--[11]; we shall denote it as $L_c$.

Like the algebra $L_o$, the Virasoro algebra $L_c$ contains the subalgebra $s\ell(2c)$ generated by the operators $L_{-1}$, $L_0$, $L_{+1}$ [note that the c number term in (2.21) vanishes for $n=0,\pm1$]. In particular, the operators $L_{-1}$ and $\bar{L}_{-1}$ generate translations, whereas $L_0$ and $\bar{L}_0$ generate infinitesimal dilatations of the co-ordinates $z$ and $\bar{z}$. In the co-ordinate system $\sigma$, $\tau$ defined by (2.15), the operator

$$H = L_0 + \bar{L}_0 \tag{2.22}$$

is the generator of the "time" shifts. So, it plays the role of the Hamiltonian. Note that the "infinite past" $\tau \to -\infty$ and the "infinite future" $\tau \to \infty$ correspond to the points $z = 0$ and $z = \infty$, respectively.

The vacuum $|0$ in (2.16) is the ground state of the Hamiltonian (2.22). The vacuum must satisfy the equations

$$L_n |0\rangle = 0 \qquad \text{if} \qquad n \geq -1 \tag{2.23}$$

because otherwise the stress energy tensor would have been singular at $z=0$. Note that the operators $L_n$ with $n$ $-1$ generate the conformal transformations which are regular at $z = 0$. Therefore Eq. (2.23) is the manifestation of the conformal invariance of the vacuum. Transformations generated by the operators $L_n$ with $n$ $-2$ are singular at $z = 0$; these operators distort the vacuum,

$$L_n |0\rangle = \text{new states} \qquad \text{if} \quad n \leq -2 \tag{2.24}$$

The field $T(z)$ should also be regular at $z = \infty$. Similarly to (2.23), this means that

$$\langle 0| L_n = 0 \qquad \text{if} \qquad n \leq 1 \tag{2.25}$$

Since in the Minkowski space-time (which can be obtained by continuation to imaginary values of $\tau$) the field T(z) must be rea., the operators $L_n$ satisfy the conjugation relation

$$L_n^+ = L_{-n} \tag{2.26}$$

Note that the generators $L_{-1}$, $L_0$, $L_1$ annihilate both the "in" and "out" vacua

$$\langle 0| L_s = L_s |0\rangle = 0 \qquad ; \; s = 0, \pm 1. \tag{2.27}$$

These equations are the manifestations of the regularity of the projective transformations mentioned in the Introduction. Equations (2.27) are self-consistent because the c number term in (2.21) vanishes for n = 0, ±1.

Equations (2.23), (2.25) and the commutation relations (2.21) enable one to compute any correlation function of the form[7)]

$$\begin{aligned} &\langle T(\zeta_1) \cdots T(\zeta_N) \bar{T}(\eta_1) \cdots \bar{T}(\eta_M) \rangle = \\ &= \langle T(\zeta_1) \cdots T(\zeta_N) \rangle \langle \bar{T}(\eta_1) \cdots \bar{T}(\eta_M) \rangle \end{aligned} \tag{2.28}$$

In particular, a two-point function is given by the formula

$$\langle T(\zeta_1) T(\zeta_2) \rangle = \frac{c}{2} (\zeta_1 - \zeta_2)^{-4} \tag{2.29}$$

which shows that $c > 0$.

## 3. WARD IDENTITIES AND CONFORMAL FAMILIES

Consider the variations $\delta_\varepsilon a_j(\xi)$ of some local field $A_j$ under the infinitesimal conformal transformation (1.11). Due to its local properties, this variation is a linear combination of the function $\varepsilon(z)$ and finite number of its derivatives taken at the point $z = \xi^1 + i\xi^2$

$$\delta_\varepsilon A_j(z) = \sum_{k=0}^{\nu_j} B_j^{(k-1)}(z) \frac{d^k}{dz^k} \varepsilon(z) \tag{3.1}$$

where $B_j^{(k-1)}$ are local fields belonging to the set $\{A_j\}$, and $\nu_j$ is a certain integer. In (3.1) we have omitted the argument $\bar{z}$ which is not important here. The study of infinitesimal translations and dilatations of the variable z shows that the first and second coefficients in (3.1) are

$$B_j^{(-1)}(z) = \frac{\partial}{\partial z} A_j(z) \quad ; \quad B_j^{(0)}(z) = \Delta_j A_j(z) \tag{3.2}$$

where $\Delta_j$ is the dimension of the field $A_j$. It is evident that the dimensions of the fields $B_j^{(k-1)}$ in (3.1) are equal to

$$\Delta_{j,(k-1)} = \Delta_j + 1 - k \quad ; \quad k = 0, 1, \ldots, \nu_j \tag{3.3}$$

Let us concentrate again on the correlation function (2.8). As has already been mentioned in the previous section, this correlator is a single-valued analytic function of z possessing poles at $z = z_k$; $k = 1,2,\ldots,N$. Because of (2.10) and (3.1), one can write down the relation

$$\begin{aligned} &\langle T(z) A_{j_1}(z_1) \cdots A_{j_N}(z_N) \rangle = \\ &= \sum_{\ell=1}^{N} \sum_{k=0}^{\nu_\ell} k!\,(z - z_\ell)^{-k-1} \langle A_{j_1}(z_1) \ldots A_{j_{\ell-1}}(z_{\ell-1}) B_{j_\ell}^{(k-1)}(z_\ell) A_{j_{\ell+1}}(z_{\ell+1}) \ldots A_{j_N}(z_N) \rangle \end{aligned} \tag{3.4}$$

This formula is a general form of the conformal Ward identities.

In a physically suitable theory, the dimensions $\Delta_j$ of all the fields $A_j$ should satisfy the inequality

$$\Delta_j \geq 0 \tag{3.5}$$

since otherwise the theory would possess correlations increasing with the distance. In what follows, we shall suppose that the only field with zero dimensions $\Delta = \bar{\Delta} = 0$ is the identity operator I. Comparing (3.3) with the condition (3.5) we see that the sum in (3.1) contains a finite number of terms $\nu_j$ $\Delta_j+1$. Another important conclusion which follows from (3.3) is that the spectrum of dimensions $\{\Delta_j\}$ in any two-dimensional conformal quantum field theory consists of the infinite integer spaced series

$$\Delta_n^{(k)} = \Delta_n + k \quad ; \quad k = 0, 1, 2, \ldots \quad . \tag{3.6}$$

Here $\Delta_n$ denotes the minimal dimension of each series, whereas the index n labels the series. The same is obviously valid for the dimensions $\bar{\Delta}_j$, i.e., the spectrum $\{\bar{\Delta}_j\}$ also consists of the series

$$\bar{\Delta}_n^{(k)} = \bar{\Delta}_n + \bar{k} \quad ; \bar{k} = 0, 1, 2, \cdots \quad . \tag{3.7}$$

Let $\phi_n$ be a field with dimensions $\Delta_n$ and $\bar{\Delta}_n$. The variation (3.1) of this field has the simplest possible form

$$\delta_\varepsilon \phi_n(z) = \varepsilon(z) \frac{\partial}{\partial z} \phi_n(z) + \Delta_n \varepsilon'(z) \phi_n(z) \tag{3.8}$$

since the corresponding fields $B^{(k-1)}$ with k 0 would have dimensions smaller than $\Delta_n$. A similar formula holds for the variation $\delta_{\bar{\varepsilon}} \phi_n$. The finite form of this conformal transformation law is given by (1.16). We shall call the operators $\phi_n$ with transformation laws (1.16) the <u>primary fields</u>. Note that Eq. (3.8) is equivalent to the commutation relations

$$[L_m, \phi_n(z)] = z^{m+1} \frac{\partial}{\partial z} \phi_n(z) + \Delta_n (m+1) z^m \phi_n(z) \tag{3.9}$$

which are satisfied by the vertex operators of dual theory [8,9].

If all the fields $A_j(\xi)$ entering the correlation function (2.8) are primary ones, the general relation (3.4) is reduced to the form

$$\langle T(z) \phi_1(z_1) \ldots \phi_N(z_N) \rangle =$$
$$= \sum_{i=1}^{N} \left\{ \frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \frac{\partial}{\partial z_i} \right\} \langle \phi_1(z_1) \ldots \phi_N(z_N) \rangle \tag{3.10}$$

where $\Delta_1, \Delta_2, \ldots, \Delta_N$ are the dimensions of the primary fields $\phi_1, \phi_2, \ldots, \phi_N$ respectively. Note that this Ward identity relates explicitly the correlation functions $T(z)\phi_1,\ldots,\phi_N$ to the correlators $\phi_1 \ldots,\phi_N$ . It is also worth noticing that the projective conformal Ward identities (A.6) can be directly obtained from (3.10) if one takes into account the asymptotic condition (2.14).

The primary fields themselves cannot form a closed operator algebra. In fact, there are infinitely many other fields associated with each of the primary fields $\phi_n$. We shall refer to these fields as secondary with respect to the primary field $\phi_n$. The dimensions of the secondaries form the integer spaced series mentioned above. These secondaries, together with the primary field $\phi_n$ , constitute the conformal family $[\phi_n]$. It is essential that under the conformal transformations every member of each conformal family transforms in terms of the representations of the same conformal family. So, each conformal family forms some irreducible representation of the conformal algebra. A complete set of fields $\{A_j\}$ consists of some number (which can be infinite) of the conformal families

$$\{A_j\} = \bigoplus_n [\phi_n] \tag{3.11}$$

To understand the nature of these secondary fields, consider the product $T(\xi)\phi_n(z,\bar{z})$. This product can be expanded according to (1.6), the coefficients $C_{ij}^k$ being single-valued analytic functions of $\xi-z$ due to the relation (2.7) and the local properties of the fields $T(\xi)$ and $\phi_n(z,\bar{z})$. Therefore, this product can be represented in the form

$$T(\zeta)\,\phi_n(z) = \sum_{k=0}^{\infty} (\zeta - z)^{-2+k}\, \phi_n^{(-k)}(z) \tag{3.12}$$

where we have again omitted the dependences of the fields on the variable $\bar{z}$. The dimensions of the fields $\phi_n^{(-k)}$ are given by (3.7). The singular terms in (3.12) are completely determined by the transformation law (3.8) [remember (2.10)]. Thus the first two coefficients in (3.12) are

$$\phi_n^{(-1)}(z) = \frac{\partial}{\partial z}\phi_n(z) \quad ; \quad \phi_n^{(0)}(z) = \Delta_n \phi_n(z) \tag{3.13}$$

The coefficients $\phi_n^{(-k)}$; $k = 2,3,\dots$, of the regular terms in (3.12) are new local fields. To prove the existence of these fields, it is possible to expand the Ward identity (3.10) in power series in, say, $z-z_1$. These new fields are representatives of the conformal family $[\phi_n]$; $\phi_n^{(-k)} \in [\phi_n]$. The conformal properties of these secondary fields $\phi_n^{(-k)}$ are more complicated than those of the primary field $\phi_n$. An infinitesimal conformal transformation and comparison of the variations of both sides of (3.12) yields

$$\delta_\varepsilon \phi_n^{(-k)}(z) = \varepsilon(z)\frac{\partial}{\partial z}\phi_n^{(-k)}(z) + (\Delta_n + k)\,\varepsilon'(z)\,\phi_n^{(-k)}(z) + \sum_{\ell=1}^{k}\frac{k+\ell}{(\ell+1)!}\left[\frac{d^{\ell+1}}{dz^{\ell+1}}\varepsilon(z)\right]\phi_n^{(\ell-k)}(z) + \frac{c}{12}\frac{1}{(k-2)!}\left[\frac{d^{k+1}}{dz^{k+1}}\varepsilon(z)\right]\phi_n(z) \tag{3.14}$$

The fields $\phi_n^{(-k)}$ do not exhaust the conformal family $[\phi_n]$. Consider, for instance, the operator product expansion

$$T(\zeta)\,\phi_n^{(-k_2)}(z) = \frac{c}{12}(\zeta - z)^{-k_2-2}(k_2^3 - k_2)\,\phi_n(z) + \sum_{\ell=1}^{k_2}(\zeta - z)^{-\ell-2}(\ell + k_2)\,\phi_n^{(\ell-k_2)}(z) + \sum_{k_1=0}^{\infty}(\zeta - z)^{-2+k_1}\,\phi_n^{(-k_1,-k_2)}(z) \tag{3.15}$$

The generators accompanying the singular terms in (3.15) are unambiguously determined by Eq. (3.14). In particular

$$\phi_n^{(-1,-k)}(z) = \frac{\partial}{\partial z}\phi_n^{(-k)}(z)\ ; \quad \phi_n^{(0,-k)}(z) = (\Delta_n + k)\,\phi_n^{(-k)}(z) \tag{3.16}$$

The new local fields $\phi_n^{(-k_1,-k_2)}$ with $k_1 \rangle 1$ also belong to the conformal family $[\phi_n]$. The variations $\delta_\varepsilon \phi_n^{(-k_1,-k_2)}$ are expressed in terms of fields $\phi_n^{(-\ell_1,-\ell_2)}$, $\phi_n^{(-\ell)}$ and $\phi_n$.

Considering the operator products $T(\xi)\phi_n^{(-k_1,-k_2)}(z),\ldots,$etc., one can discover an infinite set of secondary fields

$$\phi_n^{(-k_1,-k_2,\ldots,-k_N)}(z) \tag{3.17}$$

where $k_i \geq 1$ and $N = 1,2,\ldots$ . The fields (3.17) can be defined by the explicit formula

$$\phi_n^{(-k_1,-k_2,\ldots,-k_N)}(z) = L_{-k_1}(z)L_{-k_2}(z)\cdots L_{-k_N}(z)\,\phi_n(z) \tag{3.18}$$

where the operators $L_{-k}(z)$ are given by the contour integrals

$$L_{-k}(z) = \oint \frac{d\xi\, T(\xi)}{(\xi - z)^{k+1}} \tag{3.19}$$

The integration contours associated with each of the operators $L_{-k_i}(z_i)$ in (3.18) enclose the point z and also the points $\xi_{i+1}$, $\xi_{i+2},\ldots,\xi_N$ which are the integration variables corresponding to the operators L on the right-hand side of $L_{-k_i}$ [8]. The dimensions of the fields (3.17) are

$$\Delta_n^{(k_1,\ldots,k_N)} = \Delta_n + k_1 + k_2 + \cdots + k_N \tag{3.20}$$

The infinite set of fields (3.17) constitutes the conformal family $[\phi_n]$. These fields are not linearly independent (see below). In fact, in general, the fields (3.17) with $k_1 \leq k_2 \;,\ldots, \leq k_N$ form the basis [9]. Note that

$$\phi_n^{(-1,-k_1,-k_2,\ldots,-k_N)} = \frac{\partial}{\partial z}\phi_n^{(-k_1,-k_2,\ldots,-k_N)} \tag{3.21}$$

Therefore, the conformal family $[\phi_n]$ naturally includes all the derivatives of each field involved. One can derive from (3.18) that the variations $\delta_\varepsilon \phi_n^{\{k\}}$; $\{k\} = (-k_1,-k_2,\ldots,-k_N)$, are expressed in terms of the fields belonging to the same conformal family $[\phi_n]$ and therefore each conformal family corresponds to some representation of the conformal algebra.

To describe the structure of the representation $[\phi_n]$, it is convenient to turn again to the operator formalism. Let us introduce the vectors (primary states)

$$|n\rangle = \phi_n(0)|0\rangle \tag{3.22}$$

Using the properties (2.23) of the vacuum $|0\rangle$ and the commutation relations (3.9), one can get

$$\begin{aligned} L_m|n\rangle &= 0 \qquad \text{if} \quad m>0 \\ L_0|n\rangle &= \Delta_n|n\rangle \end{aligned} \tag{3.23}$$

It follows from (3.18) that

$$\phi_n^{(-k_1,\ldots,-k_N)}(0)|0\rangle = L_{-k_1}L_{-k_2}\ldots L_{-k_N}|n\rangle \tag{3.24}$$

So, the conformal family $[\phi_n]$ is isomorphic to the space of states generated from the primary state $|n\rangle$ by the negative components $L_m$, $m < 0$ [10]. In the representation theory this space is known as the Verma modulus $V_n$ (see for example Ref. [6]). Due to the relations (2.21) there are linear dependences between the vectors (3.24). As has been mentioned above, in all the cases excluding some special values of $\Delta_n$ (see Section 5), the states (3.24) with $k_1 \le k_2 \le, \ldots, \le k_N$ form the basis of $V_n$. Note, that the vectors (3.24) are the eigenstates of the operator $L_0$, the eigenvalues being given by (3.20).

Up to now we have only dealt with the subgroup $\Gamma$ of the conformal group G. Actually, more precise definitions are required. Since the complete conformal group is the direct product (1.10), the representations $[\phi_n]$ are in fact the direct products of the representations of $\Gamma$ and $\bar{\Gamma}$

$$[\phi_n] = V_n \otimes \overline{V_n} \tag{3.25}$$

This means that it contains not only the vectors (3.24) but all the states of the form

$$\phi_n^{\{k\}\{\bar{k}\}}(0)|0\rangle = $$
$$= L_{-k_1} L_{-k_2} \cdots L_{-k_N} \bar{L}_{-\bar{k}_1} \cdots \bar{L}_{-\bar{k}_M} |n\rangle \tag{3.26}$$
$$\{k\} = (-k_1, -k_2, \dots, -k_N) \; ; \; \{\bar{k}\} = (-\bar{k}_1, -\bar{k}_2, \dots, -\bar{k}_M)$$

where $\{k\} = (-k_1, k_2, \dots, k_N)$; $\{\bar{k}\} = (-\bar{k}_1, -\bar{k}_2, \dots, -\bar{k}_M)$; $k_i$ and $\bar{k}_j$ are independent positive integers. Remember that the operators $L$ and $\bar{L}$ are commutative. According to (1.16), the primary state $|n\rangle$ satisfies besides (3.23), the equations

$$\bar{L}_m |n\rangle = 0 \quad \text{if } m > 0,$$
$$\bar{L}_0 |n\rangle = \bar{\Delta}_n |n\rangle \tag{3.27}$$

Therefore each conformal family $[\phi_n]$ is characterized by two parameters $\Delta_n$ and $\bar{\Delta}_n$.

Because of the conformal invariance, the two-point functions $\langle \phi_n(\xi_1)\phi_m(\xi_2)\rangle$ vanish unless the fields $\phi_n$ and $\phi_m$ have the same dimensions (see Appendix A). Moreover, the system of the primary fields can always be chosen to be orthonormal

$$\langle \phi_n(z_1,\bar{z}_1)\phi_m(z_2,\bar{z}_2)\rangle = \delta_{nm}(z_1 - z_2)^{-2\Delta_n}(\bar{z}_1 - \bar{z}_2)^{-2\bar{\Delta}_n} \tag{3.28}$$

Let us define the "out" primary states by the formula

$$\langle n| = \lim_{z,\bar{z}\to\infty} \langle 0|\phi_n(z,\bar{z})\, z^{2L_0}\, \bar{z}^{2\bar{L}_0} \tag{3.29}$$

These vectors satisfy the equations

$$\langle n| L_m = 0 \quad \text{if } m < 0$$
$$\langle n| L_0 = \Delta_n \langle n| \tag{3.30}$$

and the same equations with substitution $L \to \bar{L}$, $\Delta_n \to \bar{\Delta}_n$. As in (3.26), one has

$$\lim_{z,\bar{z}\to\infty} \langle 0|\, \phi_n^{\{k\}\{\bar{k}\}}(z,\bar{z})\, z^{2L_0}\, \bar{z}^{2\bar{L}_0} = \tag{3.31}$$

$$= \langle n |\, L_{k_N} L_{k_{N-1}} \cdots L_{k_1} \bar{L}_{\bar{k}_M} \cdots \bar{L}_{\bar{k}_1}$$

The orthonormality condition (3.28) can be evidently rewritten in the form

$$\langle n | m \rangle = \delta_{nm} \tag{3.32}$$

The conformal Ward identities make it possible to express explicitly any correlation function of the form

$$\langle T(\zeta_1) T(\zeta_2) \cdots T(\zeta_M)\, \phi_1(z_1) \cdots \phi_N(z_N) \rangle \tag{3.33}$$

in terms of the correlator

$$\langle \phi_1(z_1) \cdots \phi_N(z_N) \rangle \tag{3.34}$$

Here $\phi_1,\ldots,\phi_N$ are some primary fields. This can be done by successively applying the relation

$$\langle T(\zeta) T(\zeta_1) \cdots T(\zeta_M)\, \phi_1(z_1) \cdots \phi_N(z_N) \rangle =$$

$$
\begin{aligned}
&= \left\{ \sum_{i=1}^{N} \left[ \frac{\Delta_i}{(\zeta - z_i)^2} + \frac{1}{\zeta - z_i} \frac{\partial}{\partial z_i} \right] + \sum_{j=1}^{M} \left[ \frac{2}{(\zeta - \zeta_j)^2} + \frac{1}{\zeta - \zeta_j} \frac{\partial}{\partial \zeta_j} \right] \right\} \times \\
&\times \langle T(\zeta_1) \ldots T(\zeta_M)\, \phi_1(z_1) \ldots \phi_N(z_N) \rangle \; + \qquad (3.35) \\
&+ \sum_{j=1}^{M} \frac{c/2}{(\zeta - \zeta_j)^4} \langle T(\zeta_1) \ldots T(\zeta_{j-1}) T(\zeta_{j+1}) \ldots T(\zeta_M) \phi_1(z_1) \ldots \phi_N(z_N) \rangle
\end{aligned}
$$

The first term in (3.35) is of the same origin as (3.10), whereas the second one is due to the c number term in the transformation law (2.12)[11].

Using the correlation functions (3.33) one can compute also any correlators of the form

$$\langle \phi_1^{\{k_1\}}(z_1) \ldots \phi_N^{\{k_N\}}(z_N) \rangle \qquad (3.36)$$

where $\phi_i^{\{k_i\}}$ are some secondaries of the fields $\phi_i$, because these secondaries are none other than the coefficients in the operator product expansions like (3.12), (3.15), etc. Actually, in this way the correlators (3.36) are expressed in terms of the correlation functions (3.34) by means of linear differential operators. The general expression is rather cumbersome and we present the simplest example only [12]

$$\langle \phi_n^{(-k_1,-k_2,\dots,-k_M)}(z)\, \phi_1(z_1) \dots \phi_N(z_N) \rangle = \tag{3.37}$$
$$\doteq \hat{\mathcal{L}}_{-k_M}(z,z_i)\, \hat{\mathcal{L}}_{-k_{M-1}}(z,z_i) \dots \hat{\mathcal{L}}_{-k_1}(z,z_i) \langle \phi_n(z)\, \phi_1(z_1) \dots \phi_N(z_N) \rangle$$

where the differential operators $\mathcal{L}_{-k}$ are given by the formula

$$\hat{\mathcal{L}}_{-k}(z,z_i) = \sum_{i=1}^{N} \left[ \frac{(1-k)\Delta_i}{(z-z_i)^k} - \frac{1}{(z-z_i)^{k-1}} \frac{\partial}{\partial z_i} \right] \tag{3.38}$$

So, the conformal Ward identities enable one to express any correlation functions in terms of the correlators of the primary fields (3.34). Hence, all the information about conformal quantum field theory is contained in these correlators.

## 4. CONFORMAL PROPERTIES OF THE OPERATOR ALGEBRA

In quantum field theory the correlation functions (2.1) should obey the operator algebra (1.6). Conformal symmetry imposed hard restrictions on the coefficients $C_{ij}^{k}(\xi)$. Consider the product of two primary fields $\phi_n(\xi)\phi_m(0)$. The operator product expansion can be represented as

$$\phi_n(z,\bar z)\,\phi_m(0,0) = \sum_p \sum_{\{k\}\{\bar k\}} C_{nm}^{p,\{k\},\{\bar k\}} \times$$
$$\times\, z^{\Delta_p+\Sigma k_i-\Delta_n-\Delta_m}\ \bar z^{\bar\Delta_p+\Sigma \bar k_i-\bar\Delta_n-\bar\Delta_m}\ \phi_p^{\{k\}\{\bar k\}}(0,0) \tag{4.1}$$

where $\phi_p^{\{k\}\{\bar k\}}$ are the secondary fields belonging to the conformal family $[\phi_p]$. Both sides of (4.1) should exhibit the same conformal properties. The transformation law of the left-hand side is determined by (3.8); the conformal properties of each term in the right-hand side can be derived, in principle, from (3.18). The requirement of the conformal invariance of (4.1) leads to the relations among the numerical constants $C_{nm}^{p,\{k\}\{\bar k\}}$ with different k's but with the same index p (see Appendix B). In principle, these relations can be solved recurrently, the solution can be given as follows

$$C_{nm}^{p,\{k\}\{\bar k\}} = C_{nm}^{p}\ \beta_{n,m}^{p,\{k\}}\ \bar\beta_{nm}^{p,\{\bar k\}} \tag{4.2}$$

where $C_{nm}^p$ are the constants which appear in the definition of the primary fields $\phi_p$ themselves, the factors $\beta$ ($\bar\beta$) are expressed unambiguously in terms of the dimensions $\Delta_n$, $\Delta_m$, $\Delta_p$ ($\bar\Delta_n$, $\bar\Delta_m$, $\bar\Delta_p$) only, and the condition $\beta_{nm}^{p\{0\}} = \bar\beta_{nm}^{p\{0\}} = 1$ is implied. The factorized (in terms of $\beta$'s) form of (4.2) is a consequence of (3.25). The expansion (4.1) can be rewritten as

$$\phi_n(z,\bar z)\,\phi_m(0,0) = \sum_p C_{nm}^{p}\, z^{\Delta_p-\Delta_n-\Delta_m}\,\bar z^{\bar\Delta_p-\bar\Delta_n-\bar\Delta_m}\,\Psi_p(z,\bar z|0,0) \tag{4.3}$$

where

$$\Psi_p(z,\bar z|0,0) = \sum_{\{k\}\{\bar k\}} \beta_{nm}^{p,\{k\}}\,\bar\beta_{nm}^{p,\{\bar k\}}\, z^{\Sigma k_i}\,\bar z^{\Sigma \bar k_i}\ \phi_p^{\{k\}\{\bar k\}}(0,0) \tag{4.4}$$

is the contribution of the conformal family $[\phi_p]$. Let us stress that the conformal properties of the "bilocal" operators (4.4) coincide with those of the product $\phi_n(z,\bar{z})\phi_m(0,0)$, all the coefficients in the power series (4.4) being unambiguously determined by this requirement. Unfortunately, equations determining these coefficients are too complicated to be solved exactly. The first few coefficients $\beta$ are presented in Appendix B for the particular case $\Delta_n = \Delta_m$.

The constants $C^p_{nm}$ in (4.3) and the values of the dimensions $\Delta_n$, $\bar{\Delta}_n$ are not determined by the conformal symmetry itself. These numerical parameters are the most important dynamical characteristics of the conformal quantum field theory. Note that, under the orthonormality condition (3.28), the coefficients $C^{\ell}_{nm} = C_{nm\,\ell}$ are symmetric functions of the indices n,m,$\ell$ and coincide with the numerical factors in the three-point functions

$$\langle n|\,\phi_m(z,\bar{z})\,|\ell\rangle = C_{nm\ell}\, z^{\Delta_n-\Delta_m-\Delta_\ell}\,\bar{z}^{\bar{\Delta}_n-\bar{\Delta}_m-\bar{\Delta}_\ell} \tag{4.5}$$

where, for simplicity, we put two points equal to 0 and $\infty$. To determine the parameters $C^{\ell}_{nm}$ and $\Delta_n$ it is necessary to apply some dynamical principle. In the bootstrap approach described in the Introduction, the associativity of the operator algebra (1.6) is taken to be the main dynamical principle. As is shown in Appendix C, the associativity condition is equivalent to the crossing symmetry of the four-point correlation functions

$$\langle A_{j_1}(\xi_1)\, A_{j_2}(\xi_2)\, A_{j_3}(\xi_3)\, A_{j_4}(\xi_4)\rangle \tag{4.6}$$

Thanks to the relations described at the end of the previous Section, it is sufficient to consider the four-point functions of the primary fields

$$\langle \phi_k(\xi_1)\, \phi_\ell(\xi_2)\, \phi_n(\xi_3)\, \phi_m(\xi_4)\rangle \tag{4.7}$$

Due to the projective invariance (see Appendix A), the four-point functions depend essentially only on two anharmonic quotients

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \quad ; \quad \bar{x} = \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{z}_4)}{(\bar{z}_1 - \bar{z}_3)(\bar{z}_2 - \bar{z}_4)} \tag{4.8}$$

Therefore, it is convenient to set $z_1 = \bar{z}_1 = \infty$, $z_2 = \bar{z}_2 = 1$, $z_3 = x$; $\bar{z}_3 = \bar{x}$; $z_4 = \bar{z}_4 = 0$, and to define the functions

$$G^{\ell k}_{nm}(x, \bar{x}) = \langle k | \phi_\ell(1,1) \phi_n(x, \bar{x}) | m \rangle \tag{4.9}$$

In terms of these functions the crossing symmetry condition is

$$\begin{aligned} G^{\ell k}_{nm}(x, \bar{x}) &= G^{mk}_{n\ell}(1 - x, 1 - \bar{x}) = \\ &= x^{-2\Delta_n} \bar{x}^{-2\bar{\Delta}_n} G^{\ell m}_{nk}(1/x, 1/\bar{x}) \end{aligned} \tag{4.10}$$

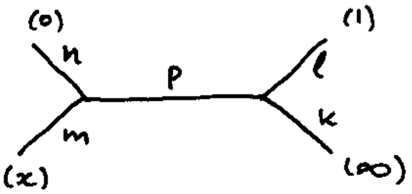
Substituting the expansion (4.3) for the product $\phi_n(x,\bar{x})\phi_m(0,0)$, one can rewrite (4.10) as

$$G^{\ell k}_{nm}(x, \bar{x}) = \sum_p C^p_{nm} C_{k\ell p} A^{\ell k}_{nm}(p | x, \bar{x}) \tag{4.11}$$

where each of the "partial waves"

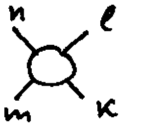
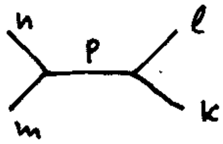
$$\begin{aligned} A^{\ell k}_{nm}(p | x, \bar{x}) &= (C^p_{k\ell})^{-1} x^{\Delta_p - \Delta_n - \Delta_m} \bar{x}^{\bar{\Delta}_p - \bar{\Delta}_n - \bar{\Delta}_m} \times \\ &\times \langle k | \phi_\ell(1,1) \Psi_p(x, \bar{x} | 0,0) | 0 \rangle \end{aligned} \tag{4.12}$$

represents the "s-channel" contribution of the conformal family $[\phi_p]$ to the four-point function (4.9). It is convenient to introduce the diagrams associated with these amplitudes

$$\mathcal{A}^{\ell k}_{nm}(p/x,\bar{x}) = $$  (4.13)



Then the "partial wave" decomposition (4.11) can be represented as

$$G^{\ell k}_{nm}(x,\bar{x}) = $$  $$= \sum_{p} C^{p}_{nm} C_{\ell k p}$$  (4.14)

It is clear from (4.4) that the amplitudes (4.12) have the following factorized form

$$\mathcal{A}^{\ell k}_{nm}(p/x,\bar{x}) = \mathcal{F}^{\ell k}_{nm}(p/x)\,\bar{\mathcal{F}}^{\ell k}_{nm}(p/\bar{x}) \tag{4.15}$$

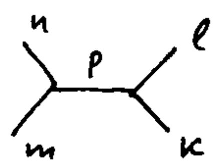
where, for instance, the function F is given by the power series

$$\mathcal{F}^{\ell k}_{nm}(p/x) = \\ = x^{\Delta_p-\Delta_n-\Delta_m} \sum_{\{k\}} \beta^{p\{k\}}_{nm}\, x^{\Sigma k_i} \frac{\langle k|\phi_\ell(1,1)L_{-k_1}\cdots L_{-k_N}|p\rangle}{\langle k|\phi_\ell(1,1)|p\rangle} \tag{4.16}$$

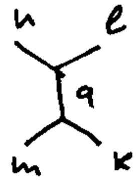
The matrix elements in the right-hand side of (4.16) can be computed exactly with the use of the commutation relations (3.9) and Eqs. (3.30). Therefore, the functions (4.16) are completely determined by the conformal symmetry; these functions depend on six parameters: five dimensions $\Delta_n$, $\Delta_m$, $\Delta_k$, $\Delta_\ell$, $\Delta_p$ and the central charge c. We shall name (4.16) the conformal blocks because any correlation function (4.7) is built out of these functions F.

The crossing symmetry conditions for the four-point functions (4.9) can be represented as the following diagrammatic equations

$$\sum_p C^p_{nm} C_{\ell k p}$$



$$= \sum_q C^q_{n\ell} C_{mkq}$$



(4.17)

The analytic form of these equations is

$$\sum_p C^p_{nm} C_{\ell k p} \mathcal{F}^{\ell k}_{nm}(p|x) \bar{\mathcal{F}}^{\ell k}_{nm}(p|\bar{x}) = \tag{4.18}$$
$$= \sum_q C^q_{n\ell} C_{mkq} \mathcal{F}^{mk}_{n\ell}(q|1-x) \bar{\mathcal{F}}^{mk}_{n\ell}(q|1-\bar{x})$$

If the conformal blocks F are known, (4.18) yields the system of equations determining the constants $\Delta_n$, $\bar{\Delta}_n$. Therefore the computation of the conformal blocks (4.16) for general values of $\Delta$'s is the problem of principal importance for conformal quantum field theory. The first few terms of the power expansion for these functions are given in Appendix B, where the case $\Delta_n=\Delta_m=\Delta_k=\Delta_\ell=\Delta$ is considered for the sake of simplicity. Although the conformal blacks are not yet known for the general case, there are special values of the dimensions $\Delta$ (associated with the degenerate representations of the Virasoro algebra; see Section 5) when the corresponding conformal blocks can be computed exactly. In these cases they are solutions of certain linear differential equations; the simplest example is given by the hypergeometric function. In these special cases the bootstrap equations (4.18) can be solved completely.

## 5. DEGENERATE CONFORMAL FAMILIES

The representation $V_\Delta$ of the Virasoro algebra is irreducible unless the dimension $\Delta$ takes some special values [6,7]. For these values the vector space $V_\Delta$ contains a special vector (the null-vector) $|\chi\rangle \in V_\Delta$ which satisfies the following equations

$$L_n|\chi\rangle = 0 \qquad \text{if} \quad n>0$$
$$L_0|\chi\rangle = (\Delta+K)|\chi\rangle \tag{5.1}$$

characteristic of the primary states; here K is some positive integer. For example, one can easily verify that the vector

$$|\chi\rangle = \left[L_{-2} + \frac{3}{2(2\Delta+1)} L_{-1}^{2}\right]|\Delta\rangle \tag{5.2}$$

(where $|\Delta\rangle$ denotes the primary state of the dimension $\Delta$) satisfies (5.1) with K = 2 provided $\Delta$ takes any one of the two values

$$\Delta = \frac{1}{16}\left[5 - c \pm \sqrt{(c-1)(c-25)}\right] \tag{5.3}$$

In general, the null-vector $|\chi\rangle$ can be considered as the primary state of its own Verma modulus $V_{\Delta+K}$. Therefore, the representation $V_\Delta$ turns out to reducible. One obtains the irreducible representation $V_\Delta^{(ir)}$ if the null-vector $|\chi\rangle$ (together with all the states belonging to $V_{\Delta+K}$) is formally put equal to zero

$$|\chi\rangle = 0 \tag{5.4}$$

Note that Eq. (5.4) does not lead to contradictions because due to (5.1) the null-vector is orthogonal to any state of $V_\Delta$ and, in particular, has zero norm

$$\langle\Psi|\chi\rangle = 0 \quad ; \quad |\Psi\rangle \in V_\Delta$$
$$\langle\chi|\chi\rangle = 0 \; . \tag{5.5}$$

In conformal quantum field theory the meaning of this phenomenon is the following. If a dimension $\Delta$ of some primary field $\phi_\Delta$ happens to take one of the special values mentioned above, then the conformal family $[\phi_\Delta]$, formally computed according to (3.18) proves to contain the special secondary field $\chi_{\Delta+K} \in [\phi_\Delta]$ which possesses the conformal properties of a primary field, i.e., satisfies the commutation relations of the type (3.9). This field corresponds to the null-vector $|\chi\rangle \in V_\Delta$ and we call it the null-field. For example, if $\Delta$ is given by (5.3) the operator

$$\chi_{\Delta+2} = \phi_{\Delta}^{(-2)} + \frac{3}{2(2\Delta+1)} \frac{\partial^2}{\partial z^2} \phi_{\Delta} \tag{5.6}$$

is the null-field.

Formally, the extra primary field $\chi_{\Delta+K}$ gives rise to the conformal family $[\chi_{\Delta+K}]$ which is embedded into $[\phi_\Delta]$. Note, however, that any correlation function of the form

$$\langle \chi_{\Delta+K}(z)\, \phi_1(z_1) \cdots \phi_N(z_N) \rangle$$

vanishes. So, the null-field $\chi_{\Delta+K}$ can be self-consistently considered to be zero

$$\chi_{\Delta+K} = 0 \tag{5.7}$$

This condition obviously kills all the secondaries of the null-field

$$[\chi_{\Delta+K}] = 0 \tag{5.8}$$

If Eq. (5.7) is applied, one gets a true irreducible conformal family $[\phi_\Delta]$ of the original primary field $\phi_\Delta$. In this case the conformal family contains "fewer" fields than usual and we call it a degenerate conformal family; we shall also call degenerate the corresponding primary field $\phi_\Delta$.

All the special values of $\Delta$ corresponding to the reducible representations $V_\Delta$ have been listed by Kac [7] (see also [6]). These values, which can be labelled by two positive integers n and m, are given by the formula

$$\Delta_{(n,m)} = \Delta_0 + \left( \frac{\alpha_+}{2} n + \frac{\alpha_-}{2} m \right)^2 \tag{5.9}$$

where

$$\Delta_0 = \frac{c-1}{24} \tag{5.10}$$

and

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \tag{5.11}$$

If $\Delta = \Delta_{(n,m)}$ then the corresponding null-vector has the dimension

$$\Delta_{(n,m)} + nm \tag{5.12}$$

Let us call $\psi_{(n,m)}$ the degenerate primary field having a dimension $\Delta_{(n,m)}$ [13]. Note that

$$\Delta_{(1,1)} = 0 \tag{5.13}$$

It can be shown that the field $\psi_{(1,1)}$ is z independent, i.e. [14],

$$\frac{\partial}{\partial z} \psi_{(1,1)} = 0 \tag{5.14}$$

The dimensions $\Delta_{(1,2)}$ and $\Delta_{(2,1)}$ are just the two values given by (5.3).

Consider the correlation functions of the form

$$\langle \psi_{(n,m)}(z)\, \phi_1(\xi_1) \cdots \phi_N(\xi_N) \rangle \tag{5.15}$$

An important property of these correlators is that they satisfy linear partial differential equations, the maximal order of derivatives being nm [15]. To make this evident, let us recall that the correlators of any secondaries

$$\langle \psi_{(n,m)}^{(-k_1,-k_2,\ldots,-k_L)}(z)\, \phi_1(\xi_1) \cdots \phi_N(\xi_N) \rangle \tag{5.16}$$

can be expressed in terms of the correlation function (5.15) by means of the linear differential operators [see (3.37)]. The null-field $\chi_{\Delta+nm}$ is a certain linear combination of the secondary fields $\psi_{(n,m)}^{(-k_1,\ldots,-k_L)}$.

Therefore, the differential equation for (5.15) follows directly from Eq. (5.7). For example, taking into account (5.6) and (3.37), one obtains for the degenerate field $\psi_{(1,2)}(z)$

$$\left\{ \frac{3}{2(2\delta+1} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^{N} \frac{\Delta_i}{(z-z_i)^2} - \sum_{i=1}^{N} \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right\} \times$$
$$\times \langle \psi_{(1,2)}(z)\, \phi_1(z_1) \cdots \phi_N(z_N) \rangle = 0 \tag{5.17}$$

where $\delta = \Delta_{(1,2)}$ and $\Delta_1, \Delta_2, \ldots, \Delta_N$ are the dimensions of the primary fields $\phi_1, \phi_2, \ldots, \phi_N$, respectively. The correlation function involving the field $\psi_{(2,1)}$ satisfies the same differential equation, the only difference being $\delta = \Delta_{(2,1)}$ [16]. The differential equation satisfied by the degenerate fields $\psi_{(1,3)}$ and $\psi_{(3,1)}$ ia presented in Appendix D as another example.

In the case of four-point functions

$$\Psi_{(n,m)}(z\,|z_1, z_2, z_3) = \langle \psi_{(n,m)}(z)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \rangle \tag{5.18}$$

the partial differential equations can be reduced to ordinary ones. Actually, in this case the relations (A.7) can be solved for the derivatives $\partial/\partial z_i$; $i = 1,2,3$. For example, substituting these derivatives into (5.17) one gets the ordinary Riemann differential equation

$$\left\{ \frac{3}{2(2\delta+1)} \frac{d^2}{dz^2} + \sum_{i=1}^{3} \left[ \frac{1}{z-z_i} \frac{d}{dz} - \frac{\Delta_i}{(z-z_i)^2} \right] + \right.$$
$$\left. + \sum_{j<i}^{3} \frac{\delta + \Delta_{ij}}{(z-z_i)(z-z_j)} \right\} \Psi(z\,|z_1, z_2, z_3) = 0 \tag{5.19}$$

where $\Delta_{12} = \Delta_1+\Delta_2-\Delta_3$, etc.; $\delta = \Delta_{(1,2)}$, $\Psi = \Psi_{(1,2)}$ or $\delta = \Delta_{(2,1)}$, $\Psi = \Psi_{(2,1)}$. So, for the cases $(n,m) = (1,2)$ or $(2,1)$ the four-point function (5.18) can be expressed in terms of the hypergeometric function.

Let us consider the operator algebra containing the degenerate fields. Some important information about this operator algebra can be obtained from the differential equations discussed above. For example, consider the product $\psi(z)\phi_\Delta(z_1)$, where $\phi_\Delta$ is some primary field of dimension $\Delta$, whereas $\psi(z)$ temporarily stands for one of the degenerate fields $\psi_{(1,2)}$ or $\psi_{(2,1)}$. Let us substitute the expansion

$$\psi(z)\,\phi_\Delta(z_1) = \\ = \mathrm{Const}\,(z-z_1)^{\varkappa}\left[\phi_{\Delta'}(z_1) + (z-z_1)\beta^{(-1)}\phi_{\Delta'}^{(-1)}(z_1) + \cdots\right] \tag{5.20}$$

into the differential equation (5.17). In (5.20) $\phi_{\Delta'}$ denotes some primary field of dimension $\Delta'$; $x = \Delta'-\Delta-\delta$ where $\delta$ is the dimension of the field $\psi$, i.e., one of the values given by (5.3). Considering the most singular term as $z \to z_1$, one immediately obtains the characteristic equation determining the exponent $x$

$$\frac{3\,\varkappa(\varkappa-1)}{2(2\delta+1)} - \Delta + \varkappa = 0 \tag{5.21}$$

To describe the solutions of this equation, it is convenient to introduce the following parametrization of the dimensions

$$\Delta(\alpha) = \Delta_0 + \frac{1}{4}\alpha^2 \tag{5.22}$$

where $\Delta_0$ is defined by (5.10). If $\Delta = \Delta(\alpha)$, then two solutions of (5.21) are given by

$$\begin{aligned} \Delta'_{(1)} &= \Delta_0 + \frac{1}{4}(\alpha + \alpha_\pm)^2 \\ \Delta'_{(2)} &= \Delta_0 + \frac{1}{4}(\alpha - \alpha_\pm)^2 \end{aligned} \tag{5.23}$$

where $\alpha_\pm$ are given by (5.11) and $\alpha_+$ ($\alpha_-$) is chosen if $\psi = \psi_{(1,2)}$ ($\psi=\psi_{(2,1)}$). Let $\phi_{(\alpha)}(z)$ be the primary field with dimension (5.22). The result of the above consideration can be represented by the following symbolic formulae

$$\begin{aligned} \psi_{(1,2)}\, \phi_{(\alpha)} &= [\, \phi_{(\alpha-\alpha_+)} \,] + [\, \phi_{(\alpha+\alpha_+)} \,] \\ \psi_{(2,1)}\, \phi_{(\alpha)} &= [\, \phi_{(\alpha-\alpha_-)} \,] + [\, \phi_{(\alpha+\alpha_-)} \,] \end{aligned} \tag{5.24}$$

Here, the square brackets denote the contributions of the corresponding conformal families to the operator product expansion of $\psi(z)\phi_{(\alpha)}(z_1)$. In (5.24) the over-all factors standing in front of these contributions are omitted. These factors certainly cannot be determined by simple calculations like the one performed above[17]. As we shall see in the next Section, some of these coefficients could vanish.

It can be shown that the "fusion rule" (5.24), when generalized to the cases of arbitrary degenerate fields $\psi_{(n,m)}$, takes the form

$$\psi_{(n,m)}\, \phi_{(\alpha)} = \sum_{\ell=1-m}^{1+m} \sum_{k=1-n}^{1+n} [\, \phi_{(\alpha+\ell\alpha_- + k\alpha_+)} \,] \tag{5.25}$$

where the variable k runs through the even (odd) values, provided the index n is odd (even); the same is valid for the variable $\ell$ and the index m. So, in the general case, the sum in (5.25) contains nm terms in agreement with the fact that the degenerate field $\psi_{(n,m)}$ satisfies the nm-order differential equation.

We see that the differential equations satisfied by the degenerate fields impose hard constraints on the operator algebra. Certainly, in the general case, these differential equations do not provide enough information to determine the correlation functions (5.15) completely. Even in the case of the four-point functions (5.18), one has to take into account the $\bar{z}$ dependence of the fields and to impose the constraints of locality. In the next Section, we shall consider the "minimal" models of conformal quantum field theory in which all primary fields involved are degenerate.

## 6. MINIMAL THEORIES

Let us consider the "fusion rule" (5.24). The substitution $\phi_{(\alpha)} = \psi_{(1,2)}$ yields

$$\psi_{(1,2)}\,\psi_{(1,2)} = [\psi_{(1,1)}] + [\psi_{(1,3)}] \qquad (6.1)$$

Here (5.9) is taken into account. Similarly, one gets for $m > 1$

$$\psi_{(1,2)}\,\psi_{(1,m)} = [\psi_{(1,m-1)}] + [\psi_{(1,m+1)}] \qquad (6.2)$$

So, if the degenerate field $\psi_{(1,2)}$ is involved in the operator algebra, in the general case, this algebra includes also all the degenerate fields $\psi_{(1,m)}$; $m = 3,4,\ldots$ . Moreover, assuming that the operator algebra includes also the degenerate field $\psi_{(2,1)}$ and, using (5.21), one can obtain all the degenerate fields $\psi_{(n,m)}$. In the "fusion rule" (5.21) the fields $\psi_{(1,2)}$ and $\psi_{(2,1)}$ act as the "shift operators"

$$\psi_{(1,2)}\,\psi_{(n,m)} = [\psi_{(n,m-1)}] + [\psi_{(n,m+1)}] \qquad (6.3a)$$

$$\psi_{(2,1)}\,\psi_{(n,m)} = [\psi_{(n-1,m)}] + [\psi_{(n+1,m)}] \qquad (6.3b)$$

The following comment is necessary. Using the rules (6.3)?? formally, one would get as a result all the fields of dimensions $\Delta_{(n,m)}$ given by (5.9) where the integers n and m take zero and negative values, as well as positive ones.

In fact, fields of dimensions $\Delta_{(n,m)}$ with zero and negative n,m drop out from the algebra, i.e., the operator algebra developed by "fusing" the fields $\psi_{(1,2)}$ and $\psi_{(2,1)}$ proves to contain the degenerate fields $\psi_{(n,m)}$ $(n,m > 0)$ only. To understand the nature of this phenomenon, consider, for instance, the product $\psi_{(1,2)}\psi_{(2,1)}$. Analyzing the differential equation for the degenerate field $\psi_{(1,2)}$ one gets, according to (6.3a)

$$\psi_{(1,2)}\,\psi_{(2,1)} = C_1\,[\phi_{(2,0)}] + C_2\,[\psi_{(2,2)}] \qquad (6.4)$$

where $\phi_{(2,0)}$ denotes the primary field of dimension $\Delta_{(2,0)} = \Delta_0 + (\alpha_+)^2$. In (6.4) we have written explicitly the numerical coefficients $C_1$ and $C_2$ of the

the corresponding primary fields in the operator product expansion; in the above symbolic formulae, like (6.1)-(6.3), such coefficients are omitted. On the other hand, the field $\phi_{(1,2)}$ being also degenerate satisfies the differential equation (5.17) which leads to the expansion

$$\psi_{(1,2)}\,\psi_{(2,1)} = C_1'\,[\phi_{(0,2)}] + C_2'\,[\psi_{(2,2)}] \tag{6.5}$$

where the field $\phi_{(0,2)}$ has dimension $\Delta_{(0,2)} = \Delta_0 + (\alpha_-)^2$ and $C_1'$, $C_2'$ are some numerical coefficients. Comparison of this formula with (6.4) gives that $C_1 = C_1' = 0$ and $C_2 = C_2'$. Hence, the expansion of the product $\psi_{(1,2)}\psi_{(2,1)}$ contains the contribution of only one conformal family

$$\psi_{(1,2)}\,\psi_{(2,1)} = [\,\psi_{(2,2)}\,] \tag{6.6}$$

We shall call the phenomenon described above the <u>truncation</u> of the operator algebra[18]. It can be shown that for the degenerate fields $\psi_{(n,m)}$ this is the general situation: the degenerate conformal families $[\psi_{(n,m)}]$ with $n > 0$, $m > 0$ actually appear only in the "fusion rules" like (6.3). The general "fusion rules" for the degenerate fields have the form[19]

$$\psi_{(n_1,m_1)}\,\psi_{(n_2,m_2)} = \sum_{k=|n_1-n_2|+1}^{n_1+n_2-1}\ \sum_{\ell=|m_1-m_2|+1}^{m_1+m_2-1} [\,\psi_{(k,\ell)}\,] \tag{6.7}$$

where the variable $k$ ($\ell$) runs over the even integers, provided $n_1+n_2$ ($m_1+m_2$) is odd and vice versa.

So, the degenerate fields (more precisely, the degenerate conformal families) form the closed operator algebra. This observation gives rise to the idea of conformal quantum field theory in which all the primary fields are degenerate. To examine this possibility, let us concentrate once more on the Kac formula (5.9). It is evident that there are three distinct domains of parameter $c$[20]. If $c \geq 25$, the second term in (5.9) is negative and the dimensions $\Delta_{(n,m)}$ become negative for sufficiently large $n$ and $m$. If $25 > c > 1$, the dimensions $\Delta_{(n,m)}$ are, in general, complex. Neither possibility is acceptable in quantum field theory[20]. Therefore, in what follows, we shall consider the domain

$$0 < c \leq 1 \tag{6.8}$$

To understand the properties of the spectrum (5.9), let us consider the "diagram of dimensions" shown in Fig. 1. The vertical and the horizontal axes in this Figure correspond to the values of the parameters n and m in (5.9); the "physical" (i.e., the positive integer) values of these parameters are shown by dots. The dotted line has the following slope :

$$tg\,\theta = -\alpha_+/\alpha_- = \frac{\sqrt{25-c} - \sqrt{1-c}}{\sqrt{25-c} + \sqrt{1-c}} \tag{6.9}$$

The value (5.22) of the dimension is associated with each point of the plane in Fig. 1, the parameter $\alpha$ being proportional to the distance between the point and the dotted line.

If the slope (6.9) takes an irrational value, the dotted line in Fig. 1 passes arbitrarily close to some of the dots. Since at $\Delta_0$ is negative at $c < 1$, we meet again with the problem of negative dimensions. Let us consider, however, the cases of the rational slope

$$tg\,\theta = -\frac{\alpha_-}{\alpha_+} = p/q \tag{6.10}$$

where p and q are positive integers. The characteristic feature of the corresponding values of c is that each degenerate representation $V_{\Delta_{(n,m)}}$ contains not only one, but infinitely many null-vectors of different dimensions; this is evident from (5.9) and (6.10). In these cases the irreducible conformal families $[\psi_{(n,m)}]$ obtained by nullification of all the null-fields contain considerably fewer fields than the usual families and we call conformal quantum field theories, corresponding to (6.10) and involving these degenerate fields $\psi_{(n,m)}$, the minimal theories. It is important that in the minimal theories the correlation functions (5.15) satisfy infinitely many differential equations obtained by the nullification of all the corresponding null-fields[21)]. This fact enables one to prove that the operator algebra of degenerate fields in the minimal theories possesses not only "truncation from below" described at the beginning of the Section, but also the "truncation from above". Namely, if one starts with the fields $\psi_{(n,m)}$ with $0 < n < p$, $0 < m < q$, then the degenerate fields with $n \geq p$ or

$m \geq q$ drop out from the "fusion rules" (6.7) [like the fields $\phi_{(2,0)}$ and $\phi_{(0,2)}$ in (6.4) and (6.5)]. In other words, the conformal families $[\psi_{(n,m)}]$ with $0 < n < p$; $0 < m < q$ form the closed algebra which can be treated as the operator algebra of quantum field theory. Note that [under the condition (6.10)] $n = p$, $m = q$ are the co-ordinates of the nearest dot in Fig. 1 through which the dotted line passes. Degenerate fields with dimensions associated with the dots inside the rectangle $0 < n < p$; $0 < m < q$, shown in Figs. 2 and 3, form the closed operator algebra. Due to the diagonal symmetry of this rectangle, there are $(p-1)(q-1)/2$ different dimensions.

Let us consider in more detail the simplest non-trivial example of the minimal theory corresponding to the case

$$p/q = 3/4 \tag{6.11}$$

which takes place if

$$c = 1/2 \tag{6.12}$$

The "diagram of dimensions" for this case is shown in Fig. 2. Let us demonstrate the "truncation from above" using this example. The dimensions corresponding to the dots in Fig. 2 are

$$\begin{aligned} \Delta_{(1,1)} &= \Delta_{(2,3)} = 0 \\ \Delta_{(2,1)} &= \Delta_{(1,3)} = 1/2 \\ \Delta_{(1,2)} &= \Delta_{(2,2)} = 1/16 \end{aligned} \tag{6.13}$$

Respectively, there are three degenerate fields[22)] which we shall denote as

$$\begin{aligned} I &= \psi_{(1,1)} = \psi_{(2,3)} \\ \varepsilon &= \psi_{(2,1)} = \psi_{(1,3)} \\ \sigma &= \psi_{(1,2)} = \psi_{(2,2)} \end{aligned} \tag{6.14}$$

Consider, for instance, the product $\varepsilon \cdot \varepsilon$. The field $\varepsilon$, being equal to $\psi_{(2,1)}$, satisfies the second order differential equation (5.17). Therefore, according to (6.3b),

$$\varepsilon \cdot \varepsilon = \psi_{(2,1)} \psi_{(2,1)} = C_1 [I] + C_2 [\psi_{(3,1)}] \tag{6.15}$$

where the field $\psi_{(3,1)}$ has dimension $\Delta_{(3,1)} = 5/3$. On the other hand, since $\varepsilon = \psi_{(1,3)}$, this field satisfies the third order differential equation (D.8) and hence

$$\varepsilon \cdot \varepsilon = \psi_{(1,3)} \psi_{(1,3)} = C_1' [I] + C_2' [\psi_{(1,3)}] + C_3' [\psi_{(1,5)}] \tag{6.16}$$

where the field $\psi_{(1,5)}$ has dimension $\Delta_{(1,5)} = 5/2$. Comparing (6.16) and (6.15), one concludes that in fact

$$\varepsilon \cdot \varepsilon = [I] \tag{6.17}$$

By a similar consideration, one can obtain the following "fusion rules" for the fields (6.14)

$$\begin{aligned} I \cdot \varepsilon &= [\varepsilon] \; ; & \varepsilon \cdot \varepsilon &= [I] \; ; \\ I \cdot \sigma &= [\sigma] \; ; & \varepsilon \cdot \sigma &= [\sigma] \; ; \\ I \cdot I &= [I] \; ; & \sigma \cdot \sigma &= [I] + [\varepsilon] . \end{aligned} \tag{6.18}$$

In Appendix E one shows that this minimal theory describes the critical point of the two-dimensional Ising model, the primary fields $\sigma$, $\varepsilon$ and I being identified with the local spin, energy density and identity operators, respectively.

In Fig. 3, the "diagram of dimensions" for the minimal theory characterized by the values

$$p/q = 4/5 \quad ; \quad c = 7/10 \tag{6.19}$$

is presented as another example. The corresponding numerical values of the dimensions are

$$\begin{aligned}
\Delta_{(1,1)} &= \Delta_{(3,4)} = 0 \\
\Delta_{(1,2)} &= \Delta_{(3,3)} = 1/10 \\
\Delta_{(1,3)} &= \Delta_{(3,2)} = 3/5 \\
\Delta_{(1,4)} &= \Delta_{(3,1)} = 3/2 \\
\Delta_{(2,2)} &= \Delta_{(2,3)} = 3/80 \\
\Delta_{(2,4)} &= \Delta_{(2,1)} = 7/16
\end{aligned} \tag{6.20}$$

Note, that due to the inequalities (6.8), the integers p and q in (6.10) are restricted as follows

$$2/3 < p/q < 1 \tag{6.21}$$

Nevertheless, there are infinitely many rational numbers satisfying (6.21) and each of them corresponds to some minimal model of conformal quantum field theory. We suppose that the minimal theories describe the second order phase transitions in the two-dimensional systems with discrete symmetry groups[23)]. In any case, each of the minimal models seems to deserve a most detailed investigation. Note that the anomalous dimensions associated with each of the minimal models are known exactly [they are given by (5.9)], whereas the correlation functions can be computed in the following way. At first, one has to derive the corresponding conformal blocks as solutions of the respective differential equations with appropriate initial conditions. Then, substituting these conformal blocks into the bootstrap equations (4.18) and taking into account the local properties of the fields, one should calculate the structure constants $C^{\ell}_{nm}$ of the operator algebra which provides enough information to construct the correlation functions. For the minimal theory (6.11), this computation is presented in Appendix E. In the general case, it has not yet been performed.

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## APPENDIX A

Let $L_{-1}$, $L_0$, $L_{+1}$ and $\bar{L}_{-1}$, $\bar{L}_0$, $\bar{L}_{+1}$ be the generators of the infinitesimal projective transformations

$$\begin{aligned} z &\to z + \varepsilon_{-1} + \varepsilon_0 z + \varepsilon_1 z^2 \; ; \\ \bar{z} &\to \bar{z} + \bar{\varepsilon}_{-1} + \bar{\varepsilon}_0 \bar{z} + \bar{\varepsilon}_1 \bar{z}^2 \, , \end{aligned} \tag{A.1}$$

where $\varepsilon$ and $\bar{\varepsilon}$ are infinitesimal parameters. The operators $L_s$; $s = 0, \pm 1$ satisfy the commutation relations

$$\begin{aligned} [L_0, L_{\pm 1}] &= \pm L_{\pm 1} \\ [L_1, L_{-1}] &= 2 L_0 \end{aligned} \tag{A.2}$$

The same relations are satisfied by $\bar{L}$'s, $L$'s and $\bar{L}$'s being commutative. The operators $P^\circ = L_{-1}+\bar{L}_{-1}$ and $P^1 = -i(L_{-1}-\bar{L}_{-1})$ are the components of the total momentum, whereas $M = i(L_0-\bar{L}_0)$ and $D = L_0+\bar{L}_0$ are generators of the rotations (Lorentz boosts in Minkowski space-time) and dilatations, respectively. The operators $L_1$ and $\bar{L}_1$ correspond to the special conformal transformations. The vacuum of conformal field theory satisfies the relations

$$\langle 0 | L_s = L_s | 0 \rangle = 0 \quad ; \quad s = 0, \pm 1 . \tag{A.3}$$

which are equivalent to the asymptotic condition (2.14).

We shall call the local field $O_\ell(z,\bar{z})$ <u>quasiprimary</u>, provided it satisfies the commutation relations

$$\begin{aligned} [L_s, O_\ell(z,\bar{z})] &= \left\{ z^{s+1} \frac{\partial}{\partial z} + (s+1)\Delta_\ell \, z^s \right\} O_\ell(z,\bar{z}) \\ [\bar{L}_s, O_\ell(z,\bar{z})] &= \left\{ \bar{z}^{s+1} \frac{\partial}{\partial \bar{z}} + (s+1)\bar{\Delta}_\ell \, \bar{z}^s \right\} O_\ell(z,\bar{z}) \end{aligned} \tag{A.4}$$

where s = 0, ±1. The constants $\Delta_\ell$ and $\bar{\Delta}_\ell$ are the dimensions of the field $O_\ell$. These relations mean that the fields $O_\ell(z,\bar{z})$ transform according to (1.16) under the projective transformations (1.15). This distinguishes them from the primary fields $\phi_n$ which transform according to (1.16) with respect to all conformal transformations (1.9)[24]. In conformal quantum field theory, the complete set of local fields $A_j$ forming the algebra (1.6) can be constituted out of an infinite number of quasiprimary fields $O_\ell$ and their co-ordinate derivatives of all orders

$$\{A_j\} = \{ O_\ell, \frac{\partial}{\partial z} O_\ell, \frac{\partial}{\partial \bar{z}} O_\ell, \frac{\partial^2}{\partial z^2} O_\ell, \frac{\partial^2}{\partial z \partial \bar{z}} O_\ell, \dots \} \qquad (A.5)$$

Consider an N-point correlation function of the quasiprimary fields. It follows from (A.3) and (A.4) that such a correlation function satisfies the equations

$$\hat{\Lambda}_s \langle O_{\ell_1}(z_1,\bar{z}_1) O_{\ell_2}(z_2,\bar{z}_2) \dots O_{\ell_N}(z_N,\bar{z}_N) \rangle = 0 \qquad (A.6)$$

where s = 0, ±1 and $\hat{\Lambda}_s$ are the differential operators

$$\begin{aligned} \hat{\Lambda}_{-1} &= \sum_{i=1}^{N} \frac{\partial}{\partial z_i} \quad ; \\ \hat{\Lambda}_0 &= \sum_{i=1}^{N} \left( z_i \frac{\partial}{\partial z_i} + \Delta_i \right) \quad ; \\ \hat{\Lambda}_1 &= \sum_{i=1}^{N} \left( z_i^2 \frac{\partial}{\partial z_i} + 2 z_i \Delta_i \right) , \end{aligned} \qquad (A.7)$$

where $\Delta_1, \Delta_2, \dots, \Delta_N$ are the dimensions of the fields $O_{\ell_1}, O_{\ell_2}, \dots, O_{\ell_N}$, respectively. Equations (A.6) are the projective Ward identities. Note that these Ward identities follow directly from the general relation (2.9). For the infinitesimal projective transformations, the function $\varepsilon(z)$ is regular in the finite part of the z-plane and, due to the asymptotic condition (2.14), the contour integral in (2.9) vanishes. Let us stress that for the general conformal transformations, the analytic function $\varepsilon(z)$

possesses singularities; this is why the corresponding Ward identities cannot be reduced to the closed equations for correlation functions like (A.6).

The general solution of Eqs. (A.6) (and the analogous equations obtained by substitution $z_i \to \bar{z}_i$; $\Delta_i \to \bar{\Delta}_i$) is

$$\langle O_{\ell_1}(z_1,\bar{z}_1)\, O_{\ell_2}(z_2,\bar{z}_2) \cdots O_{\ell_N}(z_N,\bar{z}_N)\rangle = \\ = \prod_{i<j} (z_i - z_j)^{\gamma_{ij}} (\bar{z}_i - \bar{z}_j)^{\bar{\gamma}_{ij}} \times Y(x_{ij}^{k\ell}, \bar{x}_{ij}^{k\ell}) \tag{A.8}$$

where $\gamma_{ij}$ and $\bar{\gamma}_{ij}$ are arbitrary solutions of equations

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} \gamma_{ij} = 2\Delta_i \quad ; \quad \sum_{\substack{j=1 \\ j\neq i}}^{N} \bar{\gamma}_{ij} = 2\bar{\Delta}_i \tag{A.9}$$

whereas Y is an arbitrary function of 2(N-3) anharmonic quotients

$$x_{ij}^{k\ell} = \frac{(z_i - z_j)(z_k - z_\ell)}{(z_i - z_\ell)(z_k - z_j)} \quad ; \quad \bar{x}_{ij}^{k\ell} = \frac{(\bar{z}_i - \bar{z}_j)(\bar{z}_k - \bar{z}_\ell)}{(\bar{z}_i - \bar{z}_\ell)(\bar{z}_k - \bar{z}_j)} \tag{A.10}$$

In the particular cases N = 2 and N = 3, the correlation functions are determined by Eqs. (A.8)-(A.10) completely up to a numerical factor. Namely

$$\langle O_{\ell_1}(z_1,\bar{z}_1)\, O_{\ell_2}(z_2,\bar{z}_2)\rangle =$$

$$\begin{cases} 0 & \text{if } \Delta_{\ell_1} \neq \Delta_{\ell_2} \text{ or } \bar{\Delta}_{\ell_1} \neq \bar{\Delta}_{\ell_2} \\ (z_1-z_2)^{-2\Delta_\ell}(\bar{z}_1-\bar{z}_2)^{-2\bar{\Delta}_\ell}\, D_{\ell_1\ell_2} & \text{if } \Delta_{\ell_1}=\Delta_{\ell_2}\,;\, \bar{\Delta}_{\ell_1}=\bar{\Delta}_{\ell_2} \end{cases} \tag{A.11}$$

for N = 2 and

$$\langle O_{\ell_1}(z_1,\bar{z}_1)\, O_{\ell_2}(z_2,\bar{z}_2)\, O_{\ell_3}(z_3,\bar{z}_3)\rangle =$$

$$= Y_{\ell_1\ell_2\ell_3} \prod_{i<j}^{3} (z_i - z_j)^{-\Delta_{ij}} (\bar{z}_i - \bar{z}_j)^{-\bar{\Delta}_{ij}} \tag{A.12}$$

for N = 3, where $D_{\ell_1\ell_2}$ and $Y_{\ell_1\ell_2\ell_3}$ are constants, and

$$\begin{aligned} \Delta_{12} &= \Delta_1 + \Delta_2 - \Delta_3 \;;\; etc\,, \\ \bar{\Delta}_{12} &= \bar{\Delta}_1 + \bar{\Delta}_2 - \bar{\Delta}_3 \;;\; etc\,, \end{aligned} \tag{A.13}$$

Note that the functions (A.11) and (A.12) are single-valued in Euclidean space (obtained by the substitution $\bar{z}_i = z_i^*$), provided the spins $s_\ell = \Delta_\ell - \bar{\Delta}_\ell$ of all the fields involved take integer or half-integer values.

In conformal quantum field theory, the expansion (1.6) can be represented in the form

$$\mathcal{O}_{\ell_1}(z,\bar{z})\,\mathcal{O}_{\ell_2}(0,0) = \sum_{\ell_3}\sum_{k,\bar{k}=0}^{\infty} Y_{\ell_1\ell_2}^{\ell_3,k,\bar{k}} \times$$

$$\times\, z^{\Delta_3+k-\Delta_1-\Delta_2}\,\bar{z}^{\bar{\Delta}_3+\bar{k}-\bar{\Delta}_1-\bar{\Delta}_2}\left[\frac{\partial^{k+\bar{k}}}{\partial\zeta^k\partial\bar{\zeta}^{\bar{k}}}\mathcal{O}_{\ell_3}(\zeta,\bar{\zeta})\right]_{\zeta,\bar{\zeta}=0} \tag{A.14}$$

where $Y_{\ell_1\ell_2}^{\ell_3,k,\bar{k}}$ are constants, k and $\bar{k}$ being integers. The transformation properties of both sides of this equation with respect to the projective transformations (A.1) must coincide. Commuting both sides of (A.14) with the projective generators $L_s$, $s = 0,\pm1$, and using (A.4), one gets the equations relating the coefficients $Y_{\ell_1\ell_2}^{\ell_3,k,\bar{k}}$ with different values of k. Solving these equations, one can rewrite (A.14) as

$$\mathcal{O}_{\ell}(z,\bar{z})\,\mathcal{O}_{\ell}(0,0) = \sum_{\ell'} Y_{\ell\ell}^{\ell'}\, z^{\Delta'-2\Delta}\,\bar{z}^{\bar{\Delta}'-2\bar{\Delta}} \times$$

$$F(\Delta',2\Delta',z\tfrac{\partial}{\partial\zeta})\,F(\bar{\Delta}',2\bar{\Delta}',\bar{z}\tfrac{\partial}{\partial\bar{\zeta}})\,\mathcal{O}_{\ell'}(\zeta,\bar{\zeta})\Big|_{\zeta,\bar{\zeta}=0} \tag{A.15}$$

where the case $\ell_1 = \ell_2$ is considered for the sake of simplicity; $\Delta_{\ell_1} = \Delta_{\ell_2} = \Delta$; $\Delta_{\ell'} = \Delta'$. In (A.15) $Y_{\ell\ell}^{\ell'}$ are the constants coinciding with $Y_{\ell\ell}^{\ell',0,0}$ in (A.14) and F(a,c,x) denotes the degenerate hypergeometric function.

Obviously, each conformal family $[\phi_n] = V_n \otimes \bar{V}_n$ (see Section 3) contains infinitely many quasiprimary fields. These fields correspond to the states satisfying the equations

$$\begin{aligned} L_1\,|\ell\rangle &= \bar{L}_1\,|\ell\rangle = 0 \\ L_0\,|\ell\rangle &= \Delta_\ell\,|\ell\rangle\,; \quad \bar{L}_0\,|\ell\rangle = \bar{\Delta}_\ell\,|\ell\rangle \end{aligned} \tag{A.16}$$

It can be shown that the basis in $[\phi_n]$ can be constituted out of the states

$$(L_{-1})^n(\bar{L}_{-1})^{\bar{n}}|\ell> \tag{A.17}$$

where n, $\bar{n}$ = 0,1,2,... and $|\ell$ are the quasiprimary states belonging to $[\phi_n]$. This statement is equivalent to (A.5) because the operators $L_{-1}$ and $\bar{L}_{-1}$ are obviously associated with the derivatives $\partial/\partial z$ and $\partial/\partial \bar{z}$.

## APPENDIX B

Here we shall demonstrate that the coefficients $\beta_{nm}^{\ell\{k\}}$ in (4.2) are determined completely by the requirement of the conformal symmetry of the expansion (4.1). For the sake of simplicity we consider only the particular case $\Delta_n = \Delta_m = \Delta$. Applying both sides of (4.1) to the vacuum state $|0\rangle$, one gets the equation

$$\phi_\Delta(z,\bar{z})|\Delta\rangle = \sum_\ell C_{\Delta\Delta}^{\Delta_\ell} z^{\Delta_\ell - 2\Delta} \cdot \bar{z}^{\bar{\Delta}_\ell - 2\bar{\Delta}} \times \tag{B.1}$$

$$\varphi_\Delta(z)\,\bar{\varphi}_{\bar{\Delta}}(\bar{z})\,|\Delta_\ell\rangle$$

where $|\Delta\rangle$ is the primary state of dimensions $\Delta$, $\bar{\Delta}$ and the operator $\phi_\Delta(z)$ is given by the series

$$\varphi_\Delta(z) = \sum_{\{k\}} z^{\sum k_i} \beta_{\Delta\Delta}^{\Delta_\ell,\{k\}} L_{-k_1} L_{-k_2} \dots L_{-k_N} \,. \tag{B.2}$$

The same formula, with the substitution $z \to \bar{z}$, $\beta \to \bar{\beta}$, $L \to \bar{L}$, holds for $\bar{\phi}_{\bar{\Delta}}(\bar{z})$. Let us consider the state

$$|z,\Delta'\rangle = \varphi_\Delta(z)\,|\Delta'\rangle \tag{B.3}$$

It can be represented as the power series

$$|z,\Delta'\rangle = \sum_{N=0}^{\infty} z^N\,|N,\Delta'\rangle \tag{B.4}$$

where the vectors $|N,\Delta'\rangle$ satisfy the equation

$$L_0\,|N,\Delta'\rangle = (\Delta'+N)\,|N,\Delta'\rangle \tag{B.5}$$

To compute these vectors, let us apply the operators $L_n$ to both sides of

(B.1). This leads to

$$\left[ z^{n+1} \frac{d}{dz} + \Delta (n+1) z^{n} \right] |z, \Delta'\rangle = L_n |z, \Delta'\rangle \tag{B.6}$$

Substituting the power expansion (B.4), one gets

$$L_n |N+n, \Delta'\rangle = (N + (n-1)\Delta + \Delta') |N, \Delta'\rangle \tag{B.7}$$

Actually, one can consider Eq. (B.7) with n = 1,2 only, because due to (2.21) the remaining equations follow from these two. Solving these equations, one can compute the power series (B.4) order by order. In the first three orders, the result is

$$|z, \Delta'\rangle = \Big\{ 1 + \frac{z}{2} L_{-1} + \frac{z^2}{4} \frac{\Delta'+1}{2\Delta'+1} L_{-1}^2 + \tag{B.8}$$

$$+ z^2 \frac{\Delta'(\Delta'-1) + 2\Delta(2\Delta'+1)}{C(2\Delta'+1) + 2\Delta'(8\Delta'-5)} \left( L_{-2} + \frac{3}{2(2\Delta'+1)} L_{-1}^2 \right) + \dots \Big\} |\Delta'\rangle$$

This formula gives the first three coefficients $\beta$ in (4.4).

Obviously, the conformal block $F(\Delta,\Delta',x) \equiv F^{\Delta\Delta}_{\Delta\Delta}(\Delta'|x)$ is given by the scalar product

$$\mathcal{F}(\Delta, \Delta', x) = x^{\Delta'-2\Delta} \langle 1, \Delta' | x, \Delta'\rangle \tag{B.9}$$

The first few terms of the power expansion of this function can be obtained directly from (B.8)

$$\mathcal{F}(\Delta,\Delta';x) = x^{\Delta'-2\Delta}\Big\{ 1 + \frac{\Delta'}{2}x + \frac{\Delta'(\Delta'+1)^2}{4(2\Delta'+1)}x^2 + \\ + \frac{[\Delta'(1-\Delta') - 2\Delta(2\Delta'+1)]^2}{2(2\Delta'+1)[c(2\Delta'+1)+2\Delta'(8\Delta'-5)]}x^2 + \dots \Big\} \tag{B.10}$$

## APPENDIX C

Consider the associative algebra determined by the relations

$$A_I A_J = \sum_K C_{IJ}^{K} A_K \tag{C.1}$$

Equation (1.6) is just (C.1) where each of the indices, say I, combine the space co-ordinate $\xi$ and the index i labelling the fields. Let the algebra (C.1) be supplied with the symmetric bilinear form

$$\mathcal{D}_{IJ} = \langle A_I A_J \rangle , \tag{C.2}$$

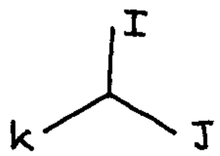
which is none other than a set of all two-point correlation functions. Let us introduce the form

$$C_{IJK} = \sum_{K'} \mathcal{D}_{KK'} C_{IJ}^{K'} \tag{C.3}$$

and suppose that it is a symmetric function of the indices I,J,K. Obviously, (C.3) coincides with the three-point correlation function

$$C_{IJK} = \langle A_I A_J A_K \rangle \tag{C.4}$$

and it can be conveniently represented by the "vertex" diagram

$C_{IJK} =$ 

(C.5)

We also introduce the diagram

$\mathcal{D}^{IJ} =$ 

(C.6)

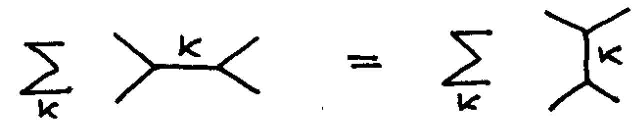
for the "inverse" propagator $D^{IJ}$ defined by the equation

$$\sum_K \mathcal{D}^{IK} \mathcal{D}_{KJ} = \delta_J^I \tag{C.7}$$

The associativity condition of the algebra (C.1)

$$\sum_K C_{IJ}^{K} C_{KL}^{M} = \sum_K C_{IK}^{M} C_{JL}^{K} \qquad \text{(C.8)}$$

can be represented as the diagrammatic equation




(C.9)

which coincides with the "crossing symmetry" condition for the four-point functions

$$\langle A_I \, A_J \, A_L \, A_M \rangle \qquad \text{(C.10)}$$

# APPENDIX D

In this Appendix we shall derive the differential equation satisfied by the correlation function

$$\langle \psi(z)\, \phi_1(z_1)\, \phi_2(z_2) \ldots \phi_N(z_N) \rangle \tag{D.1}$$

where $\psi(z)$ denotes any one of the degenerate fields $\psi_{(1,3)}(z)$ and $\psi_{(3,1)}(z)$ whereas $\phi_i(z)$ are arbitrary primary fields with dimensions $\Delta_i$, i = = 1,2,...,N. First of all, let us note that the state

$$|\chi_3\rangle = \left\{ (\Delta+2) L_{-3} - 2 L_{-1} L_{-2} + \frac{1}{\Delta+1} L_{-1}^3 \right\} |\Delta\rangle \tag{D.2}$$

(where $|\Delta\rangle$ is the primary state with dimension $\Delta$) is the null-vector (with dimension $\Delta$+3) provided $\Delta$ takes any one of the values $\Delta_{(1,3)}$ or $\Delta_{(3,1)}$, i.e.,

$$\Delta = \frac{7 - c \pm \sqrt{(c-1)(c-25)}}{6} \tag{D.3}$$

An equivalent statement is that the operator

$$\begin{aligned} \chi_{\Delta+3}(z) = {} & (\Delta+2)\, \psi^{(-3)}(z) - \\ & - 2 \frac{\partial}{\partial z} \psi^{(-2)}(z) + \frac{1}{\Delta+1} \frac{\partial^3}{\partial z^3} \psi(z) \end{aligned} \tag{D.4}$$

is the null-field of dimension $\Delta$+3. In (D.4), the $\psi^{(-k)}(z)$ are the secondaries of the degenerate field $\psi(z)$ $\left(= \psi_{(1,3)} \text{ or } \psi_{(3,1)}\right)$ and $\Delta$ is given by (D.3). The differential equation for the correlator (D.1) follows from the condition (D.1)

$$\chi_{\Delta+3} = 0 \tag{D.5}$$

According to (3.37), we have

$$\langle \psi^{(-2)}(z)\, \phi_1(z_1) \dots \phi_N(z_N) \rangle = \Big\{ \sum_{i=1}^{N} \frac{\Delta_i}{(z-z_i)^2} + \\ + \sum_{i=1}^{N} \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \Big\} \langle \psi(z)\, \phi_1(z_1) \dots \phi_N(z_N) \rangle \quad ; \tag{D.6}$$

$$\langle \psi^{(-3)}(z)\, \phi_1(z_1) \dots \phi_N(z_N) \rangle = -\Big\{ \sum_{i=1}^{N} \frac{2\Delta_i}{(z-z_i)^3} + \\ + \sum_{i=1}^{N} \frac{1}{(z-z_i)^2} \frac{\partial}{\partial z_i} \Big\} \langle \psi(z)\, \phi_1(z_1) \dots \phi_N(z_N) \rangle \tag{D.7}$$

Substituting (D.4) into (D.5), and taking into account (D.6) and (D.7), one gets the third order differential equation

$$\Big\{ \frac{1}{\Delta+1} \frac{\partial^3}{\partial z^3} - \sum_{i=1}^{N} \frac{2\Delta\Delta_i}{(z-z_i)^3} - \sum_{i=1}^{N} \frac{\Delta}{(z-z_i)^2} \frac{\partial}{\partial z_i} - \\ - \sum_{i=1}^{N} \frac{2\Delta_i}{(z-z_i)^2} \frac{\partial}{\partial z} - \sum_{i=1}^{N} \frac{2}{z-z_i} \frac{\partial^2}{\partial z \partial z_i} \Big\} \langle \psi(z)\, \phi_1(z_1) \dots \phi_N(z_N) \rangle = 0 \tag{D.8}$$

In the particular case N = 3, the derivatives $\partial/\partial z_i$ can be excluded by means of the projective Ward identities (A.7). Simple calculations lead to the following ordinary differential equation

$$\Big\{ \frac{1}{\Delta+1}\frac{d^3}{dz^3} + \sum_{i=1}^{3} \frac{1}{z-z_i}\frac{d^2}{dz^2} + \sum_{i=1}^{3} \frac{\Delta-2\Delta_i}{(z-z_i)^2}\frac{d}{dz} -$$

$$-\sum_{i=1}^{3} \frac{2\Delta\Delta_i}{(z-z_i)^3} + \sum_{i<j}^{3} \frac{2\Delta+2+\Delta_{ij}}{(z-z_i)(z-z_j)}\frac{d}{dz} + \tag{D.9}$$

$$+\sum_{i<j}^{3} \frac{\Delta+\Delta_{ij}}{(z-z_i)(z-z_j)}\Big(\frac{1}{z-z_i}+\frac{1}{z-z_j}\Big)\Big\}\langle \psi(z)\phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = 0$$

where $\Delta_{12} = \Delta_1+\Delta_2-\Delta_3$; $\Delta_{13} = \Delta_1+\Delta_3-\Delta_2$; $\Delta_{23} = \Delta_2+\Delta_3-\Delta_1$.

## APPENDIX E

As is well known (see, for instance, [15] and references therein), the two-dimensional Ising model is equivalent to the theory of free Majorana fermions. In the continuous limit, this theory is described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\psi\frac{\partial}{\partial \bar{z}}\psi + \frac{1}{2}\bar{\psi}\frac{\partial}{\partial z}\bar{\psi} + m\bar{\psi}\psi \qquad \text{(E.1)}$$

where m is the mass parameter proportional to $T-T_c$, and $(\psi,\bar{\psi})$ is the two-component Majorana field[25)]. In what follows, we shall consider the critical point only, where this field is massless

$$m = 0 \qquad \text{(E.2)}$$

According to (E.1), in this case the fields $\psi$, $\bar{\psi}$ satisfy the equation of motion

$$\frac{\partial}{\partial \bar{z}}\psi = 0 \quad ; \quad \frac{\partial}{\partial z}\bar{\psi} = 0 \qquad \text{(E.3)}$$

and therefore these fields are analytic functions of the variables z and $\bar{z}$, respectively. We shall write

$$\psi = \psi(z) \quad ; \quad \bar{\psi} = \bar{\psi}(\bar{z}) \qquad \text{(E.4)}$$

The stress energy tensor corresponding to this theory can be computed straightforwardly. In the case (E.2), it is traceless and the components (2.5) are given by

$$T(z) = -\frac{1}{2} :\psi(z)\frac{\partial}{\partial z}\psi(z): \qquad \text{(E.5)}$$
$$\bar{T}(\bar{z}) = -\frac{1}{2} :\bar{\psi}(\bar{z})\frac{\partial}{\partial \bar{z}}\bar{\psi}(\bar{z}):$$

One can easily verify that the fields (E.5) satisfy the Virasoro algebra (2.21), the central charge c being

$$c = 1/2 \qquad \text{(E.6)}$$

The fundamental fields $\psi$ and $\bar{\psi}$ satisfy the relations (1.16), i.e., these fields are primary ones. The dimensions of the field $\psi(z)$ $[\bar{\psi}(\bar{z})]$ are $\Delta = \frac{1}{2}$, $\bar{\Delta} = 0$ ($\Delta=0$, $\bar{\Delta}=\frac{1}{2}$). It can be shown that four conformal families $[I]$, $[\psi]$, $[\bar{\psi}]$, $[:\bar{\psi}\psi:]$ constitute a complete set of fields $\{A_j\}$ forming the operator algebra (1.6).

Let us take, for instance, the field $\psi(z)$. This primary field coincides with the degenerate field $\psi_{(2,1)}(z)$ [see (6.13)]. Actually, the operator product expansion for $T(\xi)\psi(z)$ [which is easily computed if (E.5) is used] is given (up to the first three terms) by

$$T(\xi)\psi(z) = \frac{1}{2}\frac{1}{(\xi-z)^2}\psi(z) + \frac{1}{\xi-z}\frac{\partial}{\partial z}\psi(z) + \qquad \text{(E.7)}$$
$$+ \frac{3}{4}\frac{\partial^2}{\partial z^2}\psi(z) + O(\xi-z)$$

which shows that its secondary field vanishes. Therefore, the correlation functions involving the degenerate field $\psi(z)$ satisfy the differential equation

$$\left\{ \frac{3}{4} \frac{\partial^2}{\partial z^2} - \sum_{i=1}^{N} \frac{\Delta_i}{(z-z_i)^2} - \sum_{i=1}^{N} \frac{1}{z-z_i} \frac{\partial}{\partial z_i} \right\} \times$$

$$\langle \psi(z)\, \phi_1(z_1) \ldots \phi_N(z_N) \rangle = 0 \tag{E.8}$$

where $\phi_i(z)$ are arbitrary primary fields [which are local themselves, but not necessarily local with respect to $\psi(z)$]. In particular, the correlation functions

$$\langle \psi(z)\psi(z_1) \ldots \psi(z_N) \rangle \tag{E.9}$$

(which can be computed using the Wick rules) satisfy (E.8).

On the other hand, the critical Ising model can be described in terms of either the order parameter field $\sigma(z,\bar{z})$ or the disorder parameter field $\mu(z,\bar{z})$ [26]. Obviously, the fields $\sigma$ and $\mu$ are primary ones. These fields have zero spins, i.e., $\Delta_\sigma = \bar{\Delta}_\sigma$; $\Delta_\mu = \bar{\Delta}_\mu$, and due to the Krammer-Wanier symmetry [15], have the same scale dimensions

$$\Delta_\sigma = \Delta_\mu = \Delta \tag{E.10}$$

The fields $\sigma(z,\bar{z})$ and $\mu(z,\bar{z})$ are neither local with respect to the fields $\psi(z)$ and $\bar{\psi}(\bar{z})$ not mutually local [15]. In fact, the correlation function

$$\langle \psi(z)\,\sigma(\xi_1) \cdots \sigma(\xi_{2N-1})\,\mu(\xi_{2N}) \cdots \mu(\xi_{2M}) \rangle \tag{E.11}$$

is a double valued analytic function of z which acquires a phase factor (-1) after the analytical continuation around any of the singular points $z_k = \xi_k^1 + i\xi_k^2$; $k = 1,2,\ldots,2M$. It follows from the definition [15] that the products $\psi(\xi)\sigma(z,\bar{z})$ can be expanded as

$$\psi(\zeta)\sigma(z,\bar{z}) = (\zeta-z)^{-1/2}\{\mu(z,\bar{z}) + O(\zeta-z)\}$$
$$\psi(\zeta)\mu(z,\bar{z}) = (\zeta-z)^{-1/2}\{\sigma(z,\bar{z}) + O(\zeta-z)\} \qquad \text{(E.12)}$$

Substituting these expansions into the differential Eq. (E.8), one gets a characteristic equation determining the parameter $\Delta$ :

$$\Delta = 1/16 \qquad \text{(E.13)}$$

in agreement with the known value of the scale dimension of the spin field $d_\sigma = 2\Delta = 1/8$ [15]. So, the differential Eq. (E.8), together with the qualitative properties (E.12) of the operator algebra, enables one to compute exactly the dimension of the field $\sigma(z,\bar{z})$.

We have now to compute the correlation functions of the order and the disorder fields

$$\langle \sigma(\xi_1)\cdots\sigma(\xi_{2N})\mu(\xi_{2N+1})\cdots\mu(\xi_{2M})\rangle \qquad \text{(E.14)}$$

Note that the double-valued function (E.11) can be represented in the form

$$\begin{aligned}\langle \psi(z)\sigma(\xi_1)\cdots\mu(\xi_{2M})\rangle = \\ = \prod_{i=1}^{2M}(z-z_i)^{-1/2} P(z/z_i,\bar{z}_i)\end{aligned} \qquad \text{(E.15)}$$

where $P(z|z_i,\bar{z}_i)$ is a polynomial in z:

$$P(z/z_i,\bar{z}_i) = \sum_{k=0}^{2M-1}(z-z_{2N})^k G_k(z_i,\bar{z}_i) \; ; \qquad \text{(E.16)}$$

the order 2M-1 of this polynomial is determined by the asymptotic condition

$$\Psi(z) \sim z^{-1} \quad ; \quad z \to \infty \tag{E.17}$$

The coefficients $G_k$ are some functions of $z_1,\ldots,z_{2M}$, $\bar{z}_1,\ldots,\bar{z}_{2M}$. Due to (E.12), the coefficient $G_0(z_i,\bar{z}_i)$ coincides with the correlation function (E.14). Substituting (E.15) into the differential Eq. (E.8), one gets the differential equations for coefficients $G_k(z_i,\bar{z}_i)$ which enable one to compute the correlator (E.14).

In fact, the differential equations for the correlation functions (E.14) can be obtained in a simpler way. One can note, comparing (E.13) with (6.13), that the field $\sigma(z,\bar{z})$ is a degenerate field $\phi_{(1,2)}$ with respect to both variables z and $\bar{z}$.

The same is valid for the field $\mu(z,\bar{z})$. Therefore, the correlation functions (E.14) satisfy the differential equations

$$\left\{\frac{4}{3}\frac{\partial^2}{\partial z_i^2} - \sum_{i\neq j}^{2M}\frac{1/16}{(z_i-z_j)^2} - \sum_{j\neq i}^{2M}\frac{1}{z_i-z_j}\frac{\partial}{\partial z_j}\right\}\times \tag{E.18}$$
$$\langle\sigma(z_1,\bar{z}_1)\ldots\sigma(z_{2N},\bar{z}_{2N})\mu(z_{2N+1},\bar{z}_{2N+1})\ldots\mu(z_{2M},\bar{z}_{2M})\rangle = 0$$

(where $i = 1,2,\ldots,2M$) and the differential equations obtained from (E.18) by the substitution $z_i \to \bar{z}_i$.

Let us consider, for example, the four-point correlation function

$$G(\xi_1,\xi_2,\xi_3,\xi_4) = \langle\sigma(\xi_1)\sigma(\xi_2)\sigma(\xi_3)\sigma(\xi_4)\rangle = \tag{E.19}$$
$$= \left[(z_1-z_3)(z_2-z_4)(\bar{z}_1-\bar{z}_3)(\bar{z}_2-\bar{z}_4)\right]^{-1/8} Y(x,\bar{x})$$

where $Y(x,\bar{x})$ is some function of the anharmonic quotients

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} \quad ; \quad \bar{x} = \frac{(\bar{z}_1 - \bar{z}_2)(\bar{z}_3 - \bar{z}_4)}{(\bar{z}_1 - \bar{z}_3)(\bar{z}_2 - \bar{z}_4)} \tag{E.20}$$

[we took into account (A.8)]. In this case the differential Eq. (E.18) is reduced to the following form

$$\Big\{ \tfrac{4}{3} \frac{\partial^2}{\partial x^2} - \frac{1}{16}\Big[\frac{1}{x^2} + \frac{1}{(x-1)^2}\Big] + \frac{1}{8}\,\frac{1}{x(x-1)} + \Big[\frac{1}{x} + \frac{1}{x-1}\Big]\frac{\partial}{\partial x}\Big\} Y(x,\bar{x}) = 0 \tag{E.21}$$

The same equation with respect to $\bar{x}$ is also valid.

Substituting

$$Y(x,\bar{x}) = [x\bar{x}(1-x)(1-\bar{x})]^{-1/8}\, u(x,\bar{x}) \tag{E.22}$$

one gets the following equation for $u(x,\bar{x})$

$$\Big\{ x(1-x)\frac{\partial^2}{\partial x^2} + \Big(\frac{1}{2} - x\Big)\frac{\partial}{\partial x} + \frac{1}{16}\Big\} u(x,\bar{x}) = 0 \tag{E.23}$$

The change of variables

$$x = \sin^2\theta \quad ; \quad \bar{x} = \sin^2\bar{\theta} \tag{E.24}$$

reduces (E.23) to

$$\left(\partial^2/\partial\theta^2 + 1/4\right) u(\theta,\bar{\theta}) = 0 \tag{E.25}$$

The equation obtained from (E.25) by the substitution $\theta \to \bar{\theta}$ is also valid. Therefore, the general solution of these differential equations has the form

$$U(\theta,\bar{\theta}) = u_{11}\cos\frac{\theta}{2}\cos\frac{\bar{\theta}}{2} + u_{12}\cos\frac{\theta}{2}\sin\frac{\bar{\theta}}{2} + + u_{21}\sin\frac{\theta}{2}\cos\frac{\bar{\theta}}{2} + u_{22}\sin\frac{\theta}{2}\sin\frac{\bar{\theta}}{2} \qquad (E.26)$$

where $u_{\alpha\beta}$ ($\alpha,\beta$ = 1,2) are arbitrary constants.

Note that two independent solutions of (E.21) coincide with the conformal blocks [see (B.9)]

$$\mathcal{F}(1/16,\ 0,\ x) = [x(1-x)]^{-1/8}\cos\theta/2$$
$$\mathcal{F}(1/16,\ 1/2,\ x) = [x(1-x)]^{-1/8}\sin\theta/2 \qquad (E.27)$$

and therefore Eq. (E.26) can be considered as the decomposition (4.11), the coefficients $u_{\alpha\beta}$ being the structure constants.

Since the field $\sigma(z,\bar{z})$ is local, the correlation function (E.20) should be single-valued in the Euclidean domain

$$\bar{x} = x^* \qquad (E.28)$$

where the asterisk denotes complex conjugation. As is clear from (E.24), the analytical continuation of the variables x and $\bar{x}$ around the singular point $x = \bar{x} = 0$ corresponds to the substitution

$$\theta \to -\theta$$
$$\bar{\theta} \to -\bar{\theta} \qquad (E.29)$$

The function (E.26) is unchanged under this transformation, provided

$$u_{12} = u_{21} = 0 \qquad (E.30)$$

The same investigation of the singular point $x = \bar{x} = 1$ (or, equivalently, imposing the crossing symmetry condition) leads to the relation

$$\mathcal{U}_{11} = \mathcal{U}_{22} \tag{E.31}$$

The over-all factor in (E.26) depends on the $\sigma$ field normalization. We shall normalize this field so that

$$\langle \sigma(z,\bar{z})\, \sigma(0,0) \rangle = [z\,\bar{z}]^{-1/8} \tag{E.32}$$

Then

$$\mathcal{U}(\theta,\bar{\theta}) = \cos\frac{\theta-\bar{\theta}}{2} \tag{E.33}$$

The four-point function given by Eqs. (E.20), (E.22) and (E.33) is in agreement with the previous result (see Ref. [16]) obtained by a different method.

Note that because of (E.27), the four-point function (E.20) can be represented as

$$\begin{aligned} G = {} & \mathcal{F}(1/16, 0, x)\, \bar{\mathcal{F}}(1/16, 0, \bar{x}) + \\ & + \mathcal{F}(1/16, 1/2, x)\, \bar{\mathcal{F}}(1/16, 1/2, \bar{x}) \end{aligned} \tag{E.34}$$

It is evident from this formula that only two conformal families contribute to the operator product expansion of $\sigma(\xi)\sigma(0)$. The corresponding primary fields have dimensions $\Delta = \bar{\Delta} = 0$ and $\Delta = \bar{\Delta} = \frac{1}{2}$. The first of them is obviously identified with the identity operator I, whereas the second is known as the energy density field

$$\mathcal{E}(z,\bar{z}) = \bar{\psi}(\bar{z})\, \psi(z) \tag{E.35}$$

The four-point correlation function

$$H(\xi_1,\xi_3;\xi_2,\xi_4) = \langle \sigma(\xi_1)\, \mu(\xi_2)\, \sigma(\xi_3)\, \mu(\xi_4) \rangle \tag{E.36}$$

can be represented in the form

$$H = [(z_1-z_3)(z_2-z_4)(\bar{z}_1-\bar{z}_3)(\bar{z}_2-\bar{z}_4)]^{-1/8}\,\tilde{Y}(x,\bar{x}) \tag{E.37}$$

where the function $\tilde{Y}$ satisfies the same differential Eq. (E.21). An investigation, similar to the one performed above, leads to the result

$$\tilde{Y}(x,\bar{x}) = [x\bar{x}(1-x)(1-\bar{x})]^{-1/8} \sin\frac{\theta+\bar{\theta}}{2} \tag{E.38}$$

Therefore the function (E.36) is

$$\begin{aligned} H &= \mathcal{F}(1/16, 0, x)\,\bar{\mathcal{F}}(1/16, 1/2, \bar{x}) + \\ &\quad + \mathcal{F}(1/16, 1/2, x)\,\bar{\mathcal{F}}(1/16, 0, \bar{x}) \end{aligned} \tag{E.39}$$

This formula corresponds to the following operator product expansion

$$\begin{aligned} \sigma(z,\bar{z})\mu(0,0) &= z^{3/8}\bar{z}^{-1/8}\{\psi(z) + O(z,\bar{z})\} + \\ &\quad + z^{-1/8}\bar{z}^{3/8}\{\bar{\psi}(\bar{z}) + O(z,\bar{z})\} \end{aligned} \tag{E.40}$$

which is in agreement with the idea of the field $\psi$ being the regularized product $:\sigma\mu_i:$.

To avoid misunderstandings, let us stress that there are three different sets of fields

$$\begin{aligned} \{A_j\} &= \{[I], [\psi], [\bar{\psi}], [\varepsilon]\}, \\ \{B_j\} &= \{[I], [\sigma], [\varepsilon]\}, \\ \{C_j\} &= \{[I], [\mu], [\varepsilon]\} \end{aligned} \tag{E.41}$$

Each of these sets forms the closed operator algebra and is appropriate to describe the critical Ising field theory. All the fields entering the same set are mutually local, whereas the fields entering different sets are in general non-local with respect to each other.

FOOTNOTES

1) Although the projective group (1.15) and the complete conformal group G are both consequences of (1.2) and therefore appear in quantum field theory together, we found it instructive to consider first the general consequences of the projective symmetry. The corresponding formulae, which are certainly nothing but the particular case D = 2 of the results of Refs. [2-4], are presented in Appendix A.

2) The spin $S_n$ of a local field can take an integer or half-integer value only.

3) The representation $V_n$ is known as the Verma modulus over the Virasoro algebra (see, for example, [6]). This representation is evidently characterized by the $\Delta_n$ parameter only.

4) Here and below, we generally consider correlation functions in the complex space $C^2$, see the Introduction.

5) Equation (2.12) corresponds to the following transformation of T(z) under the finite conformal substitution (1.9)

$$T(z) \rightarrow \left(\frac{d\xi}{dz}\right)^2 T(\xi) + \frac{c}{12}\{\xi, z\}$$

where $\{\xi,z\}$ is the Schwartz derivative [12]

$$\{\xi, z\} = \left(\frac{d^3\xi}{dz^3} \Big/ \frac{d\xi}{dz}\right) - \frac{3}{2}\left(\frac{d^2\xi}{dz^2} \Big/ \frac{d\xi}{dz}\right)^2$$

Note that the Schwartz derivative satisfies the following composition law

$$\{w, z\} = \left(\frac{d\xi}{dz}\right)^2 \{w, \xi\} + \{\xi, z\}$$

6) This central extension has been discovered by Gel'fand and Fuks [10].

7) It can be shown that these correlators coincide with those of the fields

$$T^{(0)} = \varphi_z \varphi_z + 2\alpha_0 \varphi_{zz}$$

where $\phi$ is a free massless boson field and the parameter $\alpha_0$ is defined by the formula

$$c = 1 - 24\alpha_0^2$$

8) One can easily verify that the operators (3.19), where $k = 0, \pm 1, \pm 2, \ldots$, satisfy the Virasoro algebra (2.21). Obviously, the operators $L_n$ introduced in Section 2 are none other than $L_n(0)$.

9) This statement does not hold for some special values of $\Delta_n$; see Section 5.

10) This statement is not precise, because we neglected the $\bar{z}$ dependence of the fields; the correct definition is given below.

11) Obviously, the fields $T(z)$ and $\bar{T}(\bar{z})$ are not primary fields; they belong to the conformal family $[I]$ of the identity operator.

12) To obtain (4.5) in the simplest way, one can substitute the explicit formula (3.18) and deform the integration contours so as to enclose them around the singularities $z_1, z_2, \ldots, z_N$.

13) This notation is not complete because it says nothing about the second dimension $\bar{\Delta}$ of the primary field $\psi$. This fact, which should always be kept in mind, does not violate the conclusions we make below.

14) If both dimensions $\Delta$ and $\bar{\Delta}$ of the field $\psi$ are zero, this field does not depend on the co-ordinates at all and coincides with the identity operator I.

15) The simplest example of these equations is (5.14).

16) The following interpretation of Eq. (5.17) is worth noting. Let $\psi(z)$ stand for one of the fields $\psi_{(1,2)}$ or $\psi_{(2,1)}$, $\delta$ being the corresponding dimension $\Delta_{(1,2)}$ or $\Delta_{(2,1)}$. Then the field $\psi(z)$ satisfies the operator equation

$$\frac{\partial^2}{\partial z^2}\psi(z) = \gamma :T(z)\psi(z): \qquad (*)$$

where $\gamma = 2(2\delta+1)/3$, whereas the singular operator product $T(z)\psi(z)$ is regularized by means of the subtractions

$$:T(z)\psi(z): = \lim_{\zeta\to z}\left\{T(\zeta)\psi(z) - \frac{\delta}{(\zeta-z)^2}\psi(z) - \frac{1}{\zeta-z}\frac{\partial}{\partial z}\psi(z)\right\}$$

The classical limit of Eq. (*) [which corresponds to the choice $\psi = \psi_{(1,2)}$ and $c \to \infty$] is an essential part of the classical theory of the Liouville equation (see, for example, [13]). We suppose that Eq. (*) plays an analogous role in the quantum theory of this equation, which is apparently associated with the string theory [14]. We intend to discuss this point in another paper.

17) To determine these factors in quantum field theory, one should take into account the associativity condition for the operator algebra and local properties of the fields.

18) It is interesting to understand the connection of the truncation phenomenon with monodromy properties of the differential equations satisfied by the correlation functions. This problem can be most easily investigated for the four-point functions where one deals with the ordinary differential equations. If all the fields involved are degenerate, the space of solutions of the differential equations contains the subspace invariant under the monodromy transformations. The solutions, belonging to this subspace, correspond to the degenerate fields $\psi_{(k,\ell)}$ $(k,\ell > 0)$ in (6.7) and these very solutions contribute to the correlation function.

19) The "fusion rule" (6.7) can be obtained from the following formula

$$\psi_{(n,m)} = (\psi_{(1,2)})^{m-1} (\psi_{(2,1)})^{n-1}$$

for the degenerate field $\psi_{(n,m)}$. Although this formula scarcely has a precise mathematical meaning, one can use it to derive (6.7) assuming the associativity and taking into account the truncation phenomenon.

20) To avoid misunderstandings, let us stress that these statements by no means exclude the possibility of the existence of quantum field theory at $c \quad 1$, but rather they prevent us from including the degenerate fields in the operator algebra.

21) In fact, these differential equations are not all independent; they follow from the two "basic" ones.

22) Certainly, the analysis of the dimensions (6.13) does not prove that the operator algebra contains only three primary fields. To elucidate the structure of the fields constituting the operator algebra, one should take into account the $\bar{z}$ dependence and the local properties of the fields. For the model under consideration, this is done in Appendix E.

23) V. Dotsenko has noticed that the spectrum of dimensions associated with the minimal model

$$p/q = 5/6 \quad ; \quad c = 4/5$$

contains some dimensions characteristic for the three-state Potts model.

24) Obviously, any primary field is quasiprimary whereas there are infinitely many quasiprimary fields which are secondaries.

25) The field $\bar{\phi}$ is an independent component but in general it is not complex conjugated value of the field $\phi$.

26) The fields $\sigma$ and $\mu$ are the scaling limit of the lattice spin $\sigma_{n,m}$ and the dual spin $\mu_{n+\frac{1}{2},m+\frac{1}{2}}$, respectively; see Ref. [16] for a detailed definition.

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FIGURE CAPTIONS

Figure 1: "Diagram of dimensions". The dimension $\Delta = \Delta_0 + {}^1/_4 \alpha^2$ is associated with each point of the plane, $\alpha$ being proportional to the distance between the point and the dotted line. The dots with co-ordinates (n,m) correspond to the dimensions $\Delta_{(n,m)}$ described by Kac's formula (5.9).

Figure 2: Diagram of dimensions corresponding to the case $tg\Theta = {}^3/_4$ ($c=\frac{1}{2}$). The degenerate conformal families associated with the dots inside the rectangle form the closed operator algebra.

Figure 3: Diagram of dimensions for the case $tg\Theta = {}^4/_5$ ($c={}^7/_{10}$).

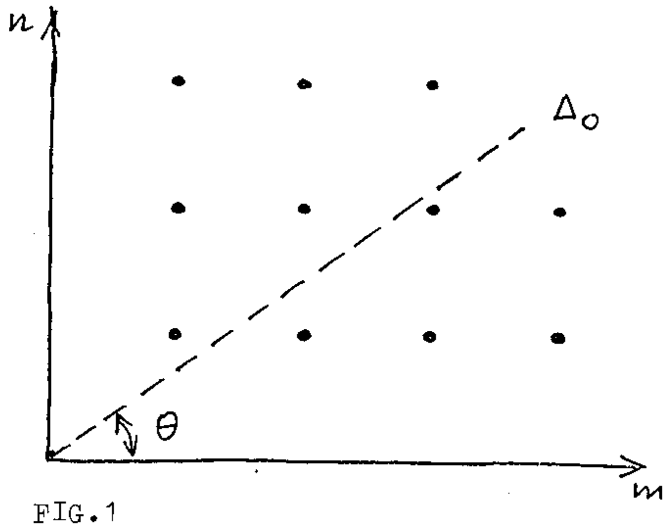


FIG.1

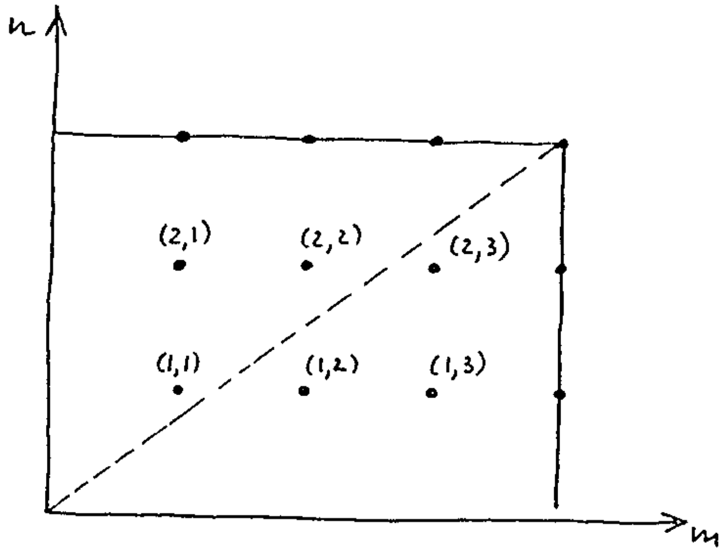


FIG. 2

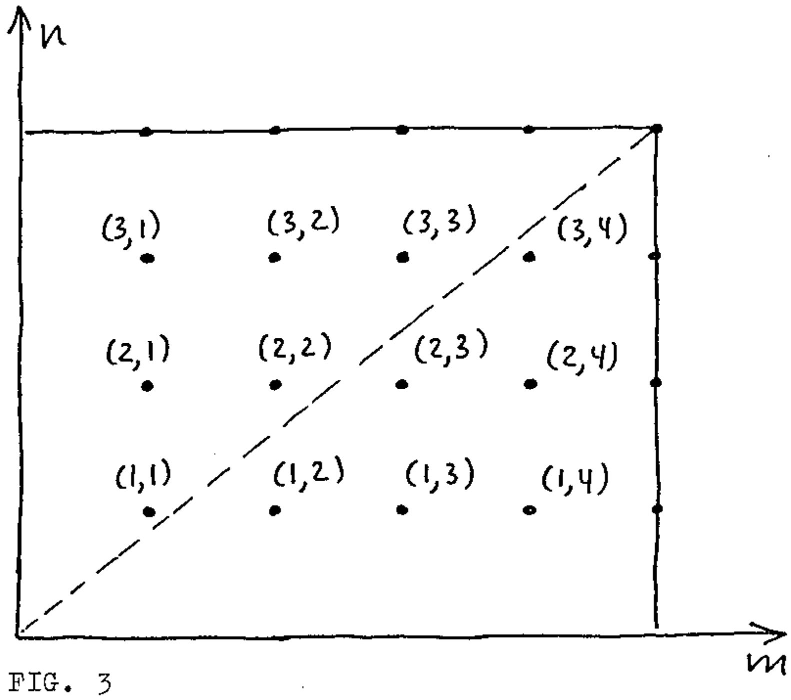


FIG. 3